Review of *Quantifying the minimum ensemble size for asymptotic accuracy of the ensemble Kalman filter using the degrees of instability* by Kota Takeda and Takemasa Miyoshi
Sumitted to Nonlin. Processes Geophys.
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Overall, this is an interesting study. But it seems to lack novelty. The novelties that I could see are (i) the downsizing of the ensemble in the spin-up of the EnKF, (ii) using Eq. (17) to define *asymptotic filter accuracy*. But in both cases, I believe their introduction should be better justified, and their added value demonstrated. Concerning (i), in my experience whatever divergence the downsizing is meant to prevent can be addressed flawlessly with an adaptive inflation ensemble Kalman filter (EnKF) such as the EnKF-N (see below) in the Lorenz 96 context. Concerning (ii), the large majority of authors on similar topics use the mean root mean square error (RMSE) which is known to also be a measure of asymptotic filter accuracy in the Lorenz 96 context. I am not sure of the added value of the criterion based on Eq. (17) in the Lorenz 96 context. In any quick numerical tests that I performed, one implied the other (i.e., mean RMSE vs Eq. (17)). A configuration where the mean RMSE is proper but would violate your definition of filter accuracy Eq. (17) would be weird, and I was not able not exhibit any. Can the authors prove me wrong?

1 Main remarks

- The title is referring to a subject that has already been addressed by several other papers in the data assimilation and dynamical systems literature. Instead, the paper should point to its novelty, unless the paper is meant to be a topic overview (see below for a selection of references).
- Similarly, in the abstract the only sentence that may point to some novelty is *Therefore, we propose an ensemble spin-up and downsizing method within data assimilation cycles* (see below for a selection of references).
- I believe that you have to make a disclaimer on the use of localisation. Most of the issues addressed in the manuscript are fixed by localisation in high-dimensional geophysical systems. I understand and support why one wants to carry on such an investigation without localisation. But this should quickly be mentioned once, as a disclaimer.
- Since using Eq. (17) as a filter accuracy criterion seems (to me) like the real novelty of the paper, you should better explain why you introduce it, how different it is from the criteria of accuracy used so far in the literature.
- You have to better explain the added value of the ensemble downsizing method, because spinning-up an ensemble with the Lorenz 96 model rarely poses any problem. Especially if one is using an adaptive inflation technique, such as Bocquet (2011); Bocquet et al. (2015); Raanes et al. (2019).
- There are merits in considering Eq. (17) rather that say the mean RMSE. But it might be highly dependent on the chosen multiplicative scheme. If this the case, how to deal with such dependence on the inflation scheme in this context? This should be discussed.

2 Related remarks, suggestions, and typos

1. l.17-18: *The degree of instability is quantified by the Lyapunov exponents (LE's), which are defined as the exponential growth or decay rates of infinitesimal perturbations in the tangent space (Beckman and Ruler, 1985)*: This is not an accurate definition of the Lyapunov exponents.

2. l.19-20: *For continuous-time dynamical systems, such as ordinary differential equations, one of the Le's is zero, corresponding to perturbations parallel to the vector field.*: The existence of a zero Lyapunov exponent comes from the fact that such continuous in time dynamical system is autonomous, i.e. it does not explicitly depends on time (Haken, 1983). It stems from the time-translation invariance of the dynamics. It is critical to emphasise this point since many geophysical systems are not autonomous. Your current statement is wrong in general, and especially for most dynamics in the geosciences.

3. l.50: Bocquet et al. (2017); Bocquet and Carrassi (2017) are not papers about AUS but about the (ensemble) Kalman filters and smoothers, even though they are connected.

4. l.53-55: *Theoretical analyses for linear systems in (Bocquet et al., 2017; Bocquet and Carrassi, 2017) suggest that the rank of the ETKF covariance is asymptotically bounded by N_+ due to the exponential decay of uncertainty in the stable subspace and the slower decay in the neutral subspace under some conditions.*: You seem to downplay the results obtained in Gurumoorthy et al. (2017); Bocquet et al. (2017); Bocquet and Carrassi (2017). In particular, Bocquet et al. (2017) proved mathematically in the linear case that the ensemble should be of size $N_0 + 1$ or greater. They went much farther than just suggestions or conjectures.

5. l.62: *However, these results do not address the time-asymptotic accuracy of the ETKF*: On the contrary, Bocquet et al. (2017); Bocquet and Carrassi (2017); Grudzien et al. (2018a,b) are mainly focused on the time-asymptotic accuracy of the EnKF (and smoother as well). In particular, they use the mean RMSE which is a measure of asymptotic accuracy in an ergodic dynamics context. Bocquet et al. (2017) even demonstrate an analytic formula for the asymptotic (forecast or analysis) error covariance matrix of rank N_0 in the linear case. Your statement seems to be wrong. What am I missing?

6. *Hence, we conjecture that the minimum ensemble size for asymptotic accuracy of the ETKF is $m^* = N_+ + 1$, where only the unstable directions are tracked by the forecast ensemble covariance.* That has already been conjectured and in some cases proven, see e.g., Gurumoorthy et al. (2017); Bocquet et al. (2017); Bocquet and Carrassi (2017); Grudzien et al. (2018a,b). Your statement implies that you are the first to make such conjecture.

7. l.78: *Although our target model is the same as that in (Bocquet and Carrassi, 2017), our objective differs in that we focus on asymptotic accuracy and its dependence on the order of the observation noise*: Bocquet and Carrassi (2017) also focus on asymptotic accuracy and they also report on the dependence onto the observation noise. So at this stage, I do not see any difference in both their objective and their focus.

8. l.93: f should be bold.

9. l.97: *Eq.(3)* should be Equation (3).

10. l.112: *…regardless of $\mathbf{x}_0$ in an invariant subset of $\mathbb{R}^{N_x}$*: is a verb missing here? I believe that I understand the sentence, but I am not so sure.

11. l.117-118: *See (Kuptsov and Parlitz, 2012; Carrassi et al., 2022) for a more comprehensive introduction to LEs and their associated vectors.* A very solid reference, especially because it was written by theoreticians of the geofluids, is Legras and Vautard (1996).

12. l.128: *…used to …*: used for?

13. p.6: Please use bold symbol for matrices (as you did for vectors) since this is the widely adopted convention for data assimilation papers, especially with journals connected to the geosciences.

14. Algorithm 2: To be rigorous, you need to tell that the SVD algorithm returns singular values and vectors by decreasing order.

15. l.195: *In this section, we assimilate observations every $n_{obs} = 5$ integration steps.*: This is a bit awkward. Why don't you just write that the time interval is 0.05 which is the standard value used in such experiments.

16. *We set* $r = 10^{-1}$: The very often used value is $r = 1$. Why this specific choice? I tested numerically the EnKF-N with $r = 1$ and $m = 15$, and I found it to achieve filter accuracy as defined by Eq. (17).

17. p.11, Fig. 2: You would obtain very similar results for the mean RMSE, questioning the relevance (or added value) of the criterion Eq. (17).

18. *This property guarantees that the squared analysis error remains of order* r^2: This seems like a tautology.

19. *The dashed line indicates the order of* r^2: I don't see the dashed line in the figure.

20. l.12, Figure 3: I used the EnKF-N and and have no issues spinning-up the ETKF with $r = 0.1$ and $m = 15$. So what would be the point of Fig. 3? What is the corresponding mean RMSE which is very common in such study? I obtain with the EnKF-N 0.017 for the mean RMSE (one single run, not tuning at all).

21. p.10: I am not sure to see the point of the experiments of Fig. 4, i.e with $r = 10^{-4}$. With the EnKF-N, I obtain in the same setup a mean RMSE of 0.46×10^{-4} which is on par with your best run ($\alpha = 1.5$).

22. p.11: What is the point of the experiments with $F = 16$? You could justify them.

23. Figures 2 and 6: some of the y-axis labels are missing.

24. l.226: *We proposed an ensemble downsizing method for the EnKF to generate an ensemble aligned with the unstable subspace of the dynamics*: Fine, but what is the added value of such scheme? Using an adaptive multiplicative inflation EnKF such as the EnKF-N does the job even without tuning the inflation.

25. l.227-228: *...we verified our conjecture that the minimum ensemble size for asymptotic accuracy is* $m^* = N_+ + 1$*, where* N_+ *is the number of positive LEs*: This conjecture has been already formulated and verified in the literature, and in some cases mathematically proven. If you meant that your statement specifically applies to the criterion Eq. (17), you have to be clearer and additionally explained why Eq. (17) would be different from using the mean RMSE as a filter accuracy indicator. I even believe that in the context of the Lorenz 96 model, one implies the other. And the few numerical tests that I performed in support of this review concur.

References

Bocquet, M., 2011. Ensemble Kalman filtering without the intrinsic need for inflation. Nonlin. Processes Geophys. 18, 735–750. doi:10.5194/npg-18-735-2011.

Bocquet, M., Carrassi, A., 2017. Four-dimensional ensemble variational data assimilation and the unstable subspace. Tellus A 69, 1304504. doi:10.1080/16000870.2017.1304504.

Bocquet, M., Gurumoorthy, K.S., Apte, A., Carrassi, A., Grudzien, C., Jones, C.K.R.T., 2017. Degenerate Kalman filter error covariances and their convergence onto the unstable subspace. SIAM/ASA J. Uncertain. Quantif. 5, 304–333. doi:10.1137/16M1068712.

Bocquet, M., Raanes, P.N., Hannart, A., 2015. Expanding the validity of the ensemble Kalman filter without the intrinsic need for inflation. Nonlin. Processes Geophys. 22, 645–662. doi:10.5194/npg-22-645-2015.

Grudzien, C., Carrassi, A., Bocquet, M., 2018a. Asymptotic forecast uncertainty and the unstable subspace in the presence of additive model error. SIAM/ASA J. Uncertain. Quantif. 6, 1335–1363. doi:10.1137/17M114073X.

Grudzien, C., Carrassi, A., Bocquet, M., 2018b. Chaotic dynamics and the role of covariance inflation for reduced rank Kalman filters with model error. Nonlin. Processes Geophys. 25, 633–648. doi:10.5194/npg-25-633-2018.

Gurumoorthy, K.S., Grudzien, C., Apte, A., Carrassi, A., Jones, C.K.R.T., 2017. Rank deficiency of Kalman error covariance matrices in linear time-varying system with deterministic evolution. SIAM J. Control Optim. 55, 741–759. doi:10.1137/15M1025839.

Haken, H., 1983. At least one Lyapunov exponent vanishes if the trajectory of an attractor does not contain a fixed point. Physics Letters A 94, 71–72. doi:10.1016/0375-9601(83)90209-8.

Legras, B., Vautard, R., 1996. A guide to lyapunov vectors, in: ECMWF Workshop on Predictability, ECMWF, Reading, United-Kingdom. pp. 135–146.

Raanes, P.N., Bocquet, M., Carrassi, A., 2019. Adaptive covariance inflation in the ensemble Kalman filter by Gaussian scale mixtures. Q. J. R. Meteorol. Soc. 145, 53–75. doi:10.1002/qj.3386.