

1 Causal inference in the presence of external forcing

Consider the linear SDE which is similar to the Equation (1) of Lohmann et al. (2025) with the nonlinear term neglected for the time being, in order to single out the effect of external forcing.

$$\frac{dx_i}{dt} = -1.25x_i + \sum_{i \neq j} A_{ij}x_j + \sigma \dot{w}_i + f(t), \quad i, j = 1, 2, 3 \quad (1)$$

where

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

is the adjacency matrix,

$$f(t) = \epsilon t$$

is the external forcing, and $\epsilon = 0.01$, $\sigma = 0.01$. A sample path in the presence of external forcing and one without are shown in Fig. 1a and b, respectively. Both are initialized with $\mathbf{x}(0) = (5, 2.5, -2)$.⁴ Correspondingly, the causal graphs inferred using the multivariate time series information flow estimate formula [Eq. (14) in Liang (2021)] are as displayed in Fig. 2. As can be seen from the adjacency matrix, the causal graph should be like $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1$, i.e., a cyclinc one as shown in the left of Fig. 2, no matter whether the external forcing applies or not. The one in the right subplot, however, does not show such a cyclic graph. Clearly, the causal relations between x_1 and x_3 and between x_3 and x_2 are wrong.

The external forcing problem has been a common one. The original IF formalism, as shown in Liang (2016), does include this nonautonomous situation. The problem arises in formulating the estimation, where autonomy has been assumed. By virtue of the property of invariance upon arbitrary coordinate transformation (Liang, 2018), this can be easily solved by adding an extra dimension to the system—a standard approach to transforming a nonautonomous system to an autonomous system. Specifically, what we need to do is to append the time series of $\{f(n)\}$ to $\{\mathbf{x}(n)\}$ to form an expanded vector time series $\{x_1(n), x_2(n), x_3(n), f(n)\}$, and redo the inference. The cyclic causal graph is hence easily reproduced. As shown in Fig. 3, the causal relations in the expanded network are accurately recovered. Particularly, the subnetwork (x_1, x_2, x_3) displays a cyclic causal graph. Also, it is

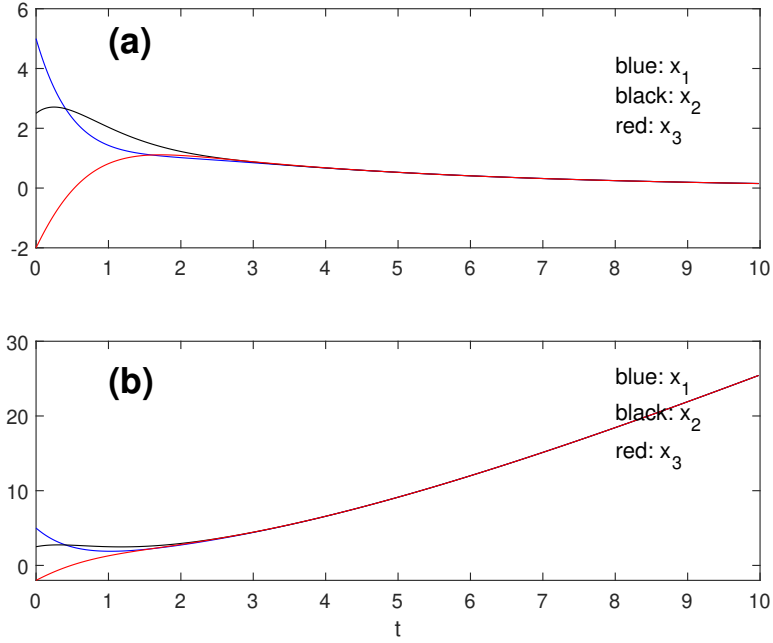


Figure 1: Sample paths generated by (1): (a) without external forcing; (b) with external forcing.

interesting to see that the forcing (frc as labeled in the plot) always causes unidirectionally to the three nodes x_i , $i = 1, 2, 3$, just as expected.

References

- [1] –, 2016: Information flow and causality as rigorous notions ab initio. PRE 94, 052201, 1-28
- [2] –, 2021: Normalized multivariate time series causality analysis and causal graph reconstruction. Entropy, 23, 679.
- [3] X.S. Liang, 2014: Unraveling the cause-effect relation between time series. Phys. Rev. E, 90, 052150.

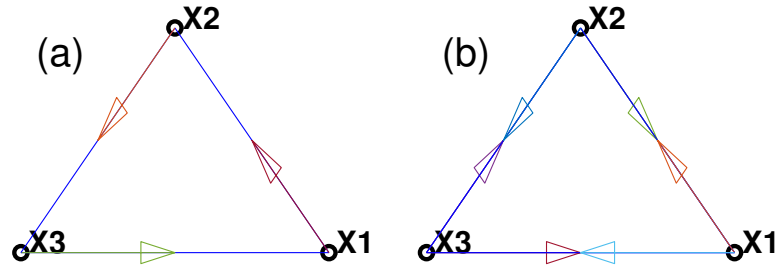


Figure 2: Causal graphs inferred from the three time series in Fig. 1 using the multivariate time series information flow estimation. Left: no external forcing; right: external forcing applied. (For the right, the inference is still with the three time series, but the result is clearly incorrect.)

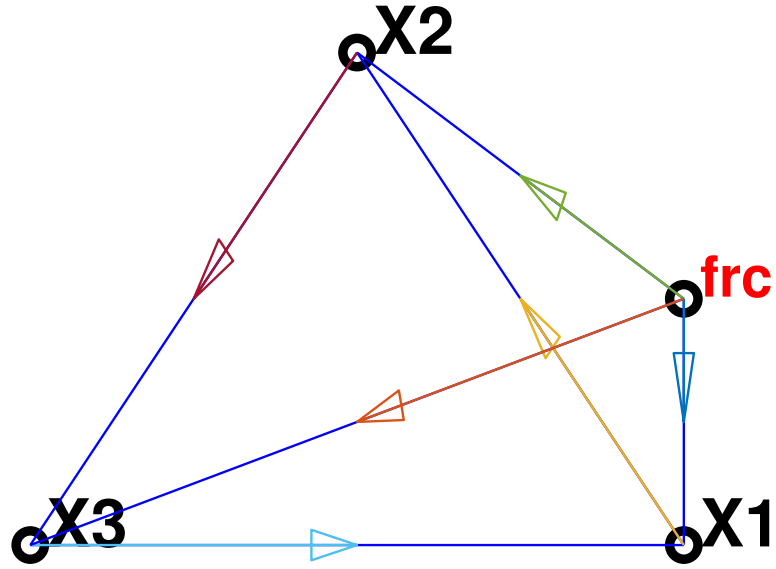


Figure 3: Causal graph inferred using the expanded vector series. The added series is the forcing (frc) as labeled in red. Note that, when forming the expanded matrix, the time series of frc should take lead by one step.