Dear Reviewer,

Thank you very much for your insightful comments and constructive suggestions on my manuscript. I have carefully considered all your feedback and made substantial revisions to address the issues raised. I believe these changes have significantly improved the quality and rigor of the paper. Please find my detailed responses to each comment below. I hope the revised manuscript meets the journal's standards and look forward to your further feedback.

Sincerely,

The author: Yaokun Li

Comments

Line 34-49: The emphasis on classical sufficient conditions for stability is outdated and needs to be rewritten to bring it up to date. In particular, a major sufficient condition for inviscid shear instability has been established, called the Hurdle Theorem. See Deguchi et al. (2024, J. Fluid Mech., 997, A25, doi:10.1017/jfm.2024.728) and Read and Dowling (2026, Encycl. Atmos. Sci. (3rd Ed.) 4, 263—283, Academic Press, doi:10.1016/B978-0-323-96026-7.00211-3).

Response: Thank you for highlighting the need to update the discussion on stability conditions and for bringing the Hurdle Theorem to my attention.

In the revised manuscript, I have added the introduction of the Hurdle Theorem, the sufficient condition for inviscid shear instability, after describing the classical sufficient conditions. The added contents are "More recently, Deguchi et al. (2024) established a sufficient condition for inviscid shear instability, called the hurdle theorem, which states that a flow is unstable if there is an interval in the flow domain for which the reciprocal Rossby Mach number (a quantity defined in terms of the zonal flow and potential vorticity distribution) surpasses a certain threshold or "hurdle". The theorem offers new insights into the theoretical understanding of pattern formation in planetary atmospheres." They are in Line 51-55.

Ray tracing is heavily used, but without first establishing the context for which it is accurate and discussing general conditions when it is inaccurate, such as inhomogeneous environments and proximity to focusing points (caustics). This can be fixed by adding a short paragraph discussing the issues early in the introduction. There is a brief mention of such an issue on Line 132, another on Line 144-152, and a workaround on Line 191+ (Section 3). These could usefully be tied into an introductory short paragraph on the general strengths and shortcomings of ray tracing.

Response: Thank you for your insightful comment. Following your comment, I have added a brief paragraph to discuss the limitations of the ray tracing method. Firstly, the concept of Rossby wave packet naturally requires large - scale, slowly varying amplitude. Secondly, ray tracing becomes inaccurate in fairly narrow and strong jets where the refractive index varies significantly so that WKB approximation becomes invalid. Another critical failure occurs in the vicinity of caustics and singularities. At caustics, where ray densities become infinite, conventional ray theory breaks down, requiring explicit phase and amplitude corrections to the computed ray tubes. Finally, in strongly anisotropic or absorbing media, rays may encounter singular directions where wave velocities coincide, resulting in numerical instabilities and indefinite expressions in ray - tracing equations.

Above revisions can be seen in Line 74-81 in the revised manuscript.

On a related note, it is not clear while reading this paper at what point or points in the evolution of an unstable shear flow the theory applies. When a shear flow is truly barotropically unstable, the end result often bears little resemblance to the initial conditions, a point made in e.g., the review by Read and Dowling (2026). It would help to add a few guideposts throughout the paper that indicate whether we are always teetering on marginal stability or not, and at what point does the evolution of unstable flow render the discussion moot.

Response: Thank you for your constructive comment. Physically speaking, the solution Eq. (2) demonstrates that we consider the initial stage of perturbation development. In this stage, the amplitude is relatively weak so that the linearization approximation is reasonable and the wave-like solution is possible. When the amplitude is such strong to induce possible instability, the end result often bears little resemblance to the initial conditions, a point made in e.g., the review by Read and Dowling (2026).

To address your concern, in the revised manuscript, I have stressed that the theory in this paper primarily applies to the linear growth stage of unstable shear flow evolution (i.e., the initial stage where perturbation amplitudes are small and the basic flow remains largely unchanged). At this stage, the flow structure is still close to the initial conditions, and the assumptions of linear stability analysis (such as the small perturbation approximation) hold.

Note that I do not specify explicit initial values for wave energy and amplitude because we are more concerned with the growth rate of the wave energy and amplitude relative to the small initial values. When the wave energy and amplitude exceed a certain value, thus violating the linearization approximation, the wave packet may enter the nonlinear evolution stage. Note that I do not specify an explicit threshold since it is hard to determine such a criterion. I predict that we may obtain an approximate criterion by analyzing the observed data, which deserves further investigation.

Above revisions are in Line 106-109, and in Line 224-227 in the revised manuscript.

It should be made clear early in the paper that the system under study has a flat bottom topography and thus misses out on an entire class of marginally stable cases that are especially relevant to Jupiter and Saturn. See Deguchi et al. (2024) for an extended discussion of this point. This limitation simply needs to be made explicit somewhere in the paper's introduction, and ties into the conclusions and future work, for example the comment on Line 472 about "real-world atmospheric flows".

Response: Thank you for highlighting this important limitation regarding the flat bottom topography assumption. I fully agree with your suggestion and have made the following revisions to address this point.

First, I have added some dedicated sentences just after Eq. (1) in the manuscript. These sentences (now in Line 97-100 in the revised manuscript) clarify that we do not consider the influence of the topography, thus missing out an entire class of marignially stable cases that are especially relevant to Jupiter and Saturn where fully dynamic weather layer overlies a layer containing a deep jet profile. One may refer to the research by Deguchi et al. (2024) for an extended discussion.

Second, I have added a short discussion in the final Conclusions and discussion Section to stress that the current investigation can be extended to explore the influence of the large-scale topography on the atmosphere. Moreover, I stress that it can also be applied o giant gas planets such as Jupiter and Saturn where a deep-layer jet, serving as a dynamical topography, also plays a significant role in modulating the weather layer above it.

The Hurdle Theorem (Deguchi et al. 2024) can be readily applied to the u = sech2(y) prototype (29) analysed in Section 4, which will significantly enhance the discussion, including the material shown in Fig. 1 c), which currently only illustrates sufficient-for-stability criteria and is lacking sufficient-for-instability criteria. Some of the main figures later in the paper are also ripe for addition of Hurdle Theorem regions.

Response: Thank you for your valuable suggestion regarding the application of the Hurdle Theorem. I fully agree that integrating this theorem will strengthen the analysis.

First, I have carefully read the paper by Deguchi et al. (2024) to study the Hurdle Theorem and have attempted to apply it. Below is my understanding (I hope it is correct).

In this investigation, I specify zonal basic flow as

$$\bar{u} = u_0 \operatorname{sech}^2 y \tag{1}$$

The corresponding meridional gradient of the absolute vorticity writes as

$$\beta^* = \beta_1 - u_0 \operatorname{sech}^2 y \left(4 - 6\operatorname{sech}^2 y\right) \tag{2}$$

When $u_0 > u_s \equiv \frac{3}{2}\beta_1 \simeq 2.428$, β^* begins to have zero points. The reciprocal Rossby Mach number $M_\alpha^{-1} = \frac{1}{k_0^2}\frac{\beta^*}{\bar{u}-\alpha}$ where $k_0 = \frac{\pi}{L}$ is the smallest meridional wavenumber associated with the largest meridional horizontal scale and L is the meridional range and α is a real number that need to be determined. Following Deguchi et al. (Deguchi et al. 2024), the zero points of β^* are labeled as $-y_i$ and $-y_j$ ($y_i = 1.7028, y_j = 0.8227$). Note that y_i and y_j are also zero points due to symmetry. The zonal basic flow speed at $-y_i$ and $-y_j$ are labeled as $\alpha_i \equiv \bar{u}(-y_i) = 0.4975$ and $\alpha_j \equiv \bar{u}(-y_j) = 2.1692$. As shown in Fig. 1 and Fig. 2 in this reply, we can see that the the zonal basic flow is case (ii) if choosing $\alpha = \alpha_j$ (both $M_\alpha^{-1}(-y_i) = 8.8256\times 10^{-5}$ and $M_\alpha^{-1}(-y_j) = 9.1467$ are larger than zero to ensure the one-signed condition although the first is quite close to zero). The hurdle, defined as

$$h = \frac{\frac{\pi^2}{(y_2-y_1)^2}}{\frac{\pi^2}{L^2}} = \frac{L^2}{(y_2-y_1)^2} \geq 1 \tag{3}$$

is always larger than one where $y_2 - y_1$ is a sub-range in the the prescribed region with the length of L. However, there is no $y_2 - y_1$ region where the reciprocal Rossby Mach number is larger than the hurdle. On the other hand, as shown in Fig. 2, there indeed are regions larger than a hurdle determined by a region.

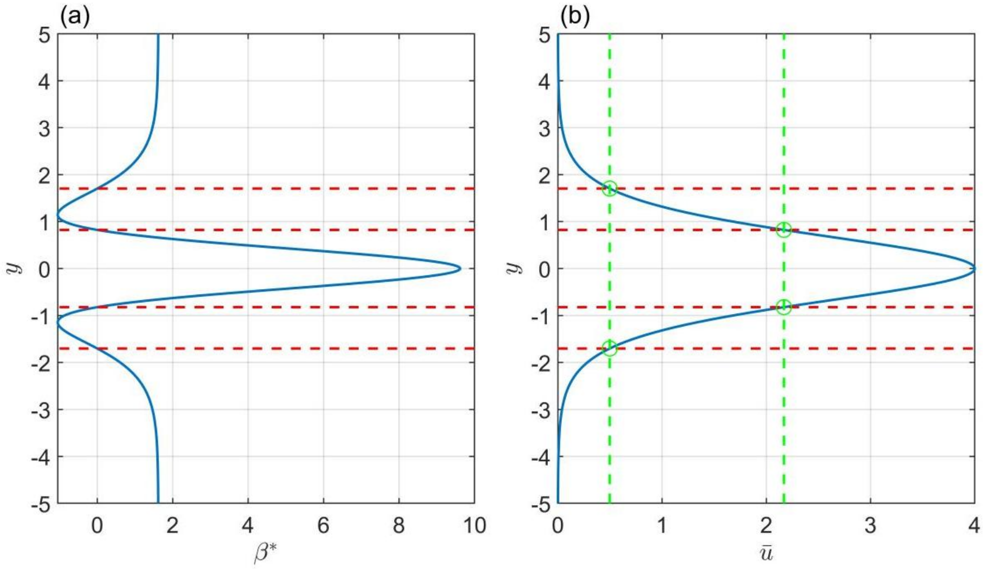

Figure 1

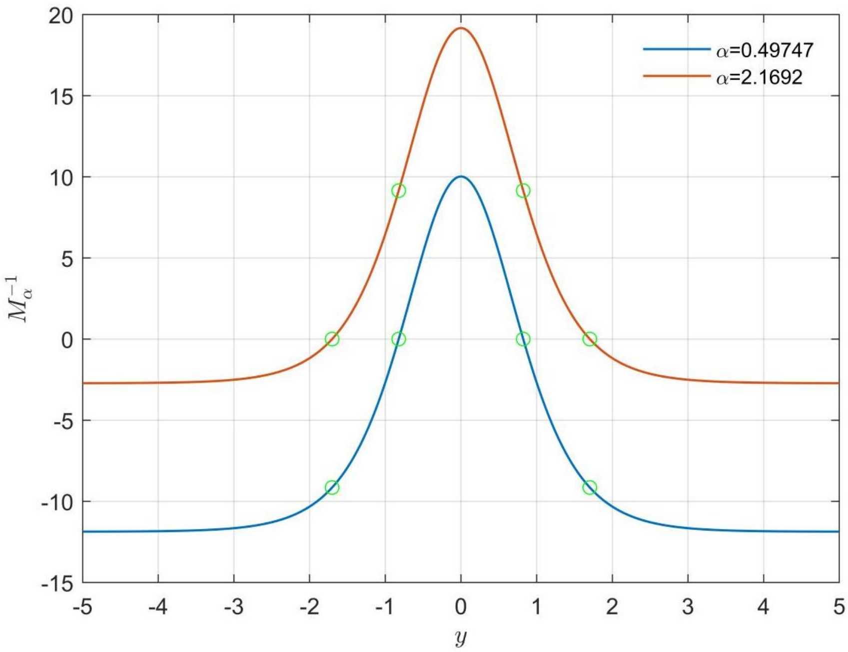

Figure 2

Second, I have added some discussions in the revised manuscript to strengthen the depth of the investigation. For example, I have added a brief discussion after describing Figure 1c in the revised manuscript. They are "According to Eq. 30, $\beta_1^* < 0$ begins to emerge when the non–dimensional magnitude of the westerly jet $u_I \equiv u_0/U > \frac{3}{2}\beta_1 \equiv u_s \approx 2.428$ (Fig. 1c) satisfies the Rayleigh–Kuo necessary condition for normal mode instability. On the other hand, according to the hurdle theorem, it is also easy to derive the sufficient condition for instability. This condition compares the hurdle with the profile of the reciprocal Rossby Mach number, which can be easily calculated from the westerly jet Eq. (29). Preliminary analysis (figures omitted) seems to suggest that $\beta_1^* < 0$ may also satisfy the hurdle theorem, implying that it may also indicate a sufficient condition for instability. In the present study, we mainly focus on the energy evolution along the ray trajectory and do not explicitly discuss the application of the hurdle theorem. This aspect deserves further investigation. Note that when $\beta_1^* < 0$, the westerly jet Eq. (29) is stable, as predicted by both the Rayleigh–Kuo necessary condition and the hurdle theorem. However, nonmodal instability caused by transient growth may still be possible. " They appear in Line 336-345 in the revised manuscript. For another example, after describing Figure 4 and Figure 5, I have added a short discussion to stress that the optimal zonal wavenumber (and correspondingly the optimal phase speed) also coincides with the reference zonal wind shift that ensures a continuous Rossby Mach number in the hurdle theorem.

All of these revisions have been colored in blue. You can easily find them in the revised manuscript.

Minor

Line 14: "by dispersion relation" -> "by the dispersion relation"

Response: Thank you for your careful reviewing. I have corrected this issue and have carefully checked the manuscript to ensure proper use of "the".

Line 17-18: the clause-isolating hyphens should be longer em-dashes, as in growth—capable

Response: Thank you for your careful reviewing. I have corrected this typo in the revised manuscript and I have also carefully checked the manuscript to avoid similar typos.

Line 26: "they may play" -> "they play"

Response: Thank you for your careful reviewing. I have removed the word "may" in the revised manuscript.

Line 59: "packet, w to" needs to be fixed

Response: Thank you for your careful reviewing. This issue occurs when I accidentally delete words while editing sentences during writing. I have corrected this issue in the revised manuscript. Now it is "Compared to the fast phase propagation, the amplitude of a Rossby wave generally varies slowly, giving rise to the so-called Rossby wave packet, which sometimes act as long-range precursors to extreme weather and presumably have an influence on the predictability of mid-latitude weather systems (Wirth et al., 2018)".

Line 70: "Li et al.(2021a) -> "Li et al. (2021a)"

Response: Thank you for your careful reviewing. I have revised the manuscript following your comment. And I have carefully checked the writing to avoid similar typos.

Line 91: The epsilon and Psi symbols are running into each other in (3). There are several spacing issues in the equations throughout the manuscript. Presumably these will at least get fixed in the typesetting.

Response: Thank you for pointing out the spacing issues in the equations. I have carefully reviewed all equations throughout the manuscript and corrected the spacing inconsistencies, ensuring proper separation between symbols and improved readability. I have also conducted a thorough check of all other equations to ensure consistent spacing and clarity, and I will confirm these adjustments during the final typesetting process to meet journal formatting standards.

Line 194 and (15): This transform needs to be better motivated and referenced in terms of how and why it is helpful and how old this strategy is.

Response: Thank you for your insightful comment. To address your concern, I firstly derive (15) in the submitted manuscript from the wave action equation,

$$\frac{\partial A}{\partial T}+\nabla\cdot\left(\mathbf{c_g}A\right)=0 \tag{4}$$

Where $A=E/\omega'$ is the wave action density. Substituting its definition into Eq. (4), we can get

$$\frac{1}{\omega'}\frac{\partial E}{\partial T}-E\frac{1}{\omega'^2}\frac{\partial\omega'}{\partial T}+\frac{1}{\omega'}\nabla\cdot\left(\mathbf{c_g}E\right)-\frac{\mathbf{c_g}E}{\omega'^2}\nabla\cdot\omega'=0 \tag{5}$$

Eq. (5) can be collected as

$$\frac{1}{\omega'}\left[\frac{\partial E}{\partial T}+\nabla\cdot\left(\mathbf{c_g}E\right)\right]-\frac{E}{\omega'^2}\left[\frac{\partial\omega'}{\partial T}+\mathbf{c_g}\nabla\cdot\omega'\right]=0 \tag{6}$$

Namely

$$\frac{\mathrm{D}_g E}{\mathrm{D}T}+E\nabla\cdot\mathbf{c_g}-\frac{E}{\omega'}\frac{\mathrm{D}_g\omega'}{\mathrm{D}T}=0 \tag{7}$$

Since $\omega'=\omega-\bar{u}k$, we have

$$\frac{\mathrm{D}_g E}{\mathrm{D}T}+E\nabla\cdot\mathbf{c_g}+\frac{E}{\omega'}\frac{\mathrm{D}_g\bar{u}}{\mathrm{D}T}k=0 \tag{8}$$

Note that we have applied the relations that frequency (ω) and zonal wavenumber (k) are constant along rays in the zonally varying basic flow $\bar{u}\left(y\right)$. Besides, $\frac{\mathrm{D}_g\bar{u}}{\mathrm{D}T}=c_{gy}\frac{\partial u}{\partial Y}$.

$\omega'=-\frac{\beta^* k}{K^2}$. $c_{g,y}=\frac{2\beta^* kl}{K^4}$. Therefore, we can finally derive

$$\frac{1}{E}\frac{\mathrm{D}_g E}{\mathrm{D}T}+\nabla\cdot\mathbf{c_g}-\frac{2kl}{K^2}\frac{\partial\bar{u}}{\partial Y}=0 \tag{9}$$

that is, Equation (15) in the submitted manuscript. We may also write Eq. (9) as

$$\frac{\partial E}{\partial T}+\nabla\cdot E\mathbf{c_g}-E\frac{2kl}{K^2}\frac{\partial\bar{u}}{\partial Y}=0 \tag{10}$$

Integrating Eq. (10) over the region S and assuming no perturbation at the boundary, we can further derive

$$\bar{E}\equiv\iint_S\frac{\partial E}{\partial T}dS=\iint_S E\frac{2kl}{K^2}\frac{\partial\bar{u}}{\partial Y}dS \tag{11}$$

Eq. (11) was derived approximately 40 years and has been widely applied to discuss the evolution of Rossby waves (Zeng 1983; Chen and Chao 1983; Pedlosky 1987). However, Eq. (11) is hard to solve due to complex form at the right hand term. Therefore, classical theory can only qualitatively predict the evolution of Rossby waves.

Since it is hard to solve Eq. (10) or Eq. (11), we turn to solve it alternative Eq. (9). As shown in Eq. (9), the only term needs to be determined it the group velocity divergence, which,however, cannot be directly solved by applying the expression of group velocity (Lighthill 2001). To overcome this difficulty, Li et al. (2021) proposed a method to solve the group velocity divergence. With the aid of it, we can easily calculate both of the wave energy and amplitude along rays. Based on my preliminary results, I further investigate in what cases both of the wave energy and amplitude can have simultaneous increase to induce possible instability. In the revised manuscript, I have added some sentences to stress that Equation (15) can be readily derived from the classic Rossby wave evolution theory without introducing extra limitation. The reason why previous investigations did not discuss Equation (15) is the difficulty in solving the group velocity divergence.

Line 215-220: There are remarkably few citations in the beginning of Section 3. Please add some citations to similar work, or add emphasis that none exists.

Response: Thank you for your observation. The theoretical derivation in this investigation is based on the classic theories of ray tracing and Rossby wave evolution. However, as demonstrated in the preceding response, classical theory essentially ceases after deriving Eq. (11), as it cannot account for the divergence of group velocity along rays. Therefore, there is no relevant works to the best of my knowledge. Following your comment, I have added a sentence to emphasize that there are no similar works.

Page 11: To this point there has been a noticeable lack of any figures. The effect is to make reading the paper more tedious than this important subject deserves. Please add a couple of examples to illustrate some of the key points being made, which will prop up the reader's morale, particularly new students.

Response: Thank you for your valuable feedback. In Section 3, I focus on a theoretical analysis for general basic flows. The conclusions theoretically derived in Section 3 has been explicitly analyzed and portrayed in Section 4 by applying a classic westerly jet prototype. Section 3 may be a little technical, particularly for early-career researchers and students. In response to this comment, I have added a brief description in Section 3: "In this section, we focus on a general theoretical analysis and thus do not prescribe an explicit profile of the westerly jet. Readers who are not familiar with the above mathematical details can find specific examples for a classical westerly jet prototype in Section 4". Readers can use the illustrations from Section 4 to better understand the theoretical analysis in Section 3. Thanks again.

Line 438: It is arguable that the explicit physical understanding of Rossby wave instability in the context of the reciprocal Rossby-Mach number (e.g., Deguchi et al. 2024) has not been well known for several decades, which undermines the point of this sentence. This needs to be updated.

Response: Thank you for your comment. I have updated the first sentence to clarify the recent advances by Deguchi et al. (2024). The revised sentences now becomes "The physical understanding behind Rossby wave instability has been investigated for several decades. Recent advances in the context of the reciprocal Rossby Mach number (Deguchi et al., 2024) yield a sufficient condition guaranteeing instability in a class of basic flows, shedding the latest insights into the classical problem."

Reference

Chen, Y. Y., and J. P. Chao, 1983: Conservation of wave action and development of spiral Rossby waves. *Sci. China Ser. B-Chem. Biol. Agric. Med. Earth Sci. Chin.*, **13**, 663–672, https://doi.org/10.1360/zb1983-13-7-663.

Deguchi, K., M. Hirota, and T. Dowling, 2024: A sufficient condition for inviscid shear instability: hurdle theorem and its application to alternating jets. *J. Fluid Mech.*, **997**, A25, https://doi.org/10.1017/jfm.2024.728.

Li, Y., J. Chao, and Y. Kang, 2021: Variations in Wave Energy and Amplitudes along the Energy Dispersion Paths of Nonstationary Barotropic Rossby Waves. *Adv. Atmospheric Sci.*, **38**, 49–64, https://doi.org/10.1007/s00376-020-0084-9.

Lighthill, J., 2001: *Waves in Fluids*. 2nd ed. Cambridge University Press 524pp.

Pedlosky, J., 1987: *Geophysical Fluid Dynamics*. Springer New York, NY 710pp., https://doi.org/10.1007/978-1-4612-4650-3.

Zeng, Q.-C., 1983: The Evolution of a Rossby-Wave Packet in a Three-Dimensional Baroclinic Atmosphere. *J. Atmospheric Sci.*, **40**, 73–84, https://doi.org/10.1175/1520-0469(1983)040%3C0073:TEOARW%3E2.0.CO;2.