

[Reply on RC1]

This paper describes several improvements to the classical
SISL method for a nonhydrostatic dycore. These include a correct
5 treatment of the free-slip boundary condition at the surface
and use a non-constant linear operator in the semi-implicit solver.
The results improve stability of the scheme especially in
the presence of steep topography, a known issue for SISL methods.
The improvements are well documented and demonstrated in challenging
10 test cases.

[Response]

We are very grateful to Referee 1 for the positive assessment of our study and the many useful comments, which have
helped us improve the paper. We respond to the specific comments below.

[Note]

During the revision, we changed $\mathcal{Q}$ to $\mathcal{P}$ in Eq. (27e) to make it consistent with Eq. (C24). We found the
corresponding bug at one location in the source code of the 2D x - z model and corrected the bug. The results
remained almost unchanged after the bug was corrected. Clearly visible differences occur only in Figs. 3c–d for case
S1, where the small oscillations present both before and after the correction make the differences more noticeable. We
20 have updated Figs. 1–5 for the test cases with mountains. Figures 6–9 have not been updated because the bug does not
affect the test cases without mountains where the no-recalculation method is used. Despite these corrections, the
overall results and conclusions of the paper remain unchanged.

25 Minor comments:

1. The results are presented in a x - z model, with Fourier series
used in the x direction. From the introduction, I concluded that
this x - z model is used for evaluation, and serves as a simplified
30 version of a 3D spherical harmonic based model, which is confirmed
in the paper's conclusions. This could be mentioned
explicitly in the introduction.

[Response]

35 Thank you for the important suggestion. We have added the following sentence to the Introduction:

"The model is essentially a two-dimensional version of the corresponding three-dimensional global model using spherical harmonics or double Fourier series and is used as a simpler framework for testing and improving numerical methods before they are applied to the global model."

We have also revised the following sentence in the Abstract:

40 "A two-dimensional x–z nonhydrostatic spectral model has been developed as a simpler framework for testing and improving numerical methods before they are applied to the corresponding global model."

2. Section 2.2: Why T and Q (or ln T and P) instead of potential temperature and

45 PHI as prognostic variables? For a mass coordinate models, the use of potential temperature and PHI is common in finite volume / finite element methods, since it avoids the need compute the D3 operator. (i.e. Dubos & Tort MWR 2014, Taylor et al. JAMES 2019)

The T/Q approach used here seems common in spectral methods.

50 I wonder if the authors can remind the reader what these reasons are and comment on if they are still important at nonhydrostatic resolutions.

[Response]

55 Thank you for the useful comment. We are interested in formulations using potential temperature and geopotential as prognostic variables.

In this study, we use prognostic equations that are often used in nonhydrostatic spectral models. We carefully select the prognostic variables used in spectral space to improve numerical stability and computational efficiency. One advantage of the present equations is that they allow similar formulations to be used in hydrostatic and nonhydrostatic models. Another advantage of these equations is that they allow the simultaneous equations associated with the semi-implicit scheme to be efficiently solved in spectral models. In this paper, the prognostic equations are discretized in time using a semi-implicit scheme with constant coefficient and non-constant coefficient linear terms. The simultaneous equations associated with the constant-coefficient linear terms can be easily solved in spectral space. This solver can also be used as a preconditioner for the equations including the non-constant-coefficient linear terms. In addition, only the non-constant-coefficient linear terms need to be recalculated in the iterations. With these features, stable calculations can be performed using the no-recalculation or single-recalculation method in many of the test cases examined in this study.

65 The use of potential temperature and geopotential avoids the explicit calculation of D_3 in the thermodynamic equation. However, it would require reformulation of the simultaneous equations associated with the semi-implicit scheme. It is not clear whether the simultaneous equations derived from this alternative formulation could be solved as easily as the

present equations in spectral space. It is also not clear whether this alternative formulation would provide comparable or better numerical stability, particularly over steep orography. We think that this alternative formulation is worth investigating. However, the present formulation provides stable and computationally efficient integrations.

We have added the following brief explanation of the reasons for the present choice of prognostic variables in Sect. 2.2:

"Almost the same prognostic equations are often used in nonhydrostatic spectral models. An advantage of these equations is that they allow similar formulations to be used in hydrostatic and nonhydrostatic models. Another advantage of these equations is that they allow the simultaneous equations associated with the semi-implicit scheme to be efficiently solved in spectral models. In this paper, the prognostic equations are discretized in time using a semi-implicit scheme with constant coefficient and non-constant coefficient linear terms (Sects. 2.4 and 2.5). The prognostic variables in spectral space are selected to improve numerical stability (Sect. 2.5). The simultaneous equations associated with the constant-coefficient linear terms can be easily solved in spectral space (Step 3 in Sect. 2.6). By utilizing this solver, the simultaneous equations including the non-constant-coefficient linear terms can also be efficiently solved (Step 4 in Sect. 2.6)."

3. Line 128: The derivation here shows that the free-slip boundary condition gives simultaneous conditions for the surface vertical velocity and a Neumann like pressure boundary condition. The authors note that this formulation gives improved results over an extrapolation (Eq. 21) used in the MRI model. It appears to this reviewer that the MRI model is using an approximation that would result in a boundary condition slightly inconsistent with the free-slip condition. I think the authors should state this.

[Response]

Thank you for the suggestion. We agree with the reviewer and have added the following sentence to Sect. 2.3:

"The approximation in Eq. (21) is not fully consistent with the surface boundary condition in Eq. (15)."

4. Line 144: The authors mention the nonconstant terms come from Eq. 7b and Eq. 13. Given that there are choices of prognostic variables which can eliminate Eq. 13, is it possible to also simplify 7b through choice of prognostic variables in order to obtain only constant coefficient linear terms?

[Response]

105 Thank you for the interesting question. We have found that the use of $\tilde{\mathbf{v}} = \mathbf{v} + \mathbf{w}/g \cdot \nabla \Phi$ instead of $\mathbf{v}$ as the prognostic variable eliminates the nonlinear term corresponding to the non-constant-coefficient linear term in Eq. (7b). However, the transformed equation contains additional terms involving $\nabla \Phi$, and Eq. (7) and other equations such as Eq. (13) become more complicated. We have not found better prognostic variables than the present ones.

110

5. Section 2.4, general comment: The authors seem to carefully consider many choices of prognostic variables and related formulation questions, and often state that choices are made because of observed improvements in stability. Sometimes, a mathematical reason is given to explain the improvements. In other cases, is it simple trial and error (trial with test cases) or can mathematical principals behind the choices be found in all cases?

[Response]

120 Thank you for the useful comments. We have performed many trial-and-error tests, and have also considered mathematical reasons why each choice of prognostic variables and formulations affects numerical stability and model results. Although we have not provided rigorous mathematical proofs, we have added mathematical explanations for many of the choices.

125 In Sect. 2.5, we have already provided the following explanation for the use of Q and $\ln T$ as prognostic variables in spectral space:

"In Eqs. (10) and (30), Q is used as a prognostic variable instead of $\mathcal{P}$, which improves the numerical stability (Bénard et al., 2010). In Eqs. (9) and (29), $\ln T$ is used as a prognostic variable instead of T , which also improves the numerical stability. When $\ln T$ is used as a prognostic variable, $\bar{T}$ is not included in the coefficient of the linear term in Eq. (29c). When T is a prognostic variable, $\bar{T}$ is included in Eq. (29c), which can reduce the accuracy of the time integration of T . Based on the results of various model tests, it appears that T should be carefully and accurately integrated in time as compared with $\mathbf{v}$ and $\mathbf{w}$ to improve numerical stability."

130 We consider the following point important: " T should be carefully and accurately integrated in time as compared with $\mathbf{v}$ and $\mathbf{w}$ to improve numerical stability." Although this has not been mathematically proved, we think that it is probably true in many atmospheric models. We have added the following sentence to the explanation above to provide a possible mathematical reason why using Q improves numerical stability:

135

" Q is approximately equal to $\mathcal{P}$ when $\mathcal{P}$ is small, but using Q instead of $\mathcal{P}$ guarantees that p/π remains positive because $p/\pi = 1 + \mathcal{P} = \exp(Q) > 0$."

We have also added the following explanation to the last paragraph of Sect. 2.5:

" D and ζ are used instead of ν to make it easier to solve the simultaneous equations associated with the semi-implicit scheme. In addition, $\ln \pi_s$ is used instead of π_s so that the constant coefficient linear term in Eq. (31) provides a better approximation to the corresponding nonlinear term. The reason for using $\hat{d}$ is explained above. The reasons for using $\ln T$ and Q are explained above Eq. (32)."

We have added the following explanation of why the combination of SETTLS and second-order decentering improves numerical stability in Sect. 2.4:

"Although the terms $F^{(+)} + \beta(-L_C^{(+)} + L_C^0) + (-L_N^{(+)} + L_N^0)$ at the departure point in Eq. (23a) contain time-extrapolated values, the combination of these terms is interpolative in time. When $F = L_C + L_N$, that is, $N = 0$, these terms become $(1 - \xi)L_C^0 + \xi L_C^- + L_N^0$. Here, $(1 - \xi)L_C^0 + \xi L_C^-$ represents a temporal interpolation between L_C^0 and L_C^- for $0 \leq \xi \leq 1$; $\xi = \beta - 1 = 0.2$ is used here. Thus, the combination of SETTLS and second-order decentering with $\beta = \xi + 1$, which is a little larger than 1.0, can avoid time extrapolation, a potential source of instability. The use of $\beta > 1$ can also enhance the stabilizing effect of the semi-implicit scheme by increasing the coefficient of L_C^+ . The results in Sect. 3.2 show that numerical stability is reduced when β is changed from 1.2 to 1.0. The factor β is applied only to the constant coefficient linear terms because its application to the non-constant coefficient linear terms does not improve numerical stability in the test cases in this paper."

We have added the following analysis of the effects of $\bar{T}$, $\bar{\bar{T}}$, and β on numerical stability in Sect. 3.2:

"When second-order decentering is not applied, that is, when β is reduced from 1.2 to 1.0, all six cases, A1–A4, D1, and D2, become unstable. For numerical stability in the constant-coefficient semi-implicit scheme, the constant-coefficient linear terms should be larger than the corresponding nonlinear terms in the original prognostic equations. Therefore, in Eq. (27) for ν^+ , the coefficient $\beta\bar{T}$ in the constant-coefficient linear term should be larger than the coefficient T of the corresponding nonlinear term. In Eq. (35) for $\hat{d}^+$, the coefficient $\beta/\bar{T}$ in the constant-coefficient linear term should also be larger than $1/T$. From these conditions, $\bar{T}/\beta < T < \beta\bar{T}$ should be satisfied for the constant-coefficient semi-implicit scheme to provide sufficient stability. Reducing β from 1.2 to 1.0 narrows the temperature range satisfying this condition. Therefore, additional experiments were performed with $\beta = 1.0$, $\bar{T} = 300 \times 1.2$ K, and $\bar{\bar{T}} = 160/1.2$ K, which retains the original bounds of this temperature range. With this setting, cases A4, D1, and D2 become stable, but cases A1–A3 remain unstable. This result indicates that the above condition alone is not sufficient for numerical stability and that the effects of $\beta > 1.0$ in Eq. (29) for $\ln T^+$, in Eq. (30) for Q^+ , and in Eq. (31) for $\ln \pi_s^+$ also contribute to numerical stability."

6. Line 249: "The quadratic truncation is used..."

170 Is this equivalent to the fully dealiased grid used in spherical harmonic
models (i.e. quadratic terms will be computed with no aliasing error)

[Response]

Yes. We have revised the sentence as follows:

175 "The quadratic truncation ($M \cong I/3$) is used to avoid aliasing errors in the evaluation of quadratic nonlinear terms."

7. Line 250: "... diffusion is not required for numerical stability,
and is not used, except that specified diffusion is used in the
180 density current test case in Sect. 3.5."

Is diffusion not used because of the improved stability properties
of the formulation? Or is it that the test cases dont require
diffusion (except for the density current test case). When this formulation
is used in a 3D spherical harmonic model, will (hyper)diffusion be used?

185

[Response]

Thank you for the comment. Diffusion could be used in these test cases. However, except for the specified diffusion used
in the density current test case, we did not use diffusion to show that stable calculations can be performed with the
present formulation in the 2D x-z model.

190 Hyperdiffusion is used in our 3D spherical harmonic model. After implementing the improvements described in this
paper, we will perform some experiments without diffusion to examine the numerical stability. However, we think that
diffusion should be used for practical prediction to ensure stable and robust integrations in the global model.