Dear Reviewer,

Thank you very much for your insightful comments and constructive suggestions on my manuscript. I have carefully considered all your feedback and made substantial revisions to address the issues raised. I believe these changes have significantly improved the quality and rigor of the paper. Please find my detailed responses to each comment below. I hope the revised manuscript meets the journal's standards and look forward to your further feedback.

Sincerely,

The author: Yaokun Li

Comments

Line 34-49: The emphasis on classical sufficient conditions for stability is outdated and needs to be rewritten to bring it up to date. In particular, a major sufficient condition for inviscid shear instability has been established, called the Hurdle Theorem. See Deguchi et al. (2024, J. Fluid Mech., 997, A25, doi:10.1017/jfm.2024.728) and Read and Dowling (2026, Encycl. Atmos. Sci. (3rd Ed.) 4, 263—283, Academic Press, doi:10.1016/B978-0-323-96026-7.00211-3).

Response: Thank you for highlighting the need to update the discussion on stability conditions and for bringing the Hurdle Theorem to my attention.

In the revised manuscript, I have added the introduction of the Hurdle Theorem, the sufficient condition for inviscid shear instability, after describing the classical sufficient conditions. The added contents are "More recently, Deguchi et al. (2024) established a sufficient condition for inviscid shear instability, called the hurdle theorem, which states that a flow is unstable if there is an interval in the flow domain for which the reciprocal Rossby Mach number (a quantity defined in terms of the zonal flow and potential vorticity distribution) surpasses a certain threshold or "hurdle". The theorem offers new insights into the theoretical understanding of pattern formation in planetary atmospheres." They are in Line 51-55.

Ray tracing is heavily used, but without first establishing the context for which it is accurate and discussing general conditions when it is inaccurate, such as inhomogeneous environments and proximity to focusing points (caustics). This can be fixed by adding a short paragraph discussing the issues early in the introduction. There is a brief mention of such an issue on Line 132, another on Line 144-152, and a workaround on Line 191+ (Section 3). These could usefully be tied into an introductory short paragraph on the general strengths and shortcomings of ray tracing.

Response: Thank you for your insightful comment. Following your comment, I have added a brief paragraph to discuss the limitations of the ray tracing method. Firstly, the concept of Rossby wave packet naturally requires large - scale, slowly varying amplitude. Secondly, ray tracing becomes inaccurate in fairly narrow and strong jets where the refractive index varies significantly so that WKB approximation becomes invalid. Another critical failure occurs in the vicinity of caustics and singularities. At caustics, where ray densities become infinite, conventional ray theory breaks down, requiring explicit phase and amplitude corrections to the computed ray tubes. Finally, in strongly anisotropic or absorbing media, rays may encounter singular directions where wave velocities coincide, resulting in numerical instabilities and indefinite expressions in ray - tracing equations.

Above revisions can be seen in Line 74-81 in the revised manuscript.

On a related note, it is not clear while reading this paper at what point or points in the evolution of an unstable shear flow the theory applies. When a shear flow is truly barotropically unstable, the end result often bears little resemblance to the initial conditions, a point made in e.g., the review by Read and Dowling (2026). It would help to add a few guideposts throughout the paper that indicate whether we are always teetering on marginal stability or not, and at what point does the evolution of unstable flow render the discussion moot.

Response: Thank you for your constructive comment. Physically speaking, the solution Eq. (2) demonstrates that we consider the initial stage of perturbation development. In this stage, the amplitude is relatively weak so that the linearization approximation is reasonable and the wave-like solution is possible. When the amplitude is such strong to induce possible instability, the end result often bears little resemblance to the initial conditions, a point made in e.g., the review by Read and Dowling (2026).

To address your concern, in the revised manuscript, I have stressed that the theory in this paper primarily applies to the linear growth stage of unstable shear flow evolution (i.e., the initial stage where perturbation amplitudes are small and the basic flow remains largely unchanged). At this stage, the flow structure is still close to the initial conditions, and the assumptions of linear stability analysis (such as the small perturbation approximation) hold.

Note that I do not specify explicit initial values for wave energy and amplitude because we are more concerned with the growth rate of the wave energy and amplitude relative to the small initial values. When the wave energy and amplitude exceed a certain value, thus violating the linearization approximation, the wave packet may enter the nonlinear evolution stage. Note that I do not specify an explicit threshold since it is hard to determine such a criterion. I predict that we may obtain an approximate criterion by analyzing the observed data, which deserves further investigation.

Above revisions are in Line 106-109, and in Line 224-227 in the revised manuscript.

It should be made clear early in the paper that the system under study has a flat bottom topography and thus misses out on an entire class of marginally stable cases that are especially relevant to Jupiter and Saturn. See Deguchi et al. (2024) for an extended discussion of this point. This limitation simply needs to be made explicit somewhere in the paper's introduction, and ties into the conclusions and future work, for example the comment on Line 472 about "real-world atmospheric flows".

Response: Thank you for highlighting this important limitation regarding the flat bottom topography assumption. I fully agree with your suggestion and have made the following revisions to address this point.

First, I have added some dedicated sentences just after Eq. (1) in the manuscript. These sentences (now in Line 97-100 in the revised manuscript) clarify that we do not consider the influence of the topography, thus missing out an entire class of marignially stable cases that are especially relevant to Jupiter and Saturn where fully dynamic weather layer overlies a layer containing a deep jet profile. One may refer to the research by Deguchi et al. (2024) for an extended discussion.

Second, I have added a short discussion in the final Conclusions and discussion Section to stress that the current investigation can be extended to explore the influence of the large-scale topography on the atmosphere. Moreover, I stress that it can also be applied o giant gas planets such as Jupiter and Saturn where a deep-layer jet, serving as a dynamical topography, also plays a significant role in modulating the weather layer above it.

The Hurdle Theorem (Deguchi et al. 2024) can be readily applied to the u = sech2(y) prototype (29) analysed in Section 4, which will significantly enhance the discussion, including the material shown in Fig. 1 c), which currently only illustrates sufficient-for-stability criteria and is lacking sufficient-for-instability criteria. Some of the main figures later in the paper are also ripe for addition of Hurdle Theorem regions.

Response: Thank you for your valuable suggestion regarding the application of the Hurdle Theorem. I fully agree that integrating this theorem will strengthen the analysis.

First, I have carefully read the paper by Deguchi et al. (2024) to study the Hurdle Theorem and have attempted to apply it. Below is my understanding (I hope it is correct).

In this investigation, I specify zonal basic flow as

$$\overline{u} = u_0 \operatorname{sech}^2 y \tag{1}$$

The corresponding meridional gradient of the absolute vorticity writes as

$$\beta^* = \beta_1 - u_0 \operatorname{sech}^2 y \left(4 - 6\operatorname{sech}^2 y\right) \tag{2}$$

When $u_0 > u_s \equiv \frac{3}{2}\beta_1 \simeq 2.428$, β^* begins to have zero points. The reciprocal Rossby Mach number $M_\alpha^{-1} = \frac{1}{k_0^2}\frac{\beta^*}{\overline{u}-\alpha}$ where $k_0 = \frac{\pi}{L}$ is the smallest meridional wavenumber associated with the largest meridional horizontal scale and L is the meridional range and α is a real number that need to be determined. Following Deguchi et al. (Deguchi et al. 2024), the zero points of β^* are labeled as $-y_i$ and $-y_j$ ($y_i = 1.7028, y_j = 0.8227$). Note that y_i and y_j are also zero points due to symmetry. The zonal basic flow speed at $-y_i$ and $-y_j$ are labeled as $\alpha_i \equiv \overline{u}(-y_i) = 0.4975$ and $\alpha_j \equiv \overline{u}(-y_j) = 2.1692$. As shown in Fig. 1 and Fig. 2 in this reply, we can see that the the zonal basic flow is case (ii) if choosing $\alpha = \alpha_j$ (both $M_\alpha^{-1}(-y_i) = 8.8256\times 10^{-5}$ and $M_\alpha^{-1}(-y_j) = 9.1467$ are larger than zero to ensure the one-signed condition although the first is quite close to zero). The hurdle, defined as

$$h = \frac{\dfrac{\pi^2}{(y_2-y_1)^2}}{\dfrac{\pi^2}{L^2}} = \frac{L^2}{(y_2-y_1)^2} \geq 1 \tag{3}$$

is always larger than one where $y_2 - y_1$ is a sub-range in the the prescribed region with the length of L. However, there is no $y_2 - y_1$ region where the reciprocal Rossby Mach number is larger than the hurdle. On the other hand, as shown in Fig. 2, there indeed are regions larger than a hurdle determined by a region.

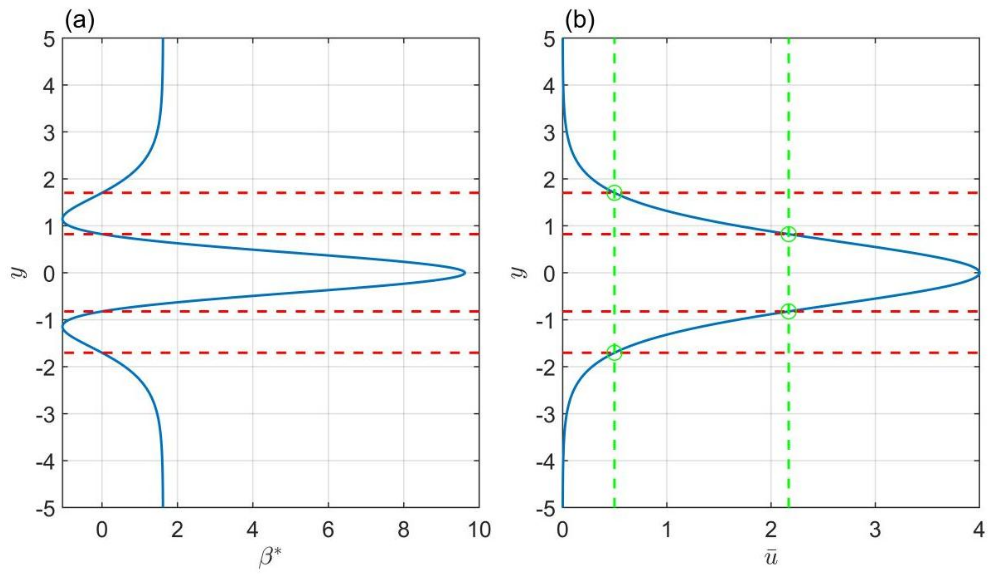


Figure 1

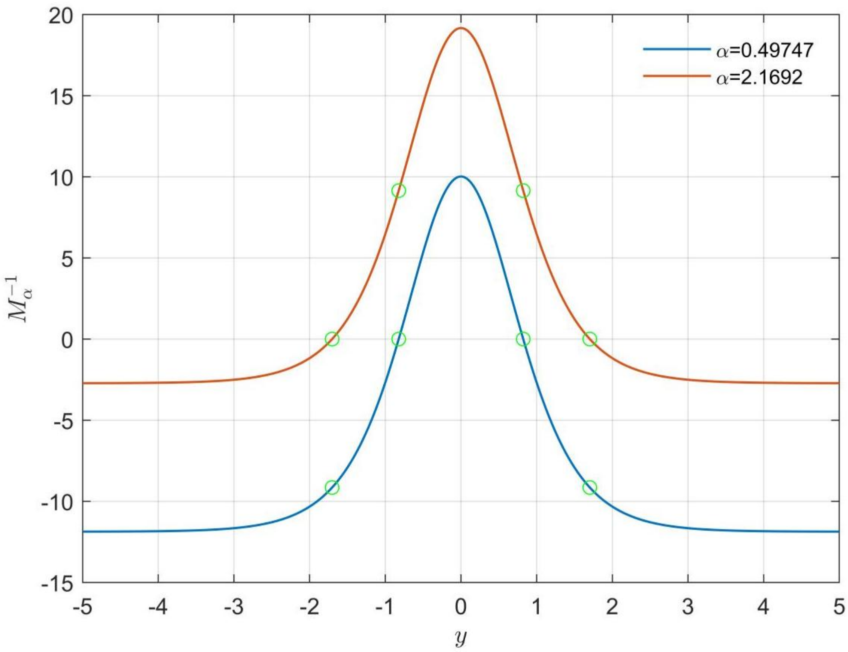


Figure 2

Second, I have added some discussions in the revised manuscript to strengthen the depth of the investigation. For example, I have added a brief discussion after describing Figure 1c in the revised manuscript. They are “According to Eq. 30, $\beta_1^* < 0$ begins to emerge when the non–dimensional magnitude of the westerly jet $u_I \equiv u_0/U > \frac{3}{2}\beta_1 \equiv u_s \approx 2.428$ (Fig. 1c) satisfies the Rayleigh–Kuo necessary condition for normal mode instability. On the other hand, according to the hurdle theorem, it is also easy to derive the sufficient condition for instability. This condition compares the hurdle with the profile of the reciprocal Rossby Mach number, which can be easily calculated from the westerly jet Eq. (29). Preliminary analysis (figures omitted) seems to suggest that $\beta_1^* < 0$ may also satisfy the hurdle theorem, implying that it may also indicate a sufficient condition for instability. In the present study, we mainly focus on the energy evolution along the ray trajectory and do not explicitly discuss the application of the hurdle theorem. This aspect deserves further investigation. Note that when $\beta_1^* < 0$, the westerly jet Eq. (29) is stable, as predicted by both the Rayleigh–Kuo necessary condition and the hurdle theorem. However, nonmodal instability caused by transient growth may still be possible. " They appear in Line 336-345 in the revised manuscript. For another example, after describing Figure 4 and Figure 5, I have added a short discussion to stress that the optimal zonal wavenumber (and correspondingly the optimal phase speed) also coincides with the reference zonal wind shift that ensures a continuous Rossby Mach number in the hurdle theorem.

All of these revisions have been colored in blue. You can easily find them in the revised manuscript.

Minor

Line 14: "by dispersion relation" -> "by the dispersion relation"

Response: Thank you for your careful reviewing. I have corrected this issue and have carefully checked the manuscript to ensure proper use of "the".

Line 17-18: the clause-isolating hyphens should be longer em-dashes, as in growth—capable

Response: Thank you for your careful reviewing. I have corrected this typo in the revised manuscript and I have also carefully checked the manuscript to avoid similar typos.

Line 26: "they may play" -> "they play"

Response: Thank you for your careful reviewing. I have removed the word "may" in the revised manuscript.

Line 59: "packet, w to" needs to be fixed

Response: Thank you for your careful reviewing. This issue occurs when I accidentally delete words while editing sentences during writing. I have corrected this issue in the revised manuscript. Now it is "Compared to the fast phase propagation, the amplitude of a Rossby wave generally varies slowly, giving rise to the so-called Rossby wave packet, which sometimes act as long-range precursors to extreme weather and presumably have an influence on the predictability of mid-latitude weather systems (Wirth et al., 2018)".

Line 70: "Li et al.(2021a) -> "Li et al. (2021a)"

Response: Thank you for your careful reviewing. I have revised the manuscript following your comment. And I have carefully checked the writing to avoid similar typos.

Line 91: The epsilon and Psi symbols are running into each other in (3). There are several spacing issues in the equations throughout the manuscript. Presumably these will at least get fixed in the typesetting.

Response: Thank you for pointing out the spacing issues in the equations. I have carefully reviewed all equations throughout the manuscript and corrected the spacing inconsistencies, ensuring proper separation between symbols and improved readability. I have also conducted a thorough check of all other equations to ensure consistent spacing and clarity, and I will confirm these adjustments during the final typesetting process to meet journal formatting standards.

Line 194 and (15): This transform needs to be better motivated and referenced in terms of how and why it is helpful and how old this strategy is.

Response: Thank you for your insightful comment. To address your concern, I firstly derive (15) in the submitted manuscript from the wave action equation,

$$\frac{\partial A}{\partial T}+\nabla\cdot\left(\mathbf{c}_{\mathbf{g}}A\right)=0 \tag{4}$$

Where $A=E/\omega'$ is the wave action density. Substituting its definition into Eq. (4), we can get

$$\frac{1}{\omega'}\frac{\partial E}{\partial T}-E\frac{1}{\omega'^2}\frac{\partial \omega'}{\partial T}+\frac{1}{\omega'}\nabla\cdot\left(\mathbf{c}_{\mathbf{g}}E\right)-\frac{\mathbf{c}_{\mathbf{g}}E}{\omega'^2}\nabla\cdot\omega'=0 \tag{5}$$

Eq. (5) can be collected as

$$\frac{1}{\omega'}\left[\frac{\partial E}{\partial T}+\nabla\cdot\left(\mathbf{c}_{\mathbf{g}}E\right)\right]-\frac{E}{\omega'^2}\left[\frac{\partial \omega'}{\partial T}+\mathbf{c}_{\mathbf{g}}\nabla\cdot\omega'\right]=0 \tag{6}$$

Namely

$$\frac{\mathrm{D}_g E}{\mathrm{D}T}+E\nabla\cdot\mathbf{c}_{\mathbf{g}}-\frac{E}{\omega'}\frac{\mathrm{D}_g \omega'}{\mathrm{D}T}=0 \tag{7}$$

Since $\omega'=\omega-\bar{u}k$, we have

$$\frac{\mathrm{D}_g E}{\mathrm{D}T}+E\nabla\cdot\mathbf{c}_{\mathbf{g}}+\frac{E}{\omega'}\frac{\mathrm{D}_g \bar{u}}{\mathrm{D}T}k=0 \tag{8}$$

Note that we have applied the relations that frequency (ω) and zonal wavenumber (k) are constant along rays in the zonally varying basic flow $\bar{u}\left(y\right)$. Besides, $\frac{\mathrm{D}_g\bar{u}}{\mathrm{D}T}=c_{gy}\frac{\partial u}{\partial Y}$.

$\omega'=-\frac{\beta^* k}{K^2}$. $c_{g,y}=\frac{2\beta^* kl}{K^4}$. Therefore, we can finally derive

$$\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T}+\nabla\cdot\mathbf{c}_{\mathbf{g}}-\frac{2kl}{K^2}\frac{\partial \bar{u}}{\partial Y}=0 \tag{9}$$

that is, Equation (15) in the submitted manuscript. We may also write Eq. (9) as

$$\frac{\partial E}{\partial T}+\nabla\cdot E\mathbf{c}_{\mathbf{g}}-E\frac{2kl}{K^2}\frac{\partial \bar{u}}{\partial Y}=0 \tag{10}$$

Integrating Eq. (10) over the region S and assuming no perturbation at the boundary, we can further derive

$$\bar{E}\equiv\iint_S\frac{\partial E}{\partial T}dS=\iint_S E\frac{2kl}{K^2}\frac{\partial \bar{u}}{\partial Y}dS \tag{11}$$

Eq. (11) was derived approximately 40 years and has been widely applied to discuss the evolution of Rossby waves (Zeng 1983; Chen and Chao 1983; Pedlosky 1987). However, Eq. (11) is hard to solve due to complex form at the right hand term. Therefore, classical theory can only qualitatively predict the evolution of Rossby waves.

Since it is hard to solve Eq. (10) or Eq. (11), we turn to solve it alternative Eq. (9). As shown in Eq. (9), the only term needs to be determined it the group velocity divergence, which,however, cannot be directly solved by applying the expression of group velocity (Lighthill 2001). To overcome this difficulty, Li et al. (2021) proposed a method to solve the group velocity divergence. With the aid of it, we can easily calculate both of the wave energy and amplitude along rays. Based on my preliminary results, I further investigate in what cases both of the wave energy and amplitude can have simultaneous increase to induce possible instability. In the revised manuscript, I have added some sentences to stress that Equation (15) can be readily derived from the classic Rossby wave evolution theory without introducing extra limitation. The reason why previous investigations did not discuss Equation (15) is the difficulty in solving the group velocity divergence.

Line 215-220: There are remarkably few citations in the beginning of Section 3. Please add some citations to similar work, or add emphasis that none exists.

Response: Thank you for your observation. The theoretical derivation in this investigation is based on the classic theories of ray tracing and Rossby wave evolution. However, as demonstrated in the preceding response, classical theory essentially ceases after deriving Eq. (11), as it cannot account for the divergence of group velocity along rays. Therefore, there is no relevant works to the best of my knowledge. Following your comment, I have added a sentence to emphasize that there are no similar works.

Page 11: To this point there has been a noticeable lack of any figures. The effect is to make reading the paper more tedious than this important subject deserves. Please add a couple of examples to illustrate some of the key points being made, which will prop up the reader's morale, particularly new students.

Response: Thank you for your valuable feedback. In Section 3, I focus on a theoretical analysis for general basic flows. The conclusions theoretically derived in Section 3 has been explicitly analyzed and portrayed in Section 4 by applying a classic westerly jet prototype. Section 3 may be a little technical, particularly for early-career researchers and students. In response to this comment, I have added a brief description in Section 3: "In this section, we focus on a general theoretical analysis and thus do not prescribe an explicit profile of the westerly jet. Readers who are not familiar with the above mathematical details can find specific examples for a classical westerly jet prototype in Section 4". Readers can use the illustrations from Section 4 to better understand the theoretical analysis in Section 3. Thanks again.

Line 438: It is arguable that the explicit physical understanding of Rossby wave instability in the context of the reciprocal Rossby-Mach number (e.g., Deguchi et al. 2024) has not been well known for several decades, which undermines the point of this sentence. This needs to be updated.

Response: Thank you for your comment. I have updated the first sentence to clarify the recent advances by Deguchi et al. (2024). The revised sentences now becomes "The physical understanding behind Rossby wave instability has been investigated for several decades. Recent advances in the context of the reciprocal Rossby Mach number (Deguchi et al., 2024) yield a sufficient condition guaranteeing instability in a class of basic flows, shedding the latest insights into the classical problem."

Reference

Chen, Y. Y., and J. P. Chao, 1983: Conservation of wave action and development of spiral Rossby waves. *Sci. China Ser. B-Chem. Biol. Agric. Med. Earth Sci. Chin.*, **13**, 663–672, https://doi.org/10.1360/zb1983-13-7-663.

Deguchi, K., M. Hirota, and T. Dowling, 2024: A sufficient condition for inviscid shear instability: hurdle theorem and its application to alternating jets. *J. Fluid Mech.*, **997**, A25, https://doi.org/10.1017/jfm.2024.728.

Li, Y., J. Chao, and Y. Kang, 2021: Variations in Wave Energy and Amplitudes along the Energy Dispersion Paths of Nonstationary Barotropic Rossby Waves. *Adv. Atmospheric Sci.*, **38**, 49–64, https://doi.org/10.1007/s00376-020-0084-9.

Lighthill, J., 2001: *Waves in Fluids*. 2nd ed. Cambridge University Press 524pp.

Pedlosky, J., 1987: *Geophysical Fluid Dynamics*. Springer New York, NY 710pp., https://doi.org/10.1007/978-1-4612-4650-3.

Zeng, Q.-C., 1983: The Evolution of a Rossby-Wave Packet in a Three-Dimensional Baroclinic Atmosphere. *J. Atmospheric Sci.*, **40**, 73–84, https://doi.org/10.1175/1520-0469(1983)040%3C0073:TEOARW%3E2.0.CO;2.

Dear Reviewer:

Thank you very much for your insightful and constructive comments on my manuscript. I greatly appreciate the time and effort you have dedicated to reviewing my work, which has significantly helped improve the quality and clarity of the paper. I have carefully addressed all your concerns and revised the manuscript accordingly. Below is a summary of the key revisions made:

1. I have compared the case discussed in the manuscript with the cases for the observed westerly jet to stress that it can represent the basic feature of the real world atmosphere.

2. I have rewritten Section 3.1 to enhance logical flow and readability. A new schematic diagram (shown in Response Figure 5) has been added to illustrate key concepts.

3. I have added a new Appendix to introduce the detailed derivation for wave energy equation along rays.

4. I have added a paragraph to explicitly discuss the ray tracing limitations regarding wave superposition and tunneling, citing Harnik (2002) and Harnik & Heifetz (2007) to strengthen theoretical rigor.

5. I have carefully checked the manuscript to avoid mistakes and typos.

I believe these revisions have substantially improved the manuscript's clarity, rigor, and accessibility. I hope the revised version meets your expectations and look forward to your further feedback.

Sincerely,

The author: Yaokun Li

1. A main point which bothers me is the relevance of the analysis in point 3.1 for the observed midlatitude flow (which I take it is what the paper is aimed at helping explain?) Specifically, the setup of a turning point south of the westerly jet, with the wave propagation to the south of the turning surface is opposite what is found in observations, for the case of waves propagating along a jet and being reflected equatorward from its poleward flank. A more realistic setup would be a turning point in a region of negative meridional shear, for example. It is also not explicitly stated if dU/dY and beta* are constant or not for this case. A schematic of the mean flow, where in the atmosphere this is relevant, and what phenomena does it represent are needed to convince the reader of the relevance of this analysis to the real atmosphere.

Response: Thank you for your insightful comment. To address your concern, I first present the distribution of the observed zonal mean zonal wind. As portrayed in Response Figure 1a, the major features of the annual, winter (December–January–February, DJF), and summer (June–July – August, JJA) mean zonal wind include two strong westerly jets dominating the subtropics in each hemisphere and a moderate easterly around the equator. Consequently, Eq. (29) in Section 4 serves as a valid theoretical prototype of the observed westerly jet. Notably, Eq. (29) is a widely applied westerly jet profile in theoretical analysis (e.g., Kuo 1973).

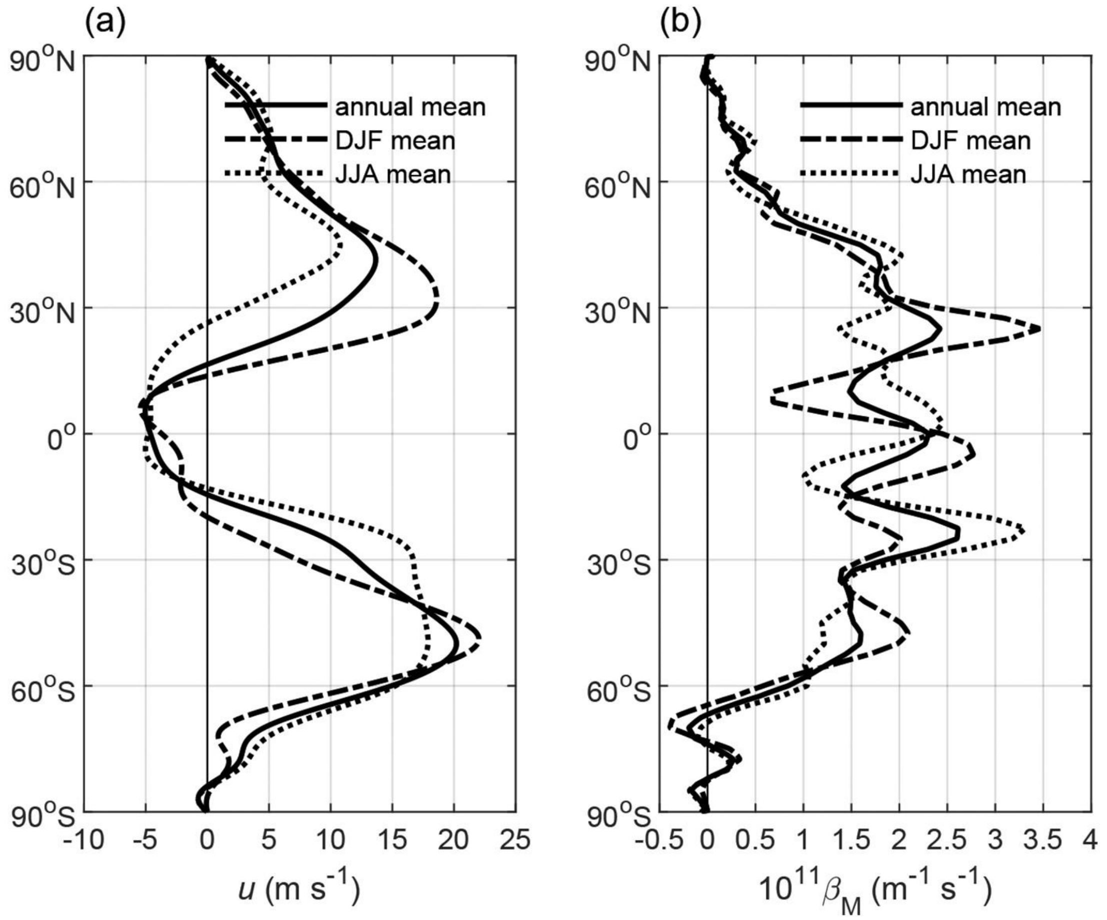
By applying the observed zonal mean zonal wind, we can readily identify the turning latitudes and critical latitudes that bound the propagative regions (see Response Figure 2). To illustrate, consider the DJF season: in the Northern Hemisphere winter, the westerly jet center is located around 30°N. The propagative regions (green-shaded in Response Figure 2) exhibit three distinct patterns: (1) for zonal wavenumbers 2 – 4, the propagative region is enclosed by two critical latitudes; (2) for wavenumbers 5 – 6, it is bounded by a southern critical latitude and a northern turning latitude (north of the jet center); and (3) for wavenumbers ⩾7, it is bounded by a southern critical latitude and a northern turning latitude (south of the jet center). Thus, the position of the turning latitude relative to the jet center depends on zonal wavenumber (Response Figure 2a), wave period (Response Figure 2b), and even the westerly jet profile (see Figure 2 in the submitted manuscript).

For the given profile of the westerly jet, we find that if the propagative regions for a ray are located south of the jet center, the turning latitude is also located south of the jet center (as shown in Figure 2 in the submitted manuscript). Therefore, we do not analyze the case in which a ray can cross the jet center. This case can, however, be theoretically analyzed as:

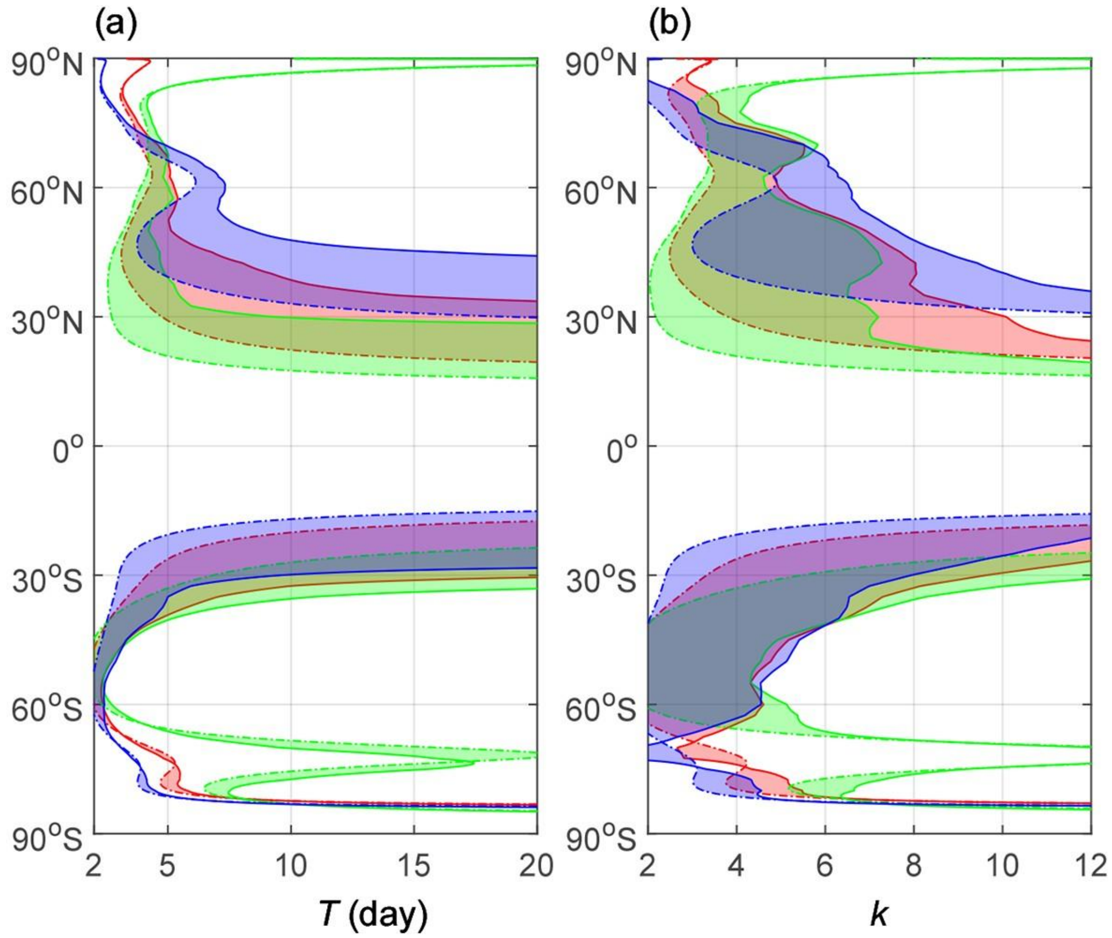
When the turning latitude lies north of the jet center, a ray (with positive initial zonal and meridional wavenumber) originating south of the jet center can cross the jet center before being reflected by the turning latitude. At the jet center, the meridional gradient of the westerly wind (dU/dY) vanishes. According to Eqs. (15) and (20) in the submitted manuscript, the wave energy density approaches an extreme value (since dU/dY > 0 south of the jet center and dU/dY < 0 north of the jet center). When continuing moving northward, the wave energy reaches another extreme value at the turning latitude (due to zero meridional wavenumber at the turning latitude). The above theoretical analysis becomes more complex when considering the modulation of λ, as indicated by Eq. (20), but the general behavior remains similar. I will explicitly analyze the case where λ varies.

Notably, in this theoretical framework, we only require the signs of dU/dY and β* rather than their specific values, confirming that these parameters are not assumed to be constant.

Following your comment and your below comments, I have carefully revised Section 3.1 to emphasize the theoretical setup is reasonable and to extend to cases where the turning latitude is located north of the jet center. I have also carefully revised the derivation to make sure the theoretical derivation is easier to follow. Thank you for your constructive comment.



Response Figure 1. (a) Meridional distribution of the annual (solid line), DJF (dash-dotted line), and JJA (dotted line) mean zonal wind, and (b) the corresponding meridional gradient of the potential vorticity. This figure is cited from Li et al. (2021).



Response Figure 2 Energy dispersion regions (shaded) bounded by a turning latitude (solid line) and by a wave trap line (dot-dashed line) for the annual (red), DJF (green) and JJA (blue) mean zonal wind: (a) for zonal wavenumber k = 8; (b) for a period of T = 10 days. This figure is cited from Li et al. (2021).

2. When the wave field consists of and incident and reflected wave, the group speed is not defined. Rather the wave consists of two waves with oppositely directed group velocities. Specifically, the sentence on lines 130-133: strictly speaking, the back and forth reflection allows the meridional wavenumber to keep constant, but only if you take into account the superposition of the two meridionally reflected waves. This means the solution is not a pure plane wave, and the relevance of the ray tracing formulation for understanding the actual amplitude is not clear. This point is discussed in Harnik (2002).

Response: Thank you for your insightful comment. I have read the paper by Harnik (2002) and regret not having consulted this work during the preparation of the manuscript. I fully agree with the conclusion that ray tracing theory fundamentally describes the trajectory of a wave packet for a quasi-plane wave with specified initial zonal/meridional wavenumbers and initial positions. In general, however, we do not observe isolated quasi-plane waves but rather superpositions of such waves. Consequently, I have added a paragraph in the ray tracing theory section to explicitly address this limitation. In the present study, following the classical theoretical framework, I focus on a single wave ray with a given wave vector propagating from a point source. Under this specific condition, the ray tracing method is equivalent to the wave packet path as Harnik (2002) suggested. Thank you again for your constructive feedback, which has strengthened the rigor of our analysis.

3. Paragraph on lines 144-152: another fundamental difference between ray tracing and the index-of refraction type of analysis that is at the heart of over-reflection theory is the inability of ray tracing to represent tunnelling - see e.g. Harnik (2002). Also, I am not sure I fully follow the sentence starting with "this contrasts" - over-reflection requires the waves to approach the critical surface via tunnelling through an evanescent region. This can happen also when the critical surface does not coincide with the turning surface (inflection point). See e.g. Harnik and Heifetz (2007).

Response: Thank you for insightful comment. I agree with Harnik's conclusion that rays cannot represent tunneling. Actually, this is consistent with my comparison here. Since rays cannot represent tunneling, it naturally cannot arrive at the critical level in a finite time. The expression here is confusing. I am sorry for that and I will made an explicit explanation below.

Firstly, the concepts such as the critical level where the zonal phase speed equals the basic flow ($\bar{u} - c = 0$ at $y = y_{\text{turn}}$), the turning level where the refractive index vanishes ($l = 0$ at $y = y_{\text{critical}}$), and the infection level where the meridional gradient of the absolute vorticity vanishes ($q_y = \beta - \partial^2 \bar{u} / \partial y^2 = 0$ at $y = y_{\text{inflection}}$) are applicable for both overreflection and ray tracing.

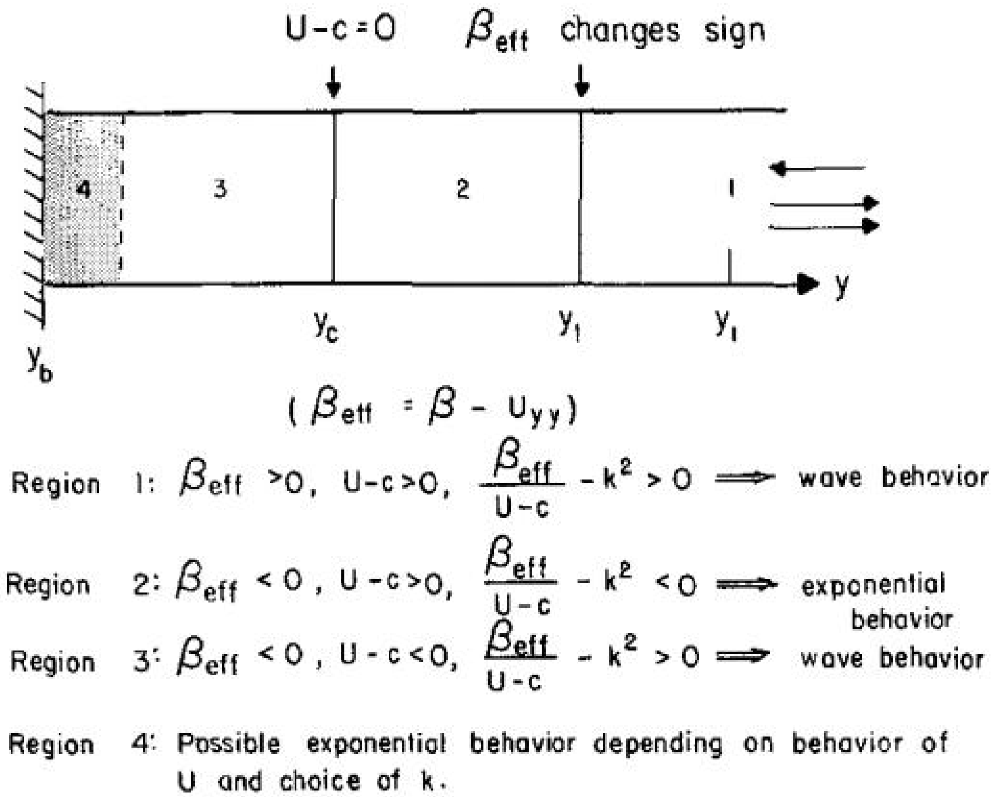
Secondly, Lindzen (1988) wrote: "there must at least be some $y = y_{\text{critical}} < y_1$ where $U = c$ in order to get overreflection. The simple existence of y_{critical} is, however, not enough. LINDZEN and TUNG (1978) showed that a necessary and sufficient condition for overreflection is that $\beta - U_{yy} < 0$ at y_{critical} (or more generally that $\beta - U_{yy}$ have the opposite sign at y_{critical} that it has at y_1). Thus, to get overreflection, we need a $y = y_{\text{inflection}}$ ($y_{\text{critical}} < y_{\text{inflection}} < y_1$) where $\beta - U_{yy} = 0$ -- i.e., an inflection point. The necessary condition for normal mode instability is a necessary and sufficient condition for wave overreflection". Correspondingly, "the geometry of wave propagation, implied by the above conditions, turns out to be important." The wave geometry can be clearly seen from Figure 2 in his paper. I also cite it here as Response Figure 3. As shown in the figure, we need a region 1 where the index of refraction is positive to enable wave propagation. Besides, we also need region 2 where the index of refraction is negative due to negative meridional gradient of the absolute vorticity. Furthermore, regions 3 where the index of refraction is positive is also important. **Note that in the wave geometry, the inflection level does not coincide with the critical level so that there exists a region 2, which is thought fundamental to overreflection.**

Thirdly, region 2, or called wave evanescence (EV) region in the investigation by Harnik and Heifetz (2007), is also necessary. This is consistent with Lindzen's conclusion. However, the turning level (TL) in the investigation by Harnik and Heifetz (2007) seems to coincide with the inflection level. To clearly show you, I also cite Figure 2 in their investigation as Response Figure 4 in this response file. Seen from the figure, the index of refraction is positive below the TL while negative above the TL. Therefore, we can infer that $n^2 = \frac{q_y}{U-c} - k^2 = 0$ at TL. On the other hand, q_y is negative below the TL while positive above the TL. Therefore, $q_y = \beta - U_{yy} = 0$ at TL. Since both two variables simultaneously equal zero at the TL, we may derive that $k = 0$ at the TL. This is a little confusing and maybe I missed something important.

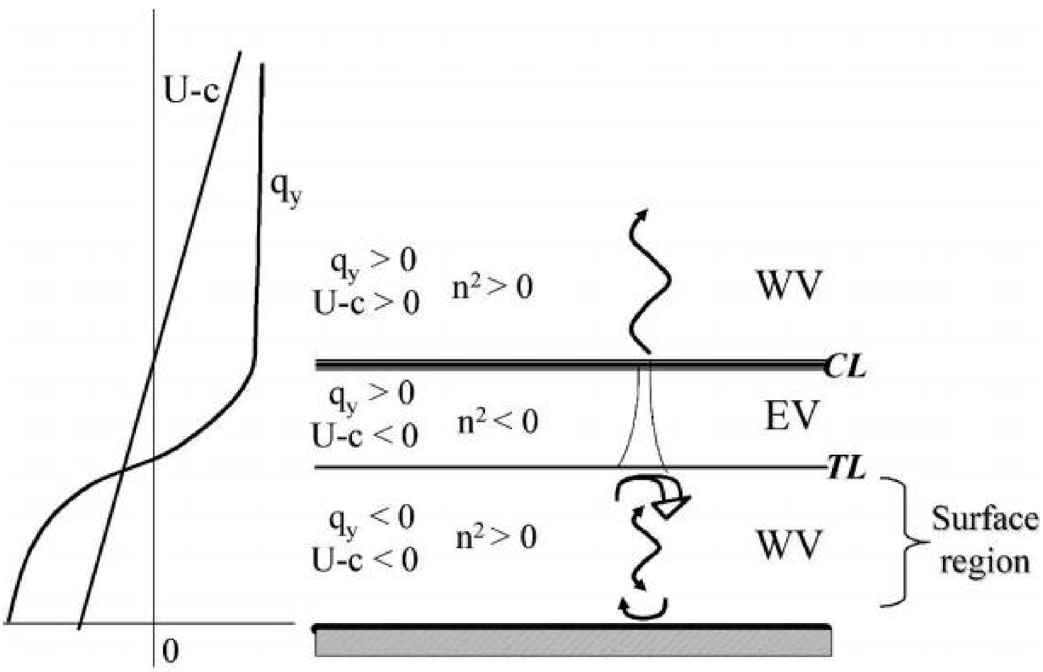
Fourthly, for the ray tracing theory, if there is no inflection level in the propagative region for a ray, the ray cannot arrive the critical level in a finite time span. In other words, the critical level is an asymptotic level for rays. In this investigation, I find that wave energy and amplitude can only have moderate increase so that instability is impossible. On the other hand, if an inflection level is in the propagative region, it must coincide with the critical level. If there is an inflection point at a location, say $y = y_{\text{inflection}}$, that is $q_y = 0$ at $y = y_{\text{inflection}}$, to make sure a positive index of refraction, that is $n^2 = \frac{q_y}{U-c} - k^2 > 0$, $U-c$ must be equal to zero at $y = y_{\text{inflection}}$ so that $\frac{q_y}{U-c}$ at $y = y_{\text{inflection}}$ can be defined. In such a case, a ray can arrive at the critical level (and the inflection level) in a finite time. Furthermore, I find that both wave energy and amplitude may have a substantial increase to induce possible instability.

To summarize, for overreflection, a region where the index of refraction is negative is necessary. However, for ray tracing, rays can never enter a region where the index of refraction is negative. I think this is the most dramatic difference between them. Actually, a ray cannot enter a region where the index of refraction is negative is equivalent to the ray cannot represent tunneling, as suggested by Harnik (2002), Harnik and Heifetz (2007).

To address your concern, I have revised the paragraph to compare with their differences through a clearer description. Now this paragraph becomes: "Note that a fundamental discrepancy exists between the propagation regimes of ray tracing theory and the wave geometry framework of overreflection theory, despite their shared dependence on critical and inflection points. In overreflection dynamics (e.g., Lindzen and Tung, 1978; Lindzen, 1988), incident waves can propagate across the evanescent region where the index of refraction is negative to realize overreflection amplification (e.g., Harnik, 2002; Harnik and Heifetz, 2007). However, ray tracing theory predicts rays can never enter such evanescent region no mater whether the inflection point exists or not. Furthermore, rays can arrive the critical point in finite time only if the critical point coincides with the inflection point. "



Response Figure 3 Barotropic flow divided into regions according to wave propagation characteristics. This figure is cited from Lindzen (1988).



Response Figure 4 A schematic description of the wave geometry for an unstable normal mode. This figure is cited from Harnik and Heifetz (2007).

4. Sentence starting on line 168 through equation 12 - this is the central point of the underlying theory on which this paper is based. I confess that I have not fully internalized the argument of why div(Cg) is not well defined and why it can be replaced with equation 12, and it would require more time than I can spare for this. Given that this is published already, and that I can vaguely see that this could hold, I have continued the review assuming this is correct.

Response: Thank you for your insightful comment. Equation (12) and the theoretical treatment $\nabla \cdot$ Cg are fundamental for the manuscript. Here I will provide an explicit explanation.

Firstly, ray tracing method deems the wave packet as a "material particle" moving with the group velocity. As Lighthill (2001) wrote in his classic book "Waves in fluids": "waves behave like particles (moving along rays); indeed, a wave packet changes position and wavenumber which can be regarded as those for a 'particle' whose energy-momentum relationship at every position exactly parallels the frequency-wavenumber relationship (dispersion relationship) for the waves." It means we may use ray tracing method to track a wave packet from one location to another without solving the full wave equations of motion (Vallis 2017). Therefore, ray tracing is of value not only because it leads to information on the spatial distribution of the wavenumber vector but also because wave energy moves along rays (Lighthill 2001).

Secondly, let us consider the basic assumption of the slowly varying wave train which requires that a wave packet carries a wave with wavelength short enough so that ***locally*** the basic fields appears spatially uniform and ***locally*** the wave amplitude is very nearly constant over one wavelength (Pedlosky 1987). With above assumption, we can derive the ***local*** dispersion relation with has the same form with the plane wave with constant amplitude.

Since the group velocity over one wavelength is also ***locally*** spatial uniform, Pedlosky (1987) suggested that ***locally*** the divergence of group velocity vanishes so that the wave action density, E/ω', does not change along rays, or, the wave energy density is proportional to the intrinsic frequency, that is, $E \sim \omega'$. It provides a simple method to evaluate the wave energy density. Furthermore, it also denotes that the wave energy density is ***locally*** determined by the intrinsic frequency which is only determined by the basic flow when the frequency and zonal wavenumber are fixed.

However, as Bretherton and Garrett (1968) commented, $E \sim \omega'$ is for the lowest (or zeroth) order approximation. It means the variation of amplitude, which is proportional to the small parameter ε (the corresponding first order terms), has been ignored, which is necessary to derive the zeroth order approximation equation (the local dispersion relation) and may not be enough for the first order approximation equation, which is just the wave action conservation.

Thirdly, to be accurate for the first order approximation, the ray divergence should be introduced to revise the above treatment. As pointed out previously, a wave packet along a moving ray has been regarded as a 'particle'. The spatial size of the 'particle' occupies is large compared to a wavelength but small compared to the slowly varying basic flow. **Therefore, viewed from the scale of the basic flow, a wave packet appears as a point with no spatial size** (e.g., Bretherton and Garrett 1968). **Actually, ray theory can only give the trajectory of a wave packet, not the detailed structure of the waves within the packet** (e.g., Vallis 2017).

Considering a particle (wave packet) with no spatial size moves with the group velocity, the divergence of group velocity is then only determined by the variation of the magnitude of the group velocity vector due to no lateral flux (Li et al. 2021), that is

$$\nabla \cdot \mathbf{c_g} = \lim_{\delta T \to 0} \frac{\left|\mathbf{c_g}\right|_{T+\delta T} - \left|\mathbf{c_g}\right|_T}{\left|\mathbf{c_g}\right| \delta T} = \frac{1}{\left|\mathbf{c_g}\right|} \frac{\mathrm{D}_g \left|\mathbf{c_g}\right|}{\mathrm{D}T} = \frac{\mathrm{D}_g \ln \left|\mathbf{c_g}\right|}{\mathrm{D}T} \tag{1}$$

where T is time and δT is a small time interval and $\left|\mathbf{c_g}\right| = \sqrt{c_{g,x}^2 + c_{g,y}^2}$ denotes the magnitude of the group velocity vector. It should be stressed again that Eq. (1) is observed from the scale of the basic flow so that the size of the wave packet is ignored. With Eq. (1), the wave action conservation equation can be written as

$$\frac{\mathrm{D}_g}{\mathrm{D}T}(E/\omega') + (E/\omega') \frac{1}{\left|\mathbf{c_g}\right|} \frac{\mathrm{D}_g \left|\mathbf{c_g}\right|}{\mathrm{D}T} = 0 \tag{2}$$

Then it is easy to derive its solution

$$E \sim \omega' / \left|\mathbf{c_g}\right| \tag{3}$$

Eq. (1) indicates that divergence of group velocity will be positive (negative) if the group velocity magnitude becomes faster and faster (slower and slower). Correspondingly, the wave action density will decrease (increase) according to Eq. (2). Physically, faster and faster (slower and slower) group velocity means the wave action density is divergent (convergent), which of course decreases (increases) the wave action density. Solution Eq. (3) means the wave energy density is proportional to the intrinsic frequency but inverse proportional to the magnitude of the group velocity.

Fourthly, if further considering the fact that a wave packet indeed occupies a certain spatial size, it seems that in principle one may directly calculate the divergence of group velocity from its expression (Bühler 2006). However, only the group velocity values on a ray are known. Neighbouring a ray, the group velocity values are unknown (Lighthill 2001). Physically, directly solving divergence of group velocity means the second order approximation should be introduced.

Finally, the solution Eq. (3) can also be derived from the multi-scale method (or WKB approximation), as emphasized by Lighthill (2001) in his seminal monograph. Under the zeroth-order approximation — valid at the slowly varying scale — the derived ***local*** dispersion relation mirrors the form of the plane wave solution. This equivalence implies that the wave is ***locally*** approximated as a plane wave with constant amplitude and wavenumbers at the slowly varying scale, leading to local conservation of wave action. Consequently, the partial derivative of wave action density with time vanishes, that is, wave action conservation equation

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega'}\right) + \nabla \cdot \left(\mathbf{c_g} \frac{E}{\omega'}\right) = 0 \tag{4}$$

can be simplified to

$$\nabla \cdot \left(\mathbf{c_g} \frac{E}{\omega'}\right) = 0, \tag{5}$$

which means $\mathbf{c_g} E/\omega'$ is a solenoidal vector (Lighthill 2001). It further demonstrates that $E/\omega'|\mathbf{c_g}| = \text{constant}$ along a ray in terms of the cross-sectional area of a thin ray tube.

To sum up, the theoretical treatment ∇・Cg has a solid mathematical and physical basis.

5. The derivation of equation 15 took some time, and the way it is phrased suggests only equation 11 is used, while actually also equation 12 and A=E/w' were used. I suggest at the very least stating this explicitly, but adding an appendix derivation will help.

Response: Thank you for your valuable feedback on the derivation of Equation (15). Here I will present the explicit derivation. The wave action conservation Eq. (4) in the above response can be expanded as

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega'}\right) + \mathbf{c_g} \cdot \nabla\left(\frac{E}{\omega'}\right) + \left(\frac{E}{\omega'}\right)\nabla \cdot \mathbf{c_g} = 0 \tag{6}$$

The first two terms are the individual derivative along rays. Therefore, Eq. (6) can be written as

$$\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right) + \left(\frac{E}{\omega'}\right)\nabla \cdot \mathbf{c_g} = 0 \tag{7}$$

Note that

$$\begin{aligned} \frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right) &= \frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T} + E\frac{\mathrm{D}_g}{\mathrm{D}T}\frac{1}{\omega'} \\ &= \frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T} - \frac{E}{\omega'^2}\frac{\mathrm{D}_g \omega'}{\mathrm{D}T} \\ &= \frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T} - \frac{E}{\omega'^2}\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\omega - \bar{u}k\right) \end{aligned} \tag{8}$$

Furthermore, the frequency and zonal wavenumber are unchanged along rays. Therefore, Eq. (8) can be further expressed as

$$\begin{aligned} \frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right) &= \frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T} + \frac{E}{\omega'^2}k\frac{\mathrm{D}_g \bar{u}}{\mathrm{D}T} \\ &= \frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T} + \frac{E}{\omega'^2}k\left(\frac{\partial \bar{u}}{\partial T} + c_{g,x}\frac{\partial \bar{u}}{\partial X} + c_{g,y}\frac{\partial \bar{u}}{\partial Y}\right) \\ &= \frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T} + \frac{E}{\omega'^2}kc_{g,y}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} \\ &= \frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T} - \frac{E}{\omega'}\frac{K^2}{\beta^* k}k\frac{2\beta^* kl}{K^4}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} \\ &= \frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T} - \frac{E}{\omega'}\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} \end{aligned} \tag{9}$$

Note that we have applied the expression the meridional group velocity and intrinsic frequency, that is,

$$c_{g,y}=\frac{2\beta^{*}kl}{K^{4}}$$
$$\omega'=\omega-\overline{u}k=-\frac{\beta^{*}k}{K^{2}} \quad (10)$$

Substituting Eq. (9) into Eq. (7), we have

$$\frac{1}{\omega'}\frac{\mathrm{D}_{g}E}{\mathrm{D}T}-\frac{E}{\omega'}\frac{2kl}{K^{2}}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y}+\frac{E}{\omega'}\nabla\cdot\mathbf{c}_{\mathbf{g}}=0 \quad (11)$$

Simplifying it, we can derive Equation (15) in the submitted manuscript, that is

$$\frac{1}{E}\frac{\mathrm{D}_{g}E}{\mathrm{D}T}=\frac{2kl}{K^{2}}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y}-\nabla\cdot\mathbf{c}_{\mathbf{g}} \quad (12)$$

In the revised manuscript, I have explicitly presented the above derivation in a new Appendix. Thank you for helping us improve the rigor and clarity of this section.

6. The explanation on lines 197-202 is a bit cumbersome. I think it is much simpler to state that the anomaly extract energy from the mean flow when the anomaly is tilted against the shear and it return energy to the flow when it is tilted with the shear. Similarly the sentence in parenthesis on line 216-217 can be changed to "mediated by structures tilting against the shear". Also the next sentence- "conversely, lambda<0 indicates a tilting with the shear... "

Response: Thank you for your constructive feedback. I have revised the relevant contents as you suggested.

7. Section 3.1: I don't understand the sentence on lines 231-232.

Response: I am sorry for the implicit expression here. In Section 3.1, I assume a simple case where an ED regime (enclosed by a critical latitude and a northern turning latitude in this case) is located south of the jet center. Therefore, the wind shear is always positive, that is $\frac{dU}{dy}>0$.

Furthermore, I assume that the meridional gradient of the absolute vorticity is always larger than zero in the ED regime, that is, $\beta^{*}>0$. I also set a ray starting from an initial location, say y_0, moves northward. According to the definition of the meridional group velocity, that is, $c_{g,y}=\frac{2\beta^{*}kl}{K^{4}}$, northward moving ray requires $kl>0$, the leading structure or tilting against the shear.

Now, if the ray is for a stationary wave (the frequency is zero and hence zonal phase speed is zero, that is, $c=0$), we can derive that

$$\xi=\frac{c^{2}-\frac{2k^{2}}{K^{2}}\left(\overline{u}-c\right)^{2}}{c^{2}+\frac{4k^{2}}{K^{2}}\left(\overline{u}-c\right)\overline{u}}\frac{2kl}{K^{2}}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y}=-\frac{kl}{K^{2}}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y}<0 \quad (13)$$

It means the change rate of the wave energy will be always smaller than until the ray arrives at the turning latitude where zero meridional wavenumber causes zero change rate of the wave energy. Therefore, the wave energy of stationary waves will decrease to a minimum value when it moves northward to arrive at the turning latitude.

To avoid confusing and to following your below comment, I have rewritten Section 3.1 to make sure it is easier to follow. Thank you for helping me improving the quality of the manuscript.

8. Please add the derivation of conditions 23 and 25 to an appendix (and remove "It is easy to derive" - on lines 235 and 241). Please add a schematic illustrating the different cases discussed in page 10 - a schematic of the flow profile, the turning surface and what the different phase peed ranges imply about the value of U-c, and the implications for the wave geometry would be helpful. Line 248 and the discussion of an upper limit of group velocity is very confusing. Is c_max in equation 26 a phase speed or group speed? The only way to understand this is as follows: You state that it is there "to make sure the ray can arrive at the northern turning point". This as far as I understand requires a real meridional wavenumber l, or a positive l^2 (I can't think of any other way by which the phase speed affect the ability of the ray to reach the turning latitude except for the condition that l^2>0 south of this latitude.) Assuming l^2>0 south of the turning latitude, you then mean to say that the maximum zonal group speed c_gx value is obtained when l^2=0 but then the group speed would be U+beta*/k^2 and not U-beta*/k^2. The phase speed for this case will actually be as written in equation 26 but what insures that this is the maximal phase speed? without knowing how beta* varies in space (assuming dU/DY>0 as stated at the beginning of this subsection). Moreover, when I try to think about any conditions on the phase speed that will insure that the meridional wavenumber is indeed real south of the turning latitude I only come up with an opposite argument: If we compare the meridional wavenumber squared for two phase speeds, c1 > c2, then l^2 will be larger for c1, as long as c<u, which implies a lower limit for c in order to insure a positive l^2, not an upper limit. Please explain this argument more clearly, and again, supporting it by a schematic of a concrete flow configuration will help.
Response: Thank you for your valuable comments. I apologize for the lack of clarity in Section 3.1, which may have caused confusion. To address your comments, I have comprehensively revised Section 3.1 to enhance logical flow and readability. Additionally, a schematic diagram has been added to further facilitate understanding of the key concepts.

To analyze wave energy evolution, the wave energy equation along rays is quite fundamental. Therefore, I also express it here as (see Eq. (15) and Eq. (20) in the manuscript)

$$\begin{aligned}\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T} &= \left(\frac{2kl}{K^2}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y} - \frac{\mathrm{D}_g \ln\left|\mathbf{c}_\mathbf{g}\right|}{\mathrm{D}T}\right) \\ &\equiv \xi(T) \\ &= \frac{c^2 - \dfrac{2k^2}{K^2}(\overline{u}-c)^2}{c^2 + \dfrac{4k^2}{K^2}(\overline{u}-c)\overline{u}}\frac{2kl}{K^2}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y} \\ &\equiv \lambda(c,\overline{u})\frac{2kl}{K^2}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y}\end{aligned} \tag{14}$$

As shown in the equation, the sign of the rate of change in wave energy depends on three terms, the coefficient λ, the wave structure kl, and the wind shear $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}$, the latter two of which are relatively easy to identify.

In section 3.1, I present a case where a ray for a wave with a fixed zonal phase speed (labeled as c_0 or c as in the revised manuscript) propagates from its initial location to a northern turning latitude in an ED regime that is located south of the westerly jet (see Response Figure 5a, b). These conditions means that $kl > 0$ and $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} > 0$. Therefore, the increase or decrease in wave energy is only determined by the sign of λ. Note that this case is the simplest. As I have responded in your first comment, this setting can also be found in observed westerly jets. Furthermore, based on this simple case, more complex cases (e.g., the turning latitude is located north of the jet center) can also be analyzed.

According to the expression of λ, its sign is determined by its numerator. Setting the numerator equals to zero, that is

$$c^2 - \frac{2k^2}{K^2}(\bar{u} - c)^2 = 0 \tag{15}$$

it is easy to solve its zero point (labeled as $c_{\lambda 0}$)

$$c_{\lambda 0}(y) = \frac{\sqrt{2}k}{K(y) + \sqrt{2}k}\bar{u}(y) \tag{16}$$

Then we know when $c_0 < c_{\lambda 0}$, $\lambda < 0$, wave energy will decrease and vice versa. On one hand, the zonal phase speed c_0 keeps unchanged along rays. On the other hand, the zero point $c_{\lambda 0}$ varies along rays since both westerly and total wavenumber varies along rays. Therefore, to compare their relative size, we should identify the variation feature of $c_{\lambda 0}$ at first.

According to our prescribed conditions, when the ray moves from its initial location (y_0) to the northern turning latitude (y_t), the wind speed is monotonically increasing while the total wavenumber is monotonically decreasing (due to decreasing meridional wavenumber). We can know that $c_{\lambda 0}$ monotonically increases when the ray moves from y_0 to y_t (see Response Figure 5c), that is

$$c_a \equiv \frac{\sqrt{2}k}{K(y_0) + \sqrt{2}k}\bar{u}(y_0) = c_{\lambda 0}(y_0) \le c_{\lambda 0}(y) \le c_{\lambda 0}(y_t) = \frac{\sqrt{2}k}{K(y_t) + \sqrt{2}k}\bar{u}(y_t) \equiv c_b$$

(17)

It means the values of $c_{\lambda 0}$ at y_0 and at y_t are its minimum and maximum. For simplicity, we have labeled the minimum and maximum as c_a and c_b. Then it would be easy to compare the relative size of c_0 and $c_{\lambda 0}$. (1) If $c_0 < c_a$, that is, the zonal phase speed is slower than the minimum of $c_{\lambda 0}$, the zonal phase speed will be smaller than all values of $c_{\lambda 0}$ in the range (y_0, y_t). Therefore, $\lambda < 0$ and hence $\frac{\mathrm{D}_g E}{\mathrm{D}T} < 0$ in (y_0, y_t). Correspondingly, wave energy will monotonically decrease from its initial value (say, equal 1) to a minimum at y_t where $\lambda = 0$ due to $l = 0$. (2) If $c_0 > c_b$, the zonal phase speed will be larger than $c_{\lambda 0}(y)$, leading $\lambda > 0$ and hence $\frac{\mathrm{D}_g E}{\mathrm{D}T} > 0$ in (y_0, y_t). Correspondingly, wave energy increases to a maximum at y_t where $\lambda = 0$ due to $l = 0$. (3) If $c_a < c_0 < c_b$, there will exist an intermediate location (say, $y_0 < y_m < y_t$) where $c_0 = c_{\lambda 0}(y_m)$. Then when the ray moves from y_0 to y_m, $\lambda > 0$; when the ray arrives at y_m, $\lambda = 0$; and when the ray continues to move from y_m to y_t, $\lambda < 0$. Correspondingly, the wave energy will increase to a maximum at y_m and then decrease to a minimum at y_t. Above features are illustrated in Response Figure 5d, e, f.

When the ray arrives at the turning latitude, the dispersion relation becomes

$$c_{yt} = \bar{u}(y_t) - \frac{\beta^*(y_t)}{k^2} \tag{18}$$

Note that c_{yt} prescribes the upper limit of the zonal phase speed of a wave whose ray can propagate in an ED regime (I used $c_{\max}$ to describe this upper limit in the submitted manuscript, which may be a little confusing. I have changed it to c_{yt} in the revised manuscript). If $c_0 > c_{yt}$, the ray will not be reflected by any turning point along its trajectory. Therefore the zonal phase speed range for a ray can propagate in an ED regime is $(0, c_{yt})$. Here qualitative analysis cannot provide the relative size of c_{yt} and c_a and c_b. Without loss of generality, we may assume that $c_{yt} > c_b$. Then the zonal phase speed range $\left(0, c_{yt}\right)$ can be divided into three parts: $\left(0, c_a\right)$, $\left(c_a, c_b\right)$, and $\left(c_b, c_{yt}\right)$. It is obvious that wave energy can have the most significant increase in the last part.

Up to now, we have identified the evolution feature of wave energy. When a westerly jet is given, and when the initial zonal and meridional wavenumber are given, we may change frequency (of course, we can not arbitrarily specify the frequency value. It must satisfy the dispersion relation) or zonal phase speed to calculate the energy evolution of rays with different zonal phase speed (e.g., see Figure 3 in the submitted manuscript). Note that here the zonal phase speed is not the propagation speed of the ray, but the phase speed of the wave.

Finally, let's consider a slightly complex case where the turning point lies north of the jet center, with other conditions remaining unchanged. Without loss of generality, we analyze the scenario where the zonal phase speed falls within the interval $\left(c_a, c_b\right)$. As the ray propagates from y_0 to y_m, wave energy increases to a maximum at y_m. When the ray travels from y_m to the jet center (say, y_z), wave energy declines from the maximum to a minimum at y_z where zero wind shear can also contribute to this minimum. As the ray proceeds from y_z to y_t, wave energy rebounds from the minimum to another maximum due to both negative wind shear and λ. Compared with the previous case, the most notable distinction is that wave energy can attain two distinct maxima as the ray propagates from the initial location to the turning point. Since analysis methodology remains consistent, we do not explicitly analyze this case.

Based on above content, I have totally rewritten Section 3.1. Since the derivation is clearer this time, I do not add them to an appendix. Thank you for helping me strengthen the presentation of this work.

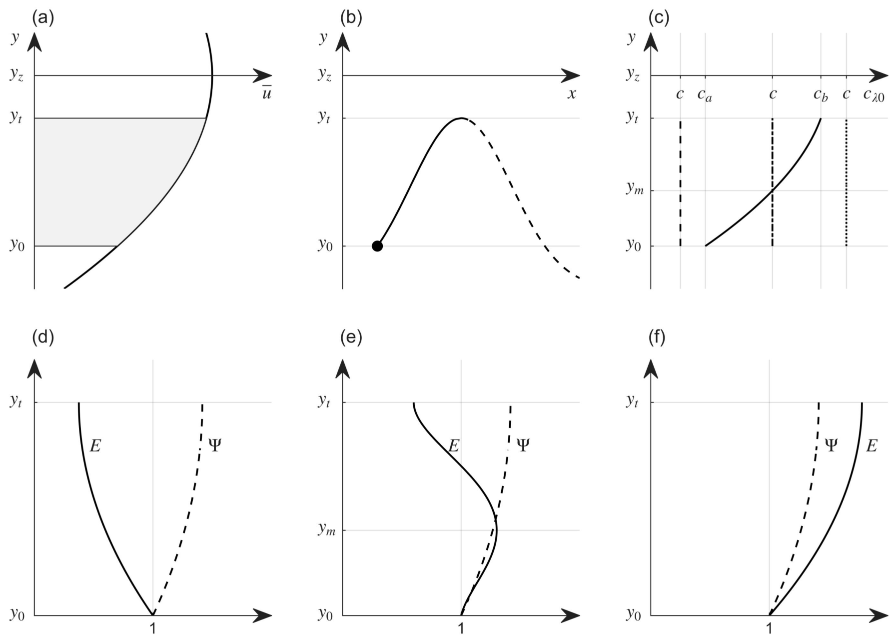
Belows are some concepts to help you understanding.

(1). When calculating ray trajectory, we should know the initial wavenumber (e.g., k0, l0), initial location (e.g., x0, y0), and initial frequency (e.g., w0). Since zonal wavenumber and frequency keep unchanged along a ray, the phase speed (c0=w0/k0) is also kept unchanged along a ray. However, the group speed varies along a ray. In above discussion, we have fixed k0, l0, x0, and westerly jet. Then we discuss the scenarios where zonal phase speed (or frequency) varies. When we give a specific value of zonal phase speed, we can calculate a specific meridional initial location (y0) according to the dispersion relation.

(2). A ray will be reflected to move toward another direction when it arrives at a location where its meridional wavenumber equals to zero. This location is called the turning latitude. A ray will move asymptotically toward the location where the meridional wavenumber square is infinity. This location is called a critical latitude. Then, there will be three types of propagative regions: (a) enclosed by two critical latitudes; (b) enclosed by two turning latitudes; and (c) enclosed by a critical latitude and a turning latitude. As we had introduced in section 2.1, these three types of propagative regions are named the bidirectional dissipation (BD) regime; the wave guide (WG) regime; and the evolution-dispersion (ED) regime, respectively. According to our previous investigations (e.g., Li et al. 2021), the wave energy can have a substantial increase in the ED regime. Therefore, We mainly focus on analyzing the theoretical wave energy evolution in the ED region in Section 3.

(3). The ED regime (see Response Figure 2) has an upper zonal phase speed limit. Saying that, we have meant the zonal wavenumber, the basic flow are both fixed. Then only the zonal phase speed (or frequency) can vary in the dispersion relation (note that meridional wavenumber equals zero at turning latitude and tends to be infinity at the critical latitude). That's the reason why I use $c_{\max}$ in the submitted manuscript (I have revised it to c_{yt} in the revised manuscript).

One of the most interesting finding in this manuscript is that we find c_{yt} corresponds with most substantial increase in both wave energy and amplitude. That's also the reason I use the notation $c_{\max}$.



Response Figure 5 Schematic illustration of an ED regime (a) that is located south of a westerly jet; a ray propagates northward from an initial location denoted by the black point (b); the distribution of the zero point of λ (c); and the cases of $c < c_a$ (d); $c_a < c < c_b$ (e); and $c > c_b$ (f), exhibited by the dashed, dash-dotted, and dotted vertical line in (c). The notation y_0, y_t and y_z denote the initial location, the turning location, and the jet center, respectively. y_m indicates a intermediate location where $c = c_{\lambda 0}(y_m)$. E and Ψ denote wave energy and amplitude, respectively.

9. Sentence on lines 256-7: I am not sure I follow. Why does dE/Dt have to be zero at the turning latitude? you said it vanishes at y_m..

Response: According to Equation (14) in this response, the variation of wave energy density along rays is (I copy it here)

$$\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T}=\left(\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}-\frac{\mathrm{D}_g \ln\left|\mathbf{c}_\mathrm{g}\right|}{\mathrm{D}T}\right)\equiv \xi(T)=\lambda(c,\bar{u})\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} \tag{19}$$

At the turning latitude ($y=y_t$), the meridional wavenumber vanishes ($l=0$) so that the wave energy density has an extreme value at the turning latitude. In some cases, this is the only one extreme value for the wave energy. In some other cases ($c_a<c<c_b$ as analyzed above), the wave energy may have other extreme values. Wave energy reaches an extreme value at $y=y_m<y_t$ due to $\lambda=0$, then we have

$$\left.\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T}\right|_{y=y_m}=0 \tag{20}$$

Since $E=\frac{1}{4}\Psi^2K^2$ where Ψ and K are amplitude and total wavenumber, we can derive that

$$\left.\frac{\mathrm{D}_g E}{\mathrm{D}T}\right|_{y=y_m}=\left.\frac{1}{4}\frac{\mathrm{D}_g K^2}{\mathrm{D}T}\Psi_0^2\right|_{y=y_m}+\left.\frac{1}{4}\frac{\mathrm{D}_g \Psi_0^2}{\mathrm{D}T}K^2\right|_{y=y_m}=0 \tag{21}$$

Above Eq. (21) is just Eq. (27) in the submitted manuscript. At $y=y_m<y_t$, the derivative of total wavenumber (the first term at the right hand) is not equal to zero. Correspondingly, the derivative of amplitude (the second term at the right hand) is also not equal to zero. Therefore, amplitude does not approach extreme value at $y=y_m<y_t$ even if wave energy approaches. That's what I want to express there in the submitted manuscript.

I am sorry for the improper expressing that confuses you. Following your comments, I have totally rewritten Section 3.1. Thanks for your constructive comment.

10. Discussion from line 260 to end of section: It will help to have schematic of the wave geometry of the different phase speed cases, drawn on the mean flow (U and beta*). I was not able to follow the point about the fast phase speed range- I have a feeling I am missing something basic in the way you view the problem. Specifically the discussion of the critical latitude starting on line 269: unless beta*=0 this line separates wave propagation from evanescence. Also, while the EP fluxes suggests wave activity emanates from the critical line for unstable waves, why do the rays that emanate from it are most important for the instability to be able to exist? Instability should occur no matter where you place a wave source, no?

Response: I am sorry for not explicitly introducing my investigation. Your opinion involves two aspects, the former of which I have already provided a detailed response. For the latter, I firstly show you the case where β*>0. In such a case, there is no normal mode instability since the necessary condition is not satisfied. Now let's focus on the critical latitude where the meridional wavenumber is infinity. It is an asymptotic location for a ray. Therefore, a ray can never arrive such an asymptotic location in a finite time. Secondly, if β*=0 at y=yi, the dispersion relation

$$c = \overline{u} - \frac{\beta^*}{K^2} \tag{22}$$

reduces to

$$c = \overline{u} \tag{23}$$

Meanwhile, at the critical latitude (where meridional wavenumber is infinity), the dispersion relation also have the same form as Equation (23) shows (it is caused by infinity K2). Therefore, if an inflection latitude consists of a boundary of a ray, it must also be the critical latitude. At the inflection latitude, there is no definition for the meridional wavenumber. However, it does not mean meridional wavenumber can be taken arbitrary. To show this, we may firstly write the dispersion relation as

$$K^2 = \frac{\beta^*}{\overline{u} - c} \tag{24}$$

Then, it can be expressed as

$$K^2 = \frac{\beta^*}{\overline{u} - c} = \frac{\frac{\mathrm{d}\beta^*}{\mathrm{d}y}}{\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}} = \frac{-\frac{\mathrm{d}^3\overline{u}}{\mathrm{d}y^3}}{\frac{\mathrm{d}\overline{u}}{\mathrm{d}y}} \equiv K_i^2 < \infty \tag{25}$$

at the inflection latitude. As shown in Equation (25), the total wavenumber can also be defined at the inflection latitude so does the meridional wavenumber (labeled as $l_i = \sqrt{K_i^2 - k^2}$). Therefore, a ray with an initial zonal and meridional wavenumber can arrive at the inflection latitude (also the critical latitude) where the meridional group velocity vanishes ($c_{g,y} = \frac{2\beta^* kl}{K^4} = 0$ due to $\beta^* = 0$). Correspondingly, the ray will move horizontally along the inflection latitude with zonal group velocity equal to the zonal phase speed ($c_{g,x} = c + \frac{2\beta^* k^2}{K^4} = c$ since $\beta^* = 0$). Above analysis means that when a ray arrives at the inflection latitude, it will move along the latitude so that it can never enter the region where $\beta^* < 0$. Similarly, if a ray propagates in the region where $\beta^* < 0$ (note that the zonal phase speed must be faster than the basic flow to make sure a positive index of refraction), when it arrives at the inflection latitude, it will also move along the latitude so that it can never enter the region where $\beta^* > 0$.

When the ray moves along the inflection latitude, energy equation (19) in this response becomes

$$\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T} = \left(\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} - \frac{\mathrm{D}_g \ln|\mathbf{c}_\mathbf{g}|}{\mathrm{D}T}\right) = \frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} \tag{26}$$

Since the group velocity will keep constant. If the inflection latitude is south of the jet center, we have $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} > 0$. Then if the meridional wavenumber is larger than zero, the right hand term will be lager than zero. If the meridional wavenumber equals l_i, the right hand term will become a constant positive value so that wave energy will increase exponentially without any limitation. This means normal mode instability.

According to Eq. (7) in the manuscript, the variation of meridional wavenumber is determined by

$$\frac{\mathrm{D}_g l}{\mathrm{D}T} \equiv -\frac{\partial \Omega}{\partial Y} = -\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}k + \frac{\mathrm{d}\beta^*}{\mathrm{d}Y}\frac{k}{K^2} = -\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}k - \frac{\mathrm{d}^3\bar{u}}{\mathrm{d}Y^3}\frac{k}{K^2} \tag{27}$$

For slowly varying basic flow, we may approximately write it as

$$\frac{\mathrm{D}_g l}{\mathrm{D}T} \approx -\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}k \tag{28}$$

It mean meridional wavenumber will monotonically decrease (we focus on discussion south of the jet center).

If a ray moves from an initial location just at the inflection latitude, the ray will always move along the inflection latitude (the trajectory is a horizontal line). Then if its initial meridional wavenumber is larger, $l_0 > l_i$, the meridional wavenumber will decrease until it decrease to equal to l_i. Then the meridional wavenumber will keep unchanged along the ray. Correspondingly, the wave energy will exponentially increase. If the initial meridional wavenumber is smaller, $l_0 < l_i$, then the meridional wavenumber will keep decreasing (since it cannot meet the critical value l_i). In such case, wave energy will eventually exponentially decrease. Therefore, whether a ray initially located at the inflection latitude can develop or not depends on the its initial wave structure. This is consistent with the critical layer problem in the classic instability theory (e.g., Pedlosky, 1987). Now I can address your concern. The ray starts to its journey from the inflection latitude is important because it is the only possibility that normal mode instability can occur in the ray tracing theory. Correspondingly, instability does not depends in the wave source where a ray starts to move. It depends on whether the wave source is on the inflection latitude or not (also the initial meridional wavenumber at the inflection latitude).

Minor:

Line 35 - the absolute vorticity (not its gradient) has an extremum, the gradient changes sign, no?

Response: Thanks for your careful reviewing. Here the gradient is redundant and should be removed. I have removed it in the revised manuscript and carefully checked the manuscript to avoid similar mistakes.

line 59 fix typo (w..)

Response: Thanks for your careful reviewing. I am sorry for the typo here. I have revised the sentence in the revised manuscript.

line 104- insert the definition of the material derivative

Response: Thanks for your careful reviewing. I have stress the definition of the material derivative along rays Line 124 in the revised manuscript.

Line 248 it should be "phase speed" not "group velocity" at the end of the line, no?

Response: Thanks for your careful reviewing. I am sorry for the typos here. I should be zonal phase speed rather than group velocity. Following your previous comments, I have rewritten Section 3.1.

line 318: it is easy to *show*

Response: Thanks for your careful reviewing. I have added the word "show" there. I have also carefully checked the manuscript to avoid similar typos.

References

Bretherton, F. P., and C. J. R. Garrett, 1968: Wavetrains in inhomogeneous moving media. *Proceedings of the Royal Society of London. Series A*, **302**, 529–554, https://doi.org/10.1098/rspa.1968.0034.

Bühler, O., 2006: *A Brief Introduction to Classical, Statistical, and Quantum Mechanics*. American Mathematical Society, https://doi.org/10.1090/cln/013.

Harnik, N., 2002: The Evolution of a Stratospheric Wave Packet. *Journal of the Atmospheric Sciences*, **59**, 202–217, https://doi.org/10.1175/1520-0469(2002)059%3C0202:TEOASW%3E2.0.CO;2.

Harnik, N., and E. Heifetz, 2007: Relating Overreflection and Wave Geometry to the Counterpropagating Rossby Wave Perspective: Toward a Deeper Mechanistic Understanding of Shear Instability. *Journal of the Atmospheric Sciences*, **64**, 2238–2261, https://doi.org/10.1175/JAS3944.1.

Kuo, H. L., 1973: On a simplified radiative-conductive heat transfer equation. *pure and applied geophysics*, **109**, 1870–1876, https://doi.org/10.1007/BF00876111.

Li, Y., J. Chao, and Y. Kang, 2021: Variations in Wave Energy and Amplitudes along the Energy Dispersion Paths of Nonstationary Barotropic Rossby Waves. *Adv. Atmos. Sci.*, **38**, 49–64, https://doi.org/10.1007/s00376-020-0084-9.

Lighthill, J., 2001: *Waves in Fluids*. 2nd ed. Cambridge University Press 524pp.

Lindzen, R. S., 1988: Instability of plane parallel shear flow (toward a mechanistic picture of how it works). *pure and applied geophysics*, **126**, 103–121, https://doi.org/10.1007/BF00876917.

Pedlosky, J., 1987: *Geophysical Fluid Dynamics*. Springer New York, NY 710pp., https://doi.org/10.1007/978-1-4612-4650-3.

Vallis, G. K., 2017: *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*. 2nd ed. Cambridge University Press, https://doi.org/10.1017/9781107588417.

Dear Prof. Gwendal Rivière:

Thank you for your careful review and valuable comments on my manuscript. In response to your questions and suggestions, I have carefully revised the manuscript and prepared a detailed point-by-point response.

I have added an new Appendix to introduce the derivative of the maximum and minimum of the turning point curve. I have also explicitly illustrate the rationality in solving the divergence of group velocity. Following the reviewer’s comment, I have also added a new schematic figure to help understanding. I have rewritten Section 3.1. I have corrected the expressing mistakes and typos and carefully checked the manuscript to avoid similar mistakes.

All revisions have been marked in blue at the corresponding positions in the revised manuscript, and the detailed responses are provided in the response file.

Thank you again for your professional guidance. I hope the revised manuscript meets the journal's publication requirements.

Sincerely,
The author: Yaokun Li

Comments

1. Line 109. About missing the expressing of the material derivative along rays.

Response: Thank you for your careful reviewing. I am sorry for missing the formula "$\frac{D_g}{DT}=\frac{\partial}{\partial T}+\mathbf{c}_g\cdot\nabla$" which denote the material derivative along group velocity vector $\mathbf{c}_g$. I have corrected this mistake in the revised manuscript.

2. Line 175-176. About the derivation of the divergence of group velocity.

Response: Thank you for your insightful comment. Equation (12) and the theoretical treatment ∇ • Cg are fundamental for the manuscript. Here I will provide an explicit explanation.

Firstly, ray tracing method deems the wave packet as a "material particle" moving with the group velocity. As Lighthill (2001) wrote in his classic book "Waves in fluids": "waves behave like particles (moving along rays); indeed, a wave packet changes position and wavenumber which can be regarded as those for a 'particle' whose energy-momentum relationship at every position exactly parallels the frequency-wavenumber relationship (dispersion relationship) for the waves." It means we may use ray tracing method to track a wave packet from one location to another without solving the full wave equations of motion (Vallis 2017). Therefore, ray tracing is of value not only because it leads to information on the spatial distribution of the wavenumber vector but also because wave energy moves along rays (Lighthill 2001).

Secondly, let us consider the basic assumption of the slowly varying wave train which requires that a wave packet carries a wave with wavelength short enough so that ***locally*** the basic fields appears spatially uniform and ***locally*** the wave amplitude is very nearly constant over one wavelength (Pedlosky 1987). With above assumption, we can derive the ***local*** dispersion relation with has the same form with the plane wave with constant amplitude.

Since the group velocity over one wavelength is also ***locally*** spatial uniform, Pedlosky (1987) suggested that ***locally*** the divergence of group velocity vanishes so that the wave action density, E/ω', does not change along rays, or, the wave energy density is proportional to the intrinsic frequency, that is, $E\sim\omega'$. It provides a simple method to evaluate the wave energy density. Furthermore, it also denotes that the wave energy density is ***locally*** determined by the intrinsic frequency which is only determined by the basic flow when the frequency and zonal wavenumber are fixed.

However, as Bretherton and Garrett (1968) commented, $E\sim\omega'$ is for the lowest (or zeroth) order approximation. It means the variation of amplitude, which is proportional to the small parameter ε (the corresponding first order terms), has been ignored, which is necessary to derive the zeroth order approximation equation (the ***local*** dispersion relation) and may not be enough for the first order approximation equation, which is just the wave action conservation.

Thirdly, to be accurate for the first order approximation, the ray divergence should be introduced to revise the above treatment. As pointed out previously, a wave packet along a moving ray has been regarded as a 'particle'. The spatial size of the 'particle' occupies is large compared to a wavelength but small compared to the slowly varying basic flow. **Therefore, viewed from the scale of the basic flow, a wave packet appears as a point with no spatial size** (e.g., Bretherton and Garrett 1968). **Actually, ray theory can only give the trajectory of a wave**

packet, not the detailed structure of the waves within the packet (e.g., Vallis 2017).

Considering a particle (wave packet) with no spatial size moves with the group velocity, the divergence of group velocity is then only determined by the variation of the magnitude of the group velocity vector due to no lateral flux (Li et al. 2021), that is

$$\nabla \cdot \mathbf{c}_{\mathrm{g}} = \lim_{\delta T \to 0} \frac{\left|\mathbf{c}_{\mathrm{g}}\right|_{T+\delta T} - \left|\mathbf{c}_{\mathrm{g}}\right|_{T}}{\left|\mathbf{c}_{\mathrm{g}}\right| \delta T} = \frac{1}{\left|\mathbf{c}_{\mathrm{g}}\right|} \frac{\mathrm{D}_{g} \left|\mathbf{c}_{\mathrm{g}}\right|}{\mathrm{D}T} = \frac{\mathrm{D}_{g} \ln \left|\mathbf{c}_{\mathrm{g}}\right|}{\mathrm{D}T} \tag{1}$$

where T is time and δT is a small time interval and $\left|\mathbf{c}_{\mathrm{g}}\right| = \sqrt{c_{g,x}^{2} + c_{g,y}^{2}}$ denotes the magnitude of the group velocity vector. It should be stressed again that Eq. (1) is observed from the scale of the basic flow so that the size of the wave packet is ignored. With Eq. (1), the wave action conservation equation can be written as

$$\frac{\mathrm{D}_{g}}{\mathrm{D}T}\left(E/\omega'\right) + \left(E/\omega'\right) \frac{1}{\left|\mathbf{c}_{\mathrm{g}}\right|} \frac{\mathrm{D}_{g} \left|\mathbf{c}_{\mathrm{g}}\right|}{\mathrm{D}T} = 0 \tag{2}$$

Then it is easy to derive its solution

$$E \sim \omega' \big/ \left|\mathbf{c}_{\mathrm{g}}\right| \tag{3}$$

Eq. (1) indicates that divergence of group velocity will be positive (negative) if the group velocity magnitude becomes faster and faster (slower and slower). Correspondingly, the wave action density will decrease (increase) according to Eq. (2). Physically, faster and faster (slower and slower) group velocity means the wave action density is divergent (convergent), which of course decreases (increases) the wave action density. Solution Eq. (3) means the wave energy density is proportional to the intrinsic frequency but inverse proportional to the magnitude of the group velocity.

Fourthly, if further considering the fact that a wave packet indeed occupies a certain spatial size, it seems that in principle one may directly calculate the divergence of group velocity from its expression (Bühler 2006). However, only the group velocity values on a ray are known. Neighbouring a ray, the group velocity values are unknown (Lighthill 2001). Physically, directly solving divergence of group velocity means the second order approximation should be introduced.

Finally, the solution Eq. (3) can also be derived from the multi-scale method (or WKB approximation), as emphasized by Lighthill (2001) in his seminal monograph. Under the zeroth-order approximation — valid at the slowly varying scale — the derived ***local*** dispersion relation mirrors the form of the plane wave solution. This equivalence implies that the wave is ***locally*** approximated as a plane wave with constant amplitude and wavenumbers at the slowly varying scale, leading to local conservation of wave action. Consequently, the partial derivative of wave action density with time vanishes, that is, wave action conservation equation

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega'}\right) + \nabla \cdot \left(\mathbf{c}_{\mathrm{g}} \frac{E}{\omega'}\right) = 0 \tag{4}$$

can be simplified to

$$\nabla\cdot\left(\mathbf{c}_{\mathrm{g}}\frac{E}{\omega'}\right)=0\,, \tag{5}$$

which means $\mathbf{c}_{\mathrm{g}}\,E/\omega'$ is a solenoidal vector (Lighthill 2001). It further demonstrates that $E/\omega'\left|\mathbf{c}_{\mathrm{g}}\right|=\text{constant}$ along a ray in terms of the cross-sectional area of a thin ray tube.

To sum up, the theoretical treatment ∇ • Cg has a solid mathematical and physical basis.

3. Line 192-193. About $\mathbf{c}_{\mathrm{g}}\,E/\omega'$.

Response: As responded above, the derivation here comes from the classic book by Lighthill (2001) who suggested Eq. (5) in the above response holds on and $\mathbf{c}_{\mathrm{g}}\,E/\omega'$ is a solenoidal vector. However, he did not further derive $E/\omega'\left|\mathbf{c}_{\mathrm{g}}\right|=\text{constant}$ along a ray. If we consider a ray as a 'particle' without spatial size, we can naturally derive $E/\omega'\left|\mathbf{c}_{\mathrm{g}}\right|=\text{constant}$ along a ray (since no spatial size, there is lateral flux across the particle so that the divergence is only determined by the variation of the group velocity magnitude. Imagine an elastic rope, where the variation in divergence depends only on the stretching of the rope).

4. Line 198. About Equation (15) in the manuscript.

Response: Thank you for your insightful comment. Here $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}$ (rather than $\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}$) is more suitable. In multi scale method, we assume the amplitude varies on a slow scale (X,Y,T) while the phase varies on a fast scale (x,y,t). The two scales are connected by a small parameter ε, that is, $\left(X,Y,T\right)=\left(\varepsilon x,\varepsilon y,\varepsilon t\right)$. Equation (15) in the manuscript (Line 198) is for wave energy which varies on the slow scale. Therefore, it is more reasonable to use the slow scale. However, applying the relation between the two scales, it is also OK to use the fast scales.

Your understanding is right. I have applied the $\frac{\mathrm{D}_g\omega'}{\mathrm{D}T}$ when deriving of the wave energy equation along the ray. Below I will explicitly show you the derivation.

The wave action conservation Eq. (4) in the above response can be expanded as

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega'}\right)+\mathbf{c}_{\mathrm{g}}\cdot\nabla\left(\frac{E}{\omega'}\right)+\left(\frac{E}{\omega'}\right)\nabla\cdot\mathbf{c}_{\mathrm{g}}=0 \tag{6}$$

The first two terms are the material derivative along rays. Therefore, Eq. (6) can be written as

$$\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right)+\left(\frac{E}{\omega'}\right)\nabla\cdot\mathbf{c}_\mathbf{g}=0 \tag{7}$$

Note that

$$\begin{aligned}\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right)&=\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}+E\frac{\mathrm{D}_g}{\mathrm{D}T}\frac{1}{\omega'}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}-\frac{E}{\omega'^2}\frac{\mathrm{D}_g\omega'}{\mathrm{D}T}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}-\frac{E}{\omega'^2}\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\omega-\bar{u}k\right)\end{aligned} \tag{8}$$

Furthermore, the frequency and zonal wavenumber are unchanged along rays. Therefore, Eq. (8) can be further expressed as

$$\begin{aligned}\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right)&=\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}+\frac{E}{\omega'^2}k\frac{\mathrm{D}_g\bar{u}}{\mathrm{D}T}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}+\frac{E}{\omega'^2}k\left(\frac{\partial\bar{u}}{\partial T}+c_{g,x}\frac{\partial\bar{u}}{\partial X}+c_{g,y}\frac{\partial\bar{u}}{\partial Y}\right)\\&=\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}+\frac{E}{\omega'^2}kc_{g,y}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}-\frac{E}{\omega'}\frac{K^2}{\beta^*k}k\frac{2\beta^*kl}{K^4}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}-\frac{E}{\omega'}\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}\end{aligned} \tag{9}$$

Note that we have applied the expression the meridional group velocity and intrinsic frequency, that is,

$$\begin{aligned}c_{g,y}&=\frac{2\beta^*kl}{K^4}\\\omega'&=\omega-\bar{u}k=-\frac{\beta^*k}{K^2}\end{aligned} \tag{10}$$

Substituting Eq. (9) into Eq. (7), we have

$$\frac{1}{\omega'}\frac{\mathrm{D}_g E}{\mathrm{D}T}-\frac{E}{\omega'}\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}+\frac{E}{\omega'}\nabla\cdot\mathbf{c}_\mathbf{g}=0 \tag{11}$$

Simplifying it, we can derive Equation (15) in the submitted manuscript, that is

$$\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T}=\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}-\nabla\cdot\mathbf{c}_\mathbf{g} \tag{12}$$

Following your comment and the other reviewer's comment, I have explicitly presented the above derivation in a new Appendix in the revised manuscript.

5. Line 217. About Equation (20) in the manuscript.

Response: Thanks for your insightful comment. Your understanding is right. In Equation (20) in the submitted manuscript (Line 217) the sign of ξ also depends on the sign of kl. That's why I choose a northward moving ray in a propagative region that is located south of the jet center when conducting theoretical analysis in Section 3. The northward moving means $c_{g,y} = \frac{2\beta^* kl}{K^4} > 0$, that is, $kl > 0$ (note that we have assumed $\beta^* > 0$). South of the jet center means $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} > 0$. Therefore, the sign of ξ will be only depended on λ. Actually, in line 221, the expression "leading structure" means $kl > 0$ while "trailing structure" means $kl < 0$. Maybe this term is not generally applied. Following the comment from the other reviewer, I have removed the expressing "leading/trailing" and to reorganize the sentence there. Thanks again.

6. Line 220. About the expressing "absorption".

Response: When $\lambda > 0$ (other terms are both positive), $\xi > 0$. Correspondingly, wave energy will increase. The increasing energy means the wave absorbs more energy from the basic flow. I think the expressing "absorption" should have the same meaning as the expressing "conversion" you used in this comment. Note that Equation (15) in the submitted manuscript indicates that there are two competing factors for wave energy variation. The first is the energy exchange with the basic flow (the wave absorbs energy from the basic flow or the wave lose energy to the basic flow). The second is the divergence of group velocity vector (when the divergence of group velocity is positive, wave energy is redistributed into a larger range so that wave energy decreases; when the divergence of group velocity is negative (or group velocity is in convergence), wave energy is redistributed into a smaller region so that wave energy increases). Therefore, if the expressing "convergence" means the convergence of group velocity, it would be not suitable. To be clearly, I used "absorbed" to replace "absorption" in the revised manuscript. I also change the corresponding sentences to "When $\lambda > 0$, energy absorbed from the basic flow (by structures tilting against the wind shear, that is, $kl > 0$ and positive wind shear) outpaces divergence of energy, resulting in net energy amplification. Conversely, $\lambda < 0$ indicates structures tilting with the wind shear ($kl < 0$) in positive wind shear regions transfer more energy to the basic flow, driving wave energy decay."

7. Line 237-238. About Equation (22) in the manuscript.

Response: These two lines are about the derivation of Equation (22) in the submitted manuscript. Section 3.1 in the submitted manuscript is a little confusing. Following the other reviewer's comment, I have totally rewritten this subsection. I also add a schematic figure to illustrate. In the rewritten subsection, the detailed and clearer derivations are present. Below I will explicitly present the process.

To analyze wave energy evolution, the wave energy equation along rays is quite fundamental. Therefore, I also express it here as (see Eq. (15) and Eq. (20) in the manuscript)

$$\begin{aligned}\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T} &= \left(\frac{2kl}{K^2}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y} - \frac{\mathrm{D}_g \ln\left|\mathbf{c}_\mathrm{g}\right|}{\mathrm{D}T}\right) \\ &\equiv \xi(T) \\ &= \frac{c^2 - \frac{2k^2}{K^2}(\overline{u}-c)^2}{c^2 + \frac{4k^2}{K^2}(\overline{u}-c)\overline{u}}\frac{2kl}{K^2}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y} \\ &\equiv \lambda(c,\overline{u})\frac{2kl}{K^2}\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y}\end{aligned} \tag{13}$$

As shown in the equation, the sign of the rate of change in wave energy depends on three terms, the coefficient λ, the wave structure kl, and the wind shear $\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y}$, the latter two of which are relatively easy to identify.

In section 3.1, I present a case where a ray for a wave with a fixed zonal phase speed (labeled as c_0 or c as in the revised manuscript) propagates from its initial location to a northern turning latitude in an ED regime that is located south of the westerly jet (see Response Figure 1a, b). These conditions means that $kl > 0$ and $\frac{\mathrm{d}\overline{u}}{\mathrm{d}Y} > 0$. Therefore, the increase or decrease in wave energy is only determined by the sign of λ. Note that this case is the simplest. As I have responded in your first comment, this setting can also be found in observed westerly jets. Furthermore, based on this simple case, more complex cases (e.g., the turning latitude is located north of the jet center) can also be analyzed.

According to the expression of λ, its sign is determined by its numerator. Setting the numerator equals to zero, that is

$$c^2 - \frac{2k^2}{K^2}(\overline{u}-c)^2 = 0 \tag{14}$$

it is easy to solve its zero point (labeled as $c_{\lambda 0}$)

$$c_{\lambda 0}(y) = \frac{\sqrt{2}k}{K(y)+\sqrt{2}k}\overline{u}(y) \tag{15}$$

Then we know when $c_0 < c_{\lambda 0}$, $\lambda < 0$, wave energy will decrease and vice versa. On one hand, the zonal phase speed c_0 keeps unchanged along rays. On the other hand, the zero point $c_{\lambda 0}$ varies along rays since both westerly and total wavenumber varies along rays. Therefore, to compare their relative size, we should identify the variation feature of $c_{\lambda 0}$ at first.

According to our prescribed conditions, when the ray moves from its initial location (y_0) to the northern turning latitude (y_t), the wind speed is monotonically increasing while the total wavenumber is monotonically decreasing (due to decreasing meridional wavenumber). We can know that $c_{\lambda 0}$ monotonically increases when the ray moves from y_0 to y_t (see Response Figure 1c), that is

$$c_a \equiv \frac{\sqrt{2}k}{K(y_0)+\sqrt{2}k}\bar{u}(y_0)=c_{\lambda 0}(y_0)\le c_{\lambda 0}(y)\le c_{\lambda 0}(y_t)=\frac{\sqrt{2}k}{K(y_t)+\sqrt{2}k}\bar{u}(y_t)\equiv c_b \tag{16}$$

It means the values of $c_{\lambda 0}$ at y_0 and at y_t are its minimum and maximum. For simplicity, we have labeled the minimum and maximum as c_a and c_b. Then it would be easy to compare the relative size of c_0 and $c_{\lambda 0}$. (1) If $c_0 < c_a$, that is, the zonal phase speed is slower than the minimum of $c_{\lambda 0}$, the zonal phase speed will be smaller than all values of $c_{\lambda 0}$ in the range (y_0, y_t). Therefore, $\lambda < 0$ and hence $\frac{\mathrm{D}_g E}{\mathrm{D}T} < 0$ in (y_0, y_t). Correspondingly, wave energy will monotonically decrease from its initial value (say, equal 1) to a minimum at y_t where $\lambda = 0$ due to $l = 0$. (2) If $c_0 > c_b$, the zonal phase speed will be larger than $c_{\lambda 0}(y)$, leading $\lambda > 0$ and hence $\frac{\mathrm{D}_g E}{\mathrm{D}T} > 0$ in (y_0, y_t). Correspondingly, wave energy increases to a maximum at y_t where $\lambda = 0$ due to $l = 0$. (3) If $c_a < c_0 < c_b$, there will exist an intermediate location (say, $y_0 < y_m < y_t$) where $c_0 = c_{\lambda 0}(y_m)$. Then when the ray moves from y_0 to y_m, $\lambda > 0$; when the ray arrives at y_m, $\lambda = 0$; and when the ray continues to move from y_m to y_t, $\lambda < 0$. Correspondingly, the wave energy will increase to a maximum at y_m and then decrease to a minimum at y_t. Above features are illustrated in Response Figure 1d, e, f.

When the ray arrives at the turning latitude, the dispersion relation becomes

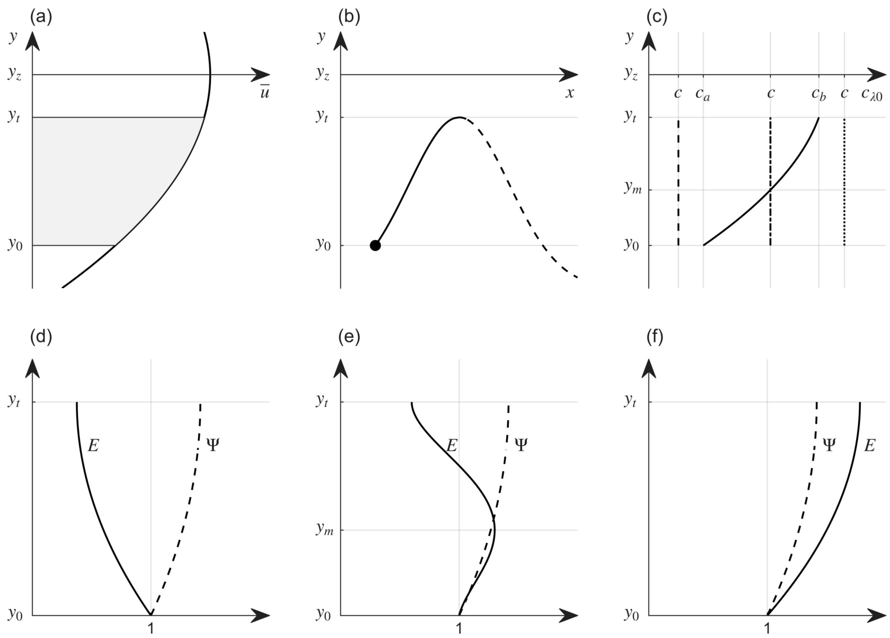
$$c_{yt} = \bar{u}(y_t) - \frac{\beta^*(y_t)}{k^2} \tag{17}$$

Note that c_{yt} prescribes the upper limit of the zonal phase speed of a wave whose ray can propagate in an ED regime (I used $c_{\max}$ to describe this upper limit in the submitted manuscript, which may be a little confusing. I have changed it to c_{yt} in the revised manuscript). If $c_0 > c_{yt}$, the ray will not be reflected by any turning point along its trajectory. Therefore the zonal phase speed range for a ray can propagate in an ED regime is $(0, c_{yt})$. Here qualitative analysis cannot provide the relative size of c_{yt} and c_a and c_b. Without loss of generality, we may assume that $c_{yt} > c_b$. Then the zonal phase speed range $(0, c_{yt})$ can be divided into three parts: $(0, c_a)$, (c_a, c_b), and (c_b, c_{yt}). It is obvious that wave energy can have the most significant increase in the last part.

Up to now, we have identified the evolution feature of wave energy. When a westerly jet is given, and when the initial zonal and meridional wavenumber are given, we may change frequency (of course, we can not arbitrarily specify the frequency value. It must satisfy the dispersion relation) or zonal phase speed to calculate the energy evolution of rays with different zonal phase speed (e.g., see Figure 3 in the submitted manuscript). Note that here the zonal phase speed is not the propagation speed of the ray, but the phase speed of the wave.

Finally, let's consider a slightly complex case where the turning point lies north of the jet center, with other conditions remaining unchanged. Without loss of generality, we analyze the scenario where the zonal phase speed falls within the interval (c_a, c_b). As the ray propagates from y_0 to y_m, wave energy increases to a maximum at y_m. When the ray travels from y_m to the jet center (say, y_z), wave energy declines from the maximum to a minimum at y_z where zero wind shear can also contribute to this minimum. As the ray proceeds from y_z to y_t, wave energy rebounds from the minimum to another maximum due to both negative wind shear and λ. Compared with the previous case, the most notable distinction is that wave energy can attain two distinct maxima as the ray propagates from the initial location to the turning point. Since analysis methodology remains consistent, we do not explicitly analyze this case.

Thank you for your valuable comments, which helps me strengthen the presentation of this work.



Response Figure 1 Schematic illustration of an ED regime (a) that is located south of a westerly jet; a ray propagates northward from an initial location denoted by the black point (b); the distribution of the zero point of λ (c); and the cases of $c < c_a$ (d); $c_a < c < c_b$ (e); and $c > c_b$ (f), exhibited by the dashed, dash-dotted, and dotted vertical line in (c). The notation y_0, y_t and y_z denote the initial location, the turning location, and the jet center, respectively. y_m indicates a intermediate location where $c = c_{\lambda 0}\left(y_m\right)$. E and Ψ denote wave energy and amplitude, respectively.

8. Line 289-290. About beta plane approximation.

Response: We are in β plane, that is, $f = f_0 + \beta_0 y$ (β_0 is a constant). The absolute vorticity of the basic flow is $f - \frac{\partial \overline{u}}{\partial y}$ and its meridional gradient $\beta^* = \beta_0 - \frac{\partial^2 \overline{u}}{\partial y^2}$ varies with y.

9. Line 310. About the notation u_I.

Response: I am sorry for choosing the similar subscript which is hard to distinguish. In the revised manuscript, I use $u_M \equiv u_0 / U$ to replace u_I to avoid confusion. Thank you for your valuable comment.

10. Line 313. About the typo.

Response: I am sorry for the typo here. I have corrected it in the revised manuscript. I have also carefully checked the manuscript to avoid similar mistakes. Thank you for your careful reviewing.

11. Line 322. About missing a word "show".

Response: Here it should be "it is easy to show that if ..." I am sorry for missing the word "show". I have corrected this mistake. Thank you for your careful reviewing.

12. Line 322-327. About Equation (34) in the manuscript.

Response: The derivation here is a bit abrupt and not easy to understand. Following your comment, I have added a new Appendix to explicit present the derivation process, which is also presented below.

The non-dimensional basic flow and meridional gradient of the absolute vorticity for basic flow can be written as

$$\begin{cases} \overline{u}_1 = u_M \operatorname{sech}^2(y_1) \\ \beta_1^* = \beta_1 - \dfrac{\mathrm{d}^2 \overline{u}_1}{\mathrm{d}y_1^2} \end{cases} \tag{18}$$

Substituting Eq. (18) into Eq. (17), we have

$$c_{yt} \equiv c_1(y_{1t}) = u_M \operatorname{sech}^2(y_{1t}) - \frac{1}{k_1^2}\left[\beta_1 - 4u_M \operatorname{sech}^2(y_{1t}) + 6u_M \operatorname{sech}^4(y_{1t})\right] \tag{19}$$

where y_{1t} denotes the non-dimensional location of the turning point. Now we consider c_{yt} as a function of y_{1t}, with its derivative being

$$\frac{\mathrm{d}c_{yt}}{\mathrm{d}y_{1t}} = -\frac{2u_M}{k_1^2}\left(k_1^2+4\right)\sinh(y_{1t})\left[\cosh^2(y_{1t}) - \frac{12}{k_1^2+4}\right]\operatorname{sech}^5(y_{1t}) \tag{20}$$

To identify the extreme values of c_{yt}, we solve the zero point of Eq. (20), which satisfies

$$\sinh(y_{1t})\left[\cosh^2(y_{1t}) - \frac{12}{k_1^2+4}\right] = 0 \tag{21}$$

It is easy to identify that $y_{1t} = 0$ is a zero point. The other zero points satisfy

$$\cosh^2(y_{1t}) = \frac{12}{k_1^2+4} \tag{22}$$

Since $\left|\cosh(y)\right| \ge 1$, To make sure Eq. (22) has real roots, we have $k_1^2 \le 8$. Setting $z = e^{y_{1t}}$ and substituting it into Eq. (22), we can obtain

$$z+\frac{1}{z}=2\sqrt{\frac{12}{k_1^2+4}} \tag{23}$$

whose solution is

$$z=\sqrt{\frac{12}{k_1^2+4}}\pm\sqrt{\frac{12}{k_1^2+4}-1} \tag{24}$$

Then it is easy to get the solution for Eq. (22), that is

$$y_{1t}=\ln\left[\sqrt{\frac{12}{k_1^2+4}}\pm\sqrt{\frac{12}{k_1^2+4}-1}\right]\equiv y_M \tag{25}$$

To identify the relative size of these extreme values, we further derive the second derivative

$$\frac{\mathrm{d}^2 c_{yt}}{\mathrm{d}y_{1t}^2}=\frac{2u_M}{k_1^2}\left[2\left(k_1^2+4\right)\sinh^4\left(y_{1t}\right)-44\sinh^2\left(y_{1t}\right)+k_1^2\sinh^2\left(y_{1t}\right)+8-k_1^2\right]\mathrm{sech}^6\left(y_{1t}\right)$$

(26)

According to Eq. (22), we know

$$\sinh^2\left(y_{1t}\right)=\frac{8-k_1^2}{k_1^2+4} \tag{27}$$

Substituting Eq. (27) into Eq. (26), we have

$$\left.\frac{\mathrm{d}^2 c_{yt}}{\mathrm{d}y_{1t}^2}\right|_{y_{1t}=y_M}=-48\frac{u_M}{k_1^2}\frac{8-k_1^2}{k_1^2+4}\mathrm{sech}^6\left(y_M\right)<0 \tag{28}$$

Therefore, we can identify c_{yt} reaches the maxima at y_M. On the other hand,

$$\left.\frac{\mathrm{d}^2 c_{yt}}{\mathrm{d}y_{1t}^2}\right|_{y_{1t}=0}=\left(8-k_1^2\right)\mathrm{sech}^6\left(y_M\right)>0 \tag{29}$$

Therefore, we can identify c_{yt} reaches a local minimum at $y_{1t}=0$. When $k_1^2>8$ this local minimum becomes the only maximum. Substituting $y_{1t}=0$ and $y_{1t}=y_M$ into Eq. (19), we can get the expressions of the local minimum and maxima, which writes as

$$\begin{aligned} c_m &\equiv c_1\left(0\right)=u_M-\frac{\beta_1+2u_M}{k_1^2} \\ c_M &\equiv c_1\left(y_M\right)=\left(k_1^2+4\right)\left[\frac{u_M}{24}-\frac{1}{k_1^2}\left(\frac{\beta_1}{k_1^2+4}-\frac{u_M}{6}\right)\right] \end{aligned} \tag{30}$$

13. Line 331. About the use of subscript p in Equation (35).

Response: Here I use c_p to denote the maximum of the critical point. We know that dispersion relation at the critical point ($y = y_c$) becomes

$$c = \overline{u}\left(y_c\right) \tag{31}$$

Therefore, the critical point is only determined by the basic flow and has no relation with the zonal and meridional wavenumber. As shown in Eq. (18), the basic flow is a westerly jet with maximum wind speed at $y = 0$. This maximum prescribes the maximum zonal phase speed for a ray. Therefore, I use

$$c_p = \max\left(\overline{u}\left(y_c\right)\right) \tag{32}$$

to denote the maximum zonal phase speed for a ray.

14. Line 334. About the propagative regimes of rays.

Response: Thank you for your insightful comment. The propagative regimes are important for understanding the propagative characteristic for rays. As I have responded above, when a ray with a fixed zonal wavenumber and a fixed zonal phase speed propagates in the fixed curved basic flow, the ray's meridional propagate is limited in regions enclosed by turning point and critical point. There are three types of propagative regime. The first is enclosed by a turning point and a critical point; the second is enclosed by two turning points; while the third is enclosed by two critical points. I have named them as evolution dispersion (ED), wave guide (WG), and bidirectional dispersion (BD) regimes, respectively. This had been explained Section 2.

Following your comment, I will explicitly describe these regimes by applying Figure 2 to start this paragraph. Thanks again.

15. Line 335-340. About the sentence in the paragraph.

Response: About the propagative regime, a ray indeed can propagate either in a WG or an ED regime when zonal phase speed varies from c_m to c_M as Figure 2a shows. Since I mainly discuss the situation in the ED case, I do not introduce the case in the WG regime. In the revised manuscript, I have revised the expressing in this paragraph. I have also divided the original paragraph into two paragraphs, following your anther comment in Line 339-340. Thanks for your constructive comments.

16. Line 345. About the gray shaded regions in Figure 2.

Response: Yes, the gray shaded zone is the propagative regions for rays in Figure 2. The gray zone is enclosed by two curves. The solid curve denotes the turning point where the meridional wavenumber vanishes. The dash-dotted curve denote the critical point where the meridional wavenumber square is infinity. Let a ray moves northward ($c_{g,y} = \dfrac{2\beta^* kl}{K^4} > 0$ note we firstly assume $\beta^* > 0$ for simplicity). The ray will stop to move northward at the turning point (where $l = 0$). Therefore the turning point is an natural boundary of a ray. When the ray moves toward the critical point ($c_{g,y} = \frac{2\beta^* kl}{K^4} \to 0$), it will gradually moves horizontally and it can never reach such a critical point in finite time. Therefore the critical point is an asymptotic location for a ray.

17. Line 367. About the prior theoretical prediction.

Response: I am sorry for the expressing here. I want to express the theoretical analysis in Section 3. I will explicitly state in Section 3 in the revised manuscript. Thank you for your valuable comment.

References

Bretherton, F. P., and C. J. R. Garrett, 1968: Wavetrains in inhomogeneous moving media. *Proceedings of the Royal Society of London. Series A*, **302**, 529–554, https://doi.org/10.1098/rspa.1968.0034.

Bühler, O., 2006: *A Brief Introduction to Classical, Statistical, and Quantum Mechanics*. American Mathematical Society, https://doi.org/10.1090/cln/013.

Li, Y., J. Chao, and Y. Kang, 2021: Variations in Wave Energy and Amplitudes along the Energy Dispersion Paths of Nonstationary Barotropic Rossby Waves. *Adv. Atmos. Sci.*, **38**, 49–64, https://doi.org/10.1007/s00376-020-0084-9.

Lighthill, J., 2001: *Waves in Fluids*. 2nd ed. Cambridge University Press 524pp.

Pedlosky, J., 1987: *Geophysical Fluid Dynamics*. Springer New York, NY 710pp., https://doi.org/10.1007/978-1-4612-4650-3.

Vallis, G. K., 2017: *Atmospheric and Oceanic Fluid Dynamics: Fundamentals and Large-Scale Circulation*. 2nd ed. Cambridge University Press, https://doi.org/10.1017/9781107588417.