Response to Reviewer 1

Manuscript: *A tempered fractional Hawkes framework for finite-memory drought dynamics*
Mauricio Herrera-Marín

Dear Reviewer,

Thank you for your detailed and mathematically rigorous report. Your comments were extremely valuable in identifying key inconsistencies in the original submission, and they have directly guided a substantial reconstruction of the manuscript.

1. Interpretation of α

You correctly pointed out that the interpretation of the exponent α was inconsistent with the kernel definition.

Response. This issue has been fully corrected. The kernel is now consistently written as

$$\phi_{\alpha,\theta}(t) = \kappa \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\theta t},$$

with explicit asymptotic interpretation:

$$\phi(t) \sim t^{\alpha-1} = t^{-(1-\alpha)}.$$

Accordingly:

- smaller $\alpha \to$ faster decay,
- larger $\alpha \to$ slower decay (stronger persistence).

All interpretations, figures, and discussion have been rewritten to reflect this.

2. Caputo vs. Riemann–Liouville formulation

You noted inconsistencies in the use of fractional operators.

Response. Section 2 has been completely rewritten. The revised manuscript:

- defines the tempered Caputo derivative explicitly,
- uses a single operator convention throughout,
- presents the Volterra equation as the primary formulation,
- derives the fractional representation as an exact dual.

Ambiguous mixing of operator definitions has been eliminated.

3. TFHP not used in estimation

You correctly observed that the original manuscript did not actually estimate the tempered model.

Response. This has been fundamentally corrected. The revised manuscript now:

- directly estimates $(\mu, \eta, \alpha, \theta)$,
- uses the tempered kernel in the likelihood,
- evaluates the compensator analytically,
- enforces subcriticality during estimation.

Thus, the theoretical model and empirical implementation are now fully aligned.

4. Conceptual clarification of memory

You pointed out the misleading use of "infinite memory".

Response. The manuscript now distinguishes clearly between:

- kernel support,
- and integrated memory (branching ratio).

The discussion has been revised accordingly.

Closing remark

Your report identified core structural issues in the original submission. Addressing them required a full redesign of the theoretical and empirical framework. The revised manuscript is therefore not a minor revision but a substantially improved and internally consistent version.

Thank you again for your careful and constructive review.

Sincerely,
Mauricio Herrera-Marín