Dear Prof. Gwendal Rivière:

Thank you for your careful review and valuable comments on my manuscript. In response to your questions and suggestions, I have carefully revised the manuscript and prepared a detailed point-by-point response.

I have added an new Appendix to introduce the derivative of the maximum and minimum of the turning point curve. I have also explicitly illustrate the rationality in solving the divergence of group velocity. Following the reviewer's comment, I have also added a new schematic figure to help understanding. I have rewritten Section 3.1. I have corrected the expressing mistakes and typos and carefully checked the manuscript to avoid similar mistakes.

All revisions have been marked in blue at the corresponding positions in the revised manuscript, and the detailed responses are provided in the response file.

Thank you again for your professional guidance. I hope the revised manuscript meets the journal's publication requirements.

Sincerely,
The author: Yaokun Li

Comments

1. Line 109. About missing the expressing of the material derivative along rays.

Response: Thank you for your careful reviewing. I am sorry for missing the formula "$\frac{\mathrm{D}_g}{\mathrm{D}T}=\frac{\partial}{\partial T}+\mathbf{c}_\mathbf{g}\cdot\nabla$" which denote the material derivative along group velocity vector $\mathbf{c}_\mathbf{g}$. I have corrected this mistake in the revised manuscript.

2. Line 175-176. About the derivation of the divergence of group velocity.

Response: Thank you for your insightful comment. Equation (12) and the theoretical treatment ∇ • Cg are fundamental for the manuscript. Here I will provide an explicit explanation.

Firstly, ray tracing method deems the wave packet as a "material particle" moving with the group velocity. As Lighthill (2001) wrote in his classic book "Waves in fluids": "waves behave like particles (moving along rays); indeed, a wave packet changes position and wavenumber which can be regarded as those for a 'particle' whose energy-momentum relationship at every position exactly parallels the frequency-wavenumber relationship (dispersion relationship) for the waves." It means we may use ray tracing method to track a wave packet from one location to another without solving the full wave equations of motion (Vallis 2017). Therefore, ray tracing is of value not only because it leads to information on the spatial distribution of the wavenumber vector but also because wave energy moves along rays (Lighthill 2001).

Secondly, let us consider the basic assumption of the slowly varying wave train which requires that a wave packet carries a wave with wavelength short enough so that ***locally*** the basic fields appears spatially uniform and ***locally*** the wave amplitude is very nearly constant over one wavelength (Pedlosky 1987). With above assumption, we can derive the ***local*** dispersion relation with has the same form with the plane wave with constant amplitude.

Since the group velocity over one wavelength is also ***locally*** spatial uniform, Pedlosky (1987) suggested that ***locally*** the divergence of group velocity vanishes so that the wave action density, E/ω', does not change along rays, or, the wave energy density is proportional to the intrinsic frequency, that is, $E\sim\omega'$. It provides a simple method to evaluate the wave energy density. Furthermore, it also denotes that the wave energy density is ***locally*** determined by the intrinsic frequency which is only determined by the basic flow when the frequency and zonal wavenumber are fixed.

However, as Bretherton and Garrett (1968) commented, $E\sim\omega'$ is for the lowest (or zeroth) order approximation. It means the variation of amplitude, which is proportional to the small parameter ε (the corresponding first order terms), has been ignored, which is necessary to derive the zeroth order approximation equation (the ***local*** dispersion relation) and may not be enough for the first order approximation equation, which is just the wave action conservation.

Thirdly, to be accurate for the first order approximation, the ray divergence should be introduced to revise the above treatment. As pointed out previously, a wave packet along a moving ray has been regarded as a 'particle'. The spatial size of the 'particle' occupies is large compared to a wavelength but small compared to the slowly varying basic flow. **Therefore, viewed from the scale of the basic flow, a wave packet appears as a point with no spatial size** (e.g., Bretherton and Garrett 1968). **Actually, ray theory can only give the trajectory of a wave**

packet, not the detailed structure of the waves within the packet (e.g., Vallis 2017).

Considering a particle (wave packet) with no spatial size moves with the group velocity, the divergence of group velocity is then only determined by the variation of the magnitude of the group velocity vector due to no lateral flux (Li et al. 2021), that is

$$\nabla \cdot \mathbf{c}_{\mathrm{g}} = \lim_{\delta T \to 0} \frac{\left|\mathbf{c}_{\mathrm{g}}\right|_{T+\delta T} - \left|\mathbf{c}_{\mathrm{g}}\right|_{T}}{\left|\mathbf{c}_{\mathrm{g}}\right| \delta T} = \frac{1}{\left|\mathbf{c}_{\mathrm{g}}\right|} \frac{\mathrm{D}_{g} \left|\mathbf{c}_{\mathrm{g}}\right|}{\mathrm{D}T} = \frac{\mathrm{D}_{g} \ln \left|\mathbf{c}_{\mathrm{g}}\right|}{\mathrm{D}T} \tag{1}$$

where T is time and δT is a small time interval and $\left|\mathbf{c}_{\mathrm{g}}\right| = \sqrt{c_{g,x}^{2} + c_{g,y}^{2}}$ denotes the magnitude of the group velocity vector. It should be stressed again that Eq. (1) is observed from the scale of the basic flow so that the size of the wave packet is ignored. With Eq. (1), the wave action conservation equation can be written as

$$\frac{\mathrm{D}_{g}}{\mathrm{D}T}\left(E/\omega'\right) + \left(E/\omega'\right) \frac{1}{\left|\mathbf{c}_{\mathrm{g}}\right|} \frac{\mathrm{D}_{g} \left|\mathbf{c}_{\mathrm{g}}\right|}{\mathrm{D}T} = 0 \tag{2}$$

Then it is easy to derive its solution

$$E \sim \omega' / \left|\mathbf{c}_{\mathrm{g}}\right| \tag{3}$$

Eq. (1) indicates that divergence of group velocity will be positive (negative) if the group velocity magnitude becomes faster and faster (slower and slower). Correspondingly, the wave action density will decrease (increase) according to Eq. (2). Physically, faster and faster (slower and slower) group velocity means the wave action density is divergent (convergent), which of course decreases (increases) the wave action density. Solution Eq. (3) means the wave energy density is proportional to the intrinsic frequency but inverse proportional to the magnitude of the group velocity.

Fourthly, if further considering the fact that a wave packet indeed occupies a certain spatial size, it seems that in principle one may directly calculate the divergence of group velocity from its expression (Bühler 2006). However, only the group velocity values on a ray are known. Neighbouring a ray, the group velocity values are unknown (Lighthill 2001). Physically, directly solving divergence of group velocity means the second order approximation should be introduced.

Finally, the solution Eq. (3) can also be derived from the multi-scale method (or WKB approximation), as emphasized by Lighthill (2001) in his seminal monograph. Under the zeroth-order approximation — valid at the slowly varying scale — the derived ***local*** dispersion relation mirrors the form of the plane wave solution. This equivalence implies that the wave is ***locally*** approximated as a plane wave with constant amplitude and wavenumbers at the slowly varying scale, leading to local conservation of wave action. Consequently, the partial derivative of wave action density with time vanishes, that is, wave action conservation equation

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega'}\right) + \nabla \cdot \left(\mathbf{c}_{\mathrm{g}} \frac{E}{\omega'}\right) = 0 \tag{4}$$

can be simplified to

$$\nabla \cdot \left(\mathbf{c}_{\mathrm{g}} \frac{E}{\omega'} \right) = 0, \tag{5}$$

which means $\mathbf{c}_{\mathrm{g}} E/\omega'$ is a solenoidal vector (Lighthill 2001). It further demonstrates that $E/\omega' |\mathbf{c}_{\mathrm{g}}| = \mathrm{constant}$ along a ray in terms of the cross-sectional area of a thin ray tube.

To sum up, the theoretical treatment ∇ • Cg has a solid mathematical and physical basis.

3. Line 192-193. About $\mathbf{c}_{\mathrm{g}} E/\omega'$.

Response: As responded above, the derivation here comes from the classic book by Lighthill (2001) who suggested Eq. (5) in the above response holds on and $\mathbf{c}_{\mathrm{g}} E/\omega'$ is a solenoidal vector. However, he did not further derive $E/\omega' |\mathbf{c}_{\mathrm{g}}| = \mathrm{constant}$ along a ray. If we consider a ray as a 'particle' without spatial size, we can naturally derive $E/\omega' |\mathbf{c}_{\mathrm{g}}| = \mathrm{constant}$ along a ray (since no spatial size, there is lateral flux across the particle so that the divergence is only determined by the variation of the group velocity magnitude. Imagine an elastic rope, where the variation in divergence depends only on the stretching of the rope).

4. Line 198. About Equation (15) in the manuscript.

Response: Thank you for your insightful comment. Here $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}$ (rather than $\frac{\mathrm{d}\bar{u}}{\mathrm{d}y}$) is more suitable. In multi scale method, we assume the amplitude varies on a slow scale (X, Y, T) while the phase varies on a fast scale (x, y, t). The two scales are connected by a small parameter ε, that is, $(X, Y, T) = (\varepsilon x, \varepsilon y, \varepsilon t)$. Equation (15) in the manuscript (Line 198) is for wave energy which varies on the slow scale. Therefore, it is more reasonable to use the slow scale. However, applying the relation between the two scales, it is also OK to use the fast scales.

Your understanding is right. I have applied the $\frac{\mathrm{D}_g \omega'}{\mathrm{D}T}$ when deriving of the wave energy equation along the ray. Below I will explicitly show you the derivation.

The wave action conservation Eq. (4) in the above response can be expanded as

$$\frac{\partial}{\partial T}\left(\frac{E}{\omega'}\right) + \mathbf{c}_{\mathrm{g}} \cdot \nabla \left(\frac{E}{\omega'}\right) + \left(\frac{E}{\omega'}\right) \nabla \cdot \mathbf{c}_{\mathrm{g}} = 0 \tag{6}$$

The first two terms are the material derivative along rays. Therefore, Eq. (6) can be written as

$$\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right)+\left(\frac{E}{\omega'}\right)\nabla\cdot\mathbf{c_g}=0 \tag{7}$$

Note that

$$\begin{aligned}\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right)&=\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}+E\frac{\mathrm{D}_g}{\mathrm{D}T}\frac{1}{\omega'}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}-\frac{E}{\omega'^2}\frac{\mathrm{D}_g\omega'}{\mathrm{D}T}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}-\frac{E}{\omega'^2}\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\omega-\bar{u}k\right)\end{aligned} \tag{8}$$

Furthermore, the frequency and zonal wavenumber are unchanged along rays. Therefore, Eq. (8) can be further expressed as

$$\begin{aligned}\frac{\mathrm{D}_g}{\mathrm{D}T}\left(\frac{E}{\omega'}\right)&=\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}+\frac{E}{\omega'^2}k\frac{\mathrm{D}_g\bar{u}}{\mathrm{D}T}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}+\frac{E}{\omega'^2}k\left(\frac{\partial\bar{u}}{\partial T}+c_{g,x}\frac{\partial\bar{u}}{\partial X}+c_{g,y}\frac{\partial\bar{u}}{\partial Y}\right)\\&=\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}+\frac{E}{\omega'^2}kc_{g,y}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}-\frac{E}{\omega'}\frac{K^2}{\beta^*k}k\frac{2\beta^*kl}{K^4}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}\\&=\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}-\frac{E}{\omega'}\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}\end{aligned} \tag{9}$$

Note that we have applied the expression the meridional group velocity and intrinsic frequency, that is,

$$\begin{aligned}c_{g,y}&=\frac{2\beta^*kl}{K^4}\\\omega'&=\omega-\bar{u}k=-\frac{\beta^*k}{K^2}\end{aligned} \tag{10}$$

Substituting Eq. (9) into Eq. (7), we have

$$\frac{1}{\omega'}\frac{\mathrm{D}_gE}{\mathrm{D}T}-\frac{E}{\omega'}\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}+\frac{E}{\omega'}\nabla\cdot\mathbf{c_g}=0 \tag{11}$$

Simplifying it, we can derive Equation (15) in the submitted manuscript, that is

$$\frac{1}{E}\frac{\mathrm{D}_gE}{\mathrm{D}T}=\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}-\nabla\cdot\mathbf{c_g} \tag{12}$$

Following your comment and the other reviewer's comment, I have explicitly presented the above derivation in a new Appendix in the revised manuscript.

5. Line 217. About Equation (20) in the manuscript.

Response: Thanks for your insightful comment. Your understanding is right. In Equation (20) in the submitted manuscript (Line 217) the sign of ξ also depends on the sign of kl. That's why I choose a northward moving ray in a propagative region that is located south of the jet center when conducting theoretical analysis in Section 3. The northward moving means $c_{g,y} = \frac{2\beta^* kl}{K^4} > 0$, that is, $kl > 0$ (note that we have assumed $\beta^* > 0$). South of the jet center means $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} > 0$. Therefore, the sign of ξ will be only depended on λ. Actually, in line 221, the expression "leading structure" means $kl > 0$ while "trailing structure" means $kl < 0$. Maybe this term is not generally applied. Following the comment from the other reviewer, I have removed the expressing "leading/trailing" and to reorganize the sentence there. Thanks again.

6. Line 220. About the expressing "absorption".

Response: When $\lambda > 0$ (other terms are both positive), $\xi > 0$. Correspondingly, wave energy will increase. The increasing energy means the wave absorbs more energy from the basic flow. I think the expressing "absorption" should have the same meaning as the expressing "conversion" you used in this comment. Note that Equation (15) in the submitted manuscript indicates that there are two competing factors for wave energy variation. The first is the energy exchange with the basic flow (the wave absorbs energy from the basic flow or the wave lose energy to the basic flow). The second is the divergence of group velocity vector (when the divergence of group velocity is positive, wave energy is redistributed into a larger range so that wave energy decreases; when the divergence of group velocity is negative (or group velocity is in convergence), wave energy is redistributed into a smaller region so that wave energy increases). Therefore, if the expressing "convergence" means the convergence of group velocity, it would be not suitable. To be clearly, I used "absorbed" to replace "absorption" in the revised manuscript. I also change the corresponding sentences to "When $\lambda > 0$, energy absorbed from the basic flow (by structures tilting against the wind shear, that is, $kl > 0$ and positive wind shear) outpaces divergence of energy, resulting in net energy amplification. Conversely, $\lambda < 0$ indicates structures tilting with the wind shear ($kl < 0$) in positive wind shear regions transfer more energy to the basic flow, driving wave energy decay."

7. Line 237-238. About Equation (22) in the manuscript.

Response: These two lines are about the derivation of Equation (22) in the submitted manuscript. Section 3.1 in the submitted manuscript is a little confusing. Following the other reviewer's comment, I have totally rewritten this subsection. I also add a schematic figure to illustrate. In the rewritten subsection, the detailed and clearer derivations are present. Below I will explicitly present the process.

To analyze wave energy evolution, the wave energy equation along rays is quite fundamental. Therefore, I also express it here as (see Eq. (15) and Eq. (20) in the manuscript)

$$\begin{aligned}\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T} &= \left(\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} - \frac{\mathrm{D}_g \ln|\mathbf{c}_\mathrm{g}|}{\mathrm{D}T}\right) \\ &\equiv \xi(T) \\ &= \frac{c^2 - \frac{2k^2}{K^2}(\bar{u}-c)^2}{c^2 + \frac{4k^2}{K^2}(\bar{u}-c)\bar{u}}\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} \\ &\equiv \lambda(c,\bar{u})\frac{2kl}{K^2}\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}\end{aligned} \tag{13}$$

As shown in the equation, the sign of the rate of change in wave energy depends on three terms, the coefficient λ, the wave structure kl, and the wind shear $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y}$, the latter two of which are relatively easy to identify.

In section 3.1, I present a case where a ray for a wave with a fixed zonal phase speed (labeled as c_0 or c as in the revised manuscript) propagates from its initial location to a northern turning latitude in an ED regime that is located south of the westerly jet (see Response Figure 1a, b). These conditions means that $kl > 0$ and $\frac{\mathrm{d}\bar{u}}{\mathrm{d}Y} > 0$. Therefore, the increase or decrease in wave energy is only determined by the sign of λ. Note that this case is the simplest. As I have responded in your first comment, this setting can also be found in observed westerly jets. Furthermore, based on this simple case, more complex cases (e.g., the turning latitude is located north of the jet center) can also be analyzed.

According to the expression of λ, its sign is determined by its numerator. Setting the numerator equals to zero, that is

$$c^2 - \frac{2k^2}{K^2}(\bar{u}-c)^2 = 0 \tag{14}$$

it is easy to solve its zero point (labeled as $c_{\lambda 0}$)

$$c_{\lambda 0}(y) = \frac{\sqrt{2}k}{K(y)+\sqrt{2}k}\bar{u}(y) \tag{15}$$

Then we know when $c_0 < c_{\lambda 0}$, $\lambda < 0$, wave energy will decrease and vice versa. On one hand, the zonal phase speed c_0 keeps unchanged along rays. On the other hand, the zero point $c_{\lambda 0}$ varies along rays since both westerly and total wavenumber varies along rays. Therefore, to compare their relative size, we should identify the variation feature of $c_{\lambda 0}$ at first.

According to our prescribed conditions, when the ray moves from its initial location (y_0) to the northern turning latitude (y_t), the wind speed is monotonically increasing while the total wavenumber is monotonically decreasing (due to decreasing meridional wavenumber). We can know that $c_{\lambda 0}$ monotonically increases when the ray moves from y_0 to y_t (see Response Figure 1c), that is

$$c_a \equiv \frac{\sqrt{2}k}{K(y_0)+\sqrt{2}k}\bar{u}(y_0)=c_{\lambda 0}(y_0)\le c_{\lambda 0}(y)\le c_{\lambda 0}(y_t)=\frac{\sqrt{2}k}{K(y_t)+\sqrt{2}k}\bar{u}(y_t)\equiv c_b \quad (16)$$

It means the values of $c_{\lambda 0}$ at y_0 and at y_t are its minimum and maximum. For simplicity, we have labeled the minimum and maximum as c_a and c_b. Then it would be easy to compare the relative size of c_0 and $c_{\lambda 0}$. (1) If $c_0 < c_a$, that is, the zonal phase speed is slower than the minimum of $c_{\lambda 0}$, the zonal phase speed will be smaller than all values of $c_{\lambda 0}$ in the range (y_0, y_t). Therefore, $\lambda < 0$ and hence $\frac{\mathrm{D}_g E}{\mathrm{D}T} < 0$ in (y_0, y_t). Correspondingly, wave energy will monotonically decrease from its initial value (say, equal 1) to a minimum at y_t where $\lambda = 0$ due to $l = 0$. (2) If $c_0 > c_b$, the zonal phase speed will be larger than $c_{\lambda 0}(y)$, leading $\lambda > 0$ and hence $\frac{\mathrm{D}_g E}{\mathrm{D}T} > 0$ in (y_0, y_t). Correspondingly, wave energy increases to a maximum at y_t where $\lambda = 0$ due to $l = 0$. (3) If $c_a < c_0 < c_b$, there will exist an intermediate location (say, $y_0 < y_m < y_t$) where $c_0 = c_{\lambda 0}(y_m)$. Then when the ray moves from y_0 to y_m, $\lambda > 0$; when the ray arrives at y_m, $\lambda = 0$; and when the ray continues to move from y_m to y_t, $\lambda < 0$. Correspondingly, the wave energy will increase to a maximum at y_m and then decrease to a minimum at y_t. Above features are illustrated in Response Figure 1d, e, f.

When the ray arrives at the turning latitude, the dispersion relation becomes

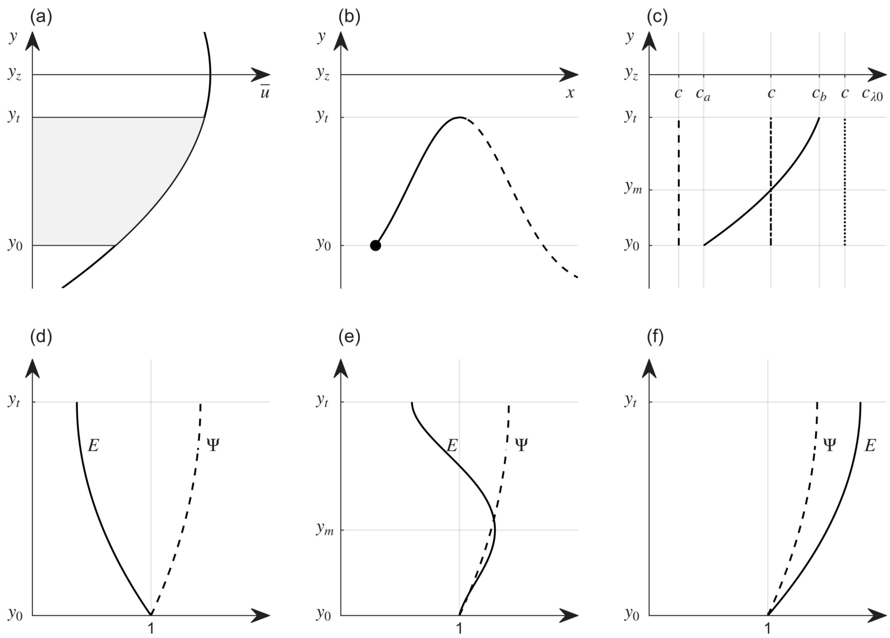
$$c_{yt} = \bar{u}(y_t) - \frac{\beta^*(y_t)}{k^2} \tag{17}$$

Note that c_{yt} prescribes the upper limit of the zonal phase speed of a wave whose ray can propagate in an ED regime (I used $c_{\max}$ to describe this upper limit in the submitted manuscript, which may be a little confusing. I have changed it to c_{yt} in the revised manuscript). If $c_0 > c_{yt}$, the ray will not be reflected by any turning point along its trajectory. Therefore the zonal phase speed range for a ray can propagate in an ED regime is $(0, c_{yt})$. Here qualitative analysis cannot provide the relative size of c_{yt} and c_a and c_b. Without loss of generality, we may assume that $c_{yt} > c_b$. Then the zonal phase speed range $(0, c_{yt})$ can be divided into three parts: $(0, c_a)$, (c_a, c_b), and (c_b, c_{yt}). It is obvious that wave energy can have the most significant increase in the last part.

Up to now, we have identified the evolution feature of wave energy. When a westerly jet is given, and when the initial zonal and meridional wavenumber are given, we may change frequency (of course, we can not arbitrarily specify the frequency value. It must satisfy the dispersion relation) or zonal phase speed to calculate the energy evolution of rays with different zonal phase speed (e.g., see Figure 3 in the submitted manuscript). Note that here the zonal phase speed is not the propagation speed of the ray, but the phase speed of the wave.

Finally, let's consider a slightly complex case where the turning point lies north of the jet center, with other conditions remaining unchanged. Without loss of generality, we analyze the scenario where the zonal phase speed falls within the interval (c_a, c_b). As the ray propagates from y_0 to y_m, wave energy increases to a maximum at y_m. When the ray travels from y_m to the jet center (say, y_z), wave energy declines from the maximum to a minimum at y_z where zero wind shear can also contribute to this minimum. As the ray proceeds from y_z to y_t, wave energy rebounds from the minimum to another maximum due to both negative wind shear and λ. Compared with the previous case, the most notable distinction is that wave energy can attain two distinct maxima as the ray propagates from the initial location to the turning point. Since analysis methodology remains consistent, we do not explicitly analyze this case.

Thank you for your valuable comments, which helps me strengthen the presentation of this work.



Response Figure 1 Schematic illustration of an ED regime (a) that is located south of a westerly jet; a ray propagates northward from an initial location denoted by the black point (b); the distribution of the zero point of λ (c); and the cases of $c < c_a$ (d); $c_a < c < c_b$ (e); and $c > c_b$ (f), exhibited by the dashed, dash-dotted, and dotted vertical line in (c). The notation y_0, y_t and y_z denote the initial location, the turning location, and the jet center, respectively. y_m indicates a intermediate location where $c = c_{\lambda 0}\left(y_m\right)$. E and Ψ denote wave energy and amplitude, respectively.

8. Line 289-290. About beta plane approximation.

Response: We are in β plane, that is, $f = f_0 + \beta_0 y$ (β_0 is a constant). The absolute vorticity of the basic flow is $f - \frac{\partial \bar{u}}{\partial y}$ and its meridional gradient $\beta^* = \beta_0 - \frac{\partial^2 \bar{u}}{\partial y^2}$ varies with y.

9. Line 310. About the notation u_I.

Response: I am sorry for choosing the similar subscript which is hard to distinguish. In the revised manuscript, I use $u_M \equiv u_0 / U$ to replace u_I to avoid confusion. Thank you for your valuable comment.

10. Line 313. About the typo.

Response: I am sorry for the typo here. I have corrected it in the revised manuscript. I have also carefully checked the manuscript to avoid similar mistakes. Thank you for your careful reviewing.

11. Line 322. About missing a word "show".

Response: Here it should be "it is easy to show that if ..." I am sorry for missing the word "show". I have corrected this mistake. Thank you for your careful reviewing.

12. Line 322-327. About Equation (34) in the manuscript.

Response: The derivation here is a bit abrupt and not easy to understand. Following your comment, I have added a new Appendix to explicit present the derivation process, which is also presented below.

The non-dimensional basic flow and meridional gradient of the absolute vorticity for basic flow can be written as

$$\begin{cases} \overline{u}_1 = u_M \operatorname{sech}^2(y_1) \\ \beta_1^* = \beta_1 - \dfrac{\mathrm{d}^2\overline{u}_1}{\mathrm{d}y_1^2} \end{cases} \tag{18}$$

Substituting Eq. (18) into Eq. (17), we have

$$c_{yt} \equiv c_1(y_{1t}) = u_M \operatorname{sech}^2(y_{1t}) - \frac{1}{k_1^2}\left[\beta_1 - 4u_M \operatorname{sech}^2(y_{1t}) + 6u_M \operatorname{sech}^4(y_{1t})\right] \tag{19}$$

where y_{1t} denotes the non-dimensional location of the turning point. Now we consider c_{yt} as a function of y_{1t}, with its derivative being

$$\frac{\mathrm{d}c_{yt}}{\mathrm{d}y_{1t}} = -\frac{2u_M}{k_1^2}\left(k_1^2+4\right)\sinh(y_{1t})\left[\cosh^2(y_{1t}) - \frac{12}{k_1^2+4}\right]\operatorname{sech}^5(y_{1t}) \tag{20}$$

To identify the extreme values of c_{yt}, we solve the zero point of Eq. (20), which satisfies

$$\sinh(y_{1t})\left[\cosh^2(y_{1t}) - \frac{12}{k_1^2+4}\right] = 0 \tag{21}$$

It is easy to identify that $y_{1t} = 0$ is a zero point. The other zero points satisfy

$$\cosh^2(y_{1t}) = \frac{12}{k_1^2+4} \tag{22}$$

Since $\left|\cosh(y)\right| \geq 1$, To make sure Eq. (22) has real roots, we have $k_1^2 \leq 8$. Setting $z = e^{y_{1t}}$ and substituting it into Eq. (22), we can obtain

$$z+\frac{1}{z}=2\sqrt{\frac{12}{k_1^2+4}} \tag{23}$$

whose solution is

$$z=\sqrt{\frac{12}{k_1^2+4}}\pm\sqrt{\frac{12}{k_1^2+4}-1} \tag{24}$$

Then it is easy to get the solution for Eq. (22), that is

$$y_{1t}=\ln\left[\sqrt{\frac{12}{k_1^2+4}}\pm\sqrt{\frac{12}{k_1^2+4}-1}\right]\equiv y_M \tag{25}$$

To identify the relative size of these extreme values, we further derive the second derivative

$$\frac{\mathrm{d}^2 c_{yt}}{\mathrm{d}y_{1t}^2}=\frac{2u_M}{k_1^2}\left[2\left(k_1^2+4\right)\sinh^4\left(y_{1t}\right)-44\sinh^2\left(y_{1t}\right)+k_1^2\sinh^2\left(y_{1t}\right)+8-k_1^2\right]\mathrm{sech}^6\left(y_{1t}\right)$$

(26)

According to Eq. (22), we know

$$\sinh^2\left(y_{1t}\right)=\frac{8-k_1^2}{k_1^2+4} \tag{27}$$

Substituting Eq. (27) into Eq. (26), we have

$$\left.\frac{\mathrm{d}^2 c_{yt}}{\mathrm{d}y_{1t}^2}\right|_{y_{1t}=y_M}=-48\frac{u_M}{k_1^2}\frac{8-k_1^2}{k_1^2+4}\mathrm{sech}^6\left(y_M\right)<0 \tag{28}$$

Therefore, we can identify c_{yt} reaches the maxima at y_M. On the other hand,

$$\left.\frac{\mathrm{d}^2 c_{yt}}{\mathrm{d}y_{1t}^2}\right|_{y_{1t}=0}=\left(8-k_1^2\right)\mathrm{sech}^6\left(y_M\right)>0 \tag{29}$$

Therefore, we can identify c_{yt} reaches a local minimum at $y_{1t}=0$. When $k_1^2>8$ this local minimum becomes the only maximum. Substituting $y_{1t}=0$ and $y_{1t}=y_M$ into Eq. (19), we can get the expressions of the local minimum and maxima, which writes as

$$\begin{aligned} c_m &\equiv c_1\left(0\right)=u_M-\frac{\beta_1+2u_M}{k_1^2} \\ c_M &\equiv c_1\left(y_M\right)=\left(k_1^2+4\right)\left[\frac{u_M}{24}-\frac{1}{k_1^2}\left(\frac{\beta_1}{k_1^2+4}-\frac{u_M}{6}\right)\right] \end{aligned} \tag{30}$$

13. Line 331. About the use of subscript p in Equation (35).

Response: Here I use c_p to denote the maximum of the critical point. We know that dispersion relation at the critical point ($y = y_c$) becomes

$$c = \overline{u}\left(y_c\right) \tag{31}$$

Therefore, the critical point is only determined by the basic flow and has no relation with the zonal and meridional wavenumber. As shown in Eq. (18), the basic flow is a westerly jet with maximum wind speed at $y = 0$. This maximum prescribes the maximum zonal phase speed for a ray. Therefore, I use

$$c_p = \max\left(\overline{u}\left(y_c\right)\right) \tag{32}$$

to denote the maximum zonal phase speed for a ray.

14. Line 334. About the propagative regimes of rays.

Response: Thank you for your insightful comment. The propagative regimes are important for understanding the propagative characteristic for rays. As I have responded above, when a ray with a fixed zonal wavenumber and a fixed zonal phase speed propagates in the fixed curved basic flow, the ray's meridional propagate is limited in regions enclosed by turning point and critical point. There are three types of propagative regime. The first is enclosed by a turning point and a critical point; the second is enclosed by two turning points; while the third is enclosed by two critical points. I have named them as evolution dispersion (ED), wave guide (WG), and bidirectional dispersion (BD) regimes, respectively. This had been explained Section 2.

Following your comment, I will explicitly describe these regimes by applying Figure 2 to start this paragraph. Thanks again.

15. Line 335-340. About the sentence in the paragraph.

Response: About the propagative regime, a ray indeed can propagate either in a WG or an ED regime when zonal phase speed varies from c_m to c_M as Figure 2a shows. Since I mainly discuss the situation in the ED case, I do not introduce the case in the WG regime. In the revised manuscript, I have revised the expressing in this paragraph. I have also divided the original paragraph into two paragraphs, following your anther comment in Line 339-340. Thanks for your constructive comments.

16. Line 345. About the gray shaded regions in Figure 2.

Response: Yes, the gray shaded zone is the propagative regions for rays in Figure 2. The gray zone is enclosed by two curves. The solid curve denotes the turning point where the meridional wavenumber vanishes. The dash-dotted curve denote the critical point where the meridional wavenumber square is infinity. Let a ray moves northward ($c_{g,y} = \dfrac{2\beta^{*}kl}{K^4} > 0$ note we firstly assume $\beta^{*}>0$ for simplicity). The ray will stop to move northward at the turning point (where $l=0$). Therefore the turning point is an natural boundary of a ray. When the ray moves toward the critical point ($c_{g,y}=\frac{2\beta^{*}kl}{K^{4}}\to 0$), it will gradually moves horizontally and it can never reach such a critical point in finite time. Therefore the critical point is an asymptotic location for a ray.

17. Line 367. About the prior theoretical prediction.

Response: I am sorry for the expressing here. I want to express the theoretical analysis in Section 3. I will explicitly state in Section 3 in the revised manuscript. Thank you for your valuable comment.

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