Responses to two reviewers:

Response to RC1

We thank Reviewer 1 for their careful reading of our manuscript and for their constructive comments. The suggestions have helped to improve the clarity and presentation of the paper. As a result, the title of the paper is revised.

Below, we provide point-by-point responses to all reviewer comments. Reviewer comments are reproduced in italics, followed by our responses.

Note: In the revised manuscript, Eq. (17) in the previous version corresponds to Eq. (19).

In this manuscript, two goals are introduced: 1) the manuscript tries to demonstrate that the EnKF can asymptotically converge to a below observation errors as long as the ensemble member is more than the dimension of the unstable subspace. 2) the manuscript proposes an ensemble downsizing approach, which is supposed to accelerate the convergence of the ensemble Kalman filter (EnKF) with smaller ensemble size. However, I find this manuscript can be benefited from a more in-depth analysis, and therefore recommend for at least a major revision.

Response: We thank the reviewer for the careful assessment of the manuscript and for the valuable comments regarding the objectives of the paper. We agree that a more in-depth analysis can further strengthen the paper, and we addressed this point in detail in our responses to the specific comments below.

Major Comments

1. In L61 - 62, the manuscripts claims that it addresses the time-asymptotic accuracy of the ETKF instead of the time-averaged analysis error within a finite-time interval. However, I find it difficult to understand the differences between them. In the referenced literature, the time-average is typically calculated after a spin-up/burn-in time when the filters are converged, which is very similar to time-asymptotic accuracy. In fact, the results in Fig.2 are in general consistent with previous studies, e.g., with an observation error of 1, in Fig. 3 of Bocquet, M.: Ensemble Kalman filtering without the intrinsic need for inflation, Nonlin. Processes Geophys., 18, 735–750, *https://doi.org/10.5194/npg-18-735-2011, 2011.*

Response: Thank you for your accurate comment. Indeed, the time-asymptotic accuracy defined in Eq. (19), is closely related to the time-averaged RMSE after a spin-up period. Therefore, we do not claim this point. Instead, we revised lines 61-62 to explain the distinction in criteria to formulate the minimum ensemble size between our approach using Eq. (19) and existing approaches such as *Bocquet, M.: Ensemble Kalman filtering without the intrinsic need for inflation, Nonlin. Processes Geophys., 18, 735–750, https://doi.org/10.5194/npg-18-735-2011, 2011*. Specifically, the key distinction concerns whether the observation noise level is fixed or treated as an asymptotic parameter. We aim to remove dependence on the choice of observation noise level to define the minimum ensemble size (see L51-58, L82-83).

However, Fig.2 is still an interesting figure. The filter accuracy, as defined by Eq.(17), is violated for large observation errors. This accuracy only starts to appear from ensemble size larger than the unstable subspace (m = 14). These results show clear importance of the accuracy definition of Eq. (17) and the observation noises. Could the authors provide references and explanations of Eq.(17)?

Response: We added appropriate references for Eq. (19), see L45-46. We use the notion of filter accuracy defined in Eq. (19) for the following two reasons explained in L54-58.

It would be also of interests to have an explanation for the sensitivity to observation noises, which is absent from the manuscript. One possible explanation might be the observation noises affecting the linearity which is briefly demonstrate in the Appendix of in Marc Bocquet & Alberto Carrassi (2017) Four-dimensional ensemble variational data assimilation and the unstable subspace, Tellus A: Dynamic Meteorology and Oceanography, 69:1, 1304504, DOI: 10.1080/16000870.2017.1304504. If the linearity is the cause, would smaller observation interval allow for larger observation noises? Nevertheless, these results seem to suggest that to get accurate analysis, an ensemble size of 14 requires very stringent condition.

Response: We incorporated a short discussion on this point in L248-249. Moreover, we added a numerical example with a small observation interval in Appendix B.

2. The manuscript claims that the ensembled downsizing method can "mitigate the slow convergence of uncertainty". However, this is not clearly shown in available Figures. In the comparison between N_spinup = 0 and N_spinup = 720 in Fig.3, it is not clear that N_spinup=720 converges faster. Moreover, the figure presents m=15 which is different from m=14 as claimed in the conclusion section. However, this figure indeed shows only a smaller inflation factor is needed. However, it is not clear if this effect is related to the unstable-neutral subspace. Is it possible that the initial ensemble size mitigates the sampling errors? I wonder if it would be useful to investigate the alignment of error covariance matrix and Lyapunov vectors over time for the ensemble downsizing method.

Response: Thank you for your detailed comments. We performed additional targeted numerical experiments:

(i) To clarify faster convergence with N_spinup=720 than that without N_spinup, we changed Figure 3 by that with m=14.

(ii) To clarify the effect of the ensemble downsizing method related to the unstable subspace (which we are interested in), we performed experiments with initial ensemble which has accurate mean close to the true state and inaccurate perturbations not aligned in the unstable subspace, and compare results for N_spinup=0, 720 (see Figure 4). These experiments can distinguish whether the ensemble downsizing method improves alignments of ensemble perturbations (as well as faster convergence to the true state).

A detailed quantitative analysis of the alignment between ensemble covariance eigenvectors and Lyapunov vectors is an important topic, but we consider it beyond the scope of the present manuscript and leave it for future work (see L304-306).

3. The motivation of F=16 is not very clear to me. Is it just used to validate the results for F=8?

Response: That is correct. This choice was made to confirm that the proposed approach remains effective even when N+ varies as a result of changes in the external forcing.

Minor Comments

L19: "For continuous-time non-autonomous dynamical system, ..., at least one of the LEs is zero". With this condition, this is still not completely correct for certain dissipative systems, I think.

Response: Thank you for your detailed comment. In the manuscript, we focus on autonomous systems. Therefore, we rephrased the sentence to specify autonomous dynamical systems only (see L21).

L28: (ETKF, Bishop et al., 2001). If LaTex is used, try ¥citep[ETKF,][]{ref}. The same follows for cases in L140, L146.

Response: We revised them accordingly (see L31, L172, L178).

L77, L117, L175: No bracket for the reference.

Response: We revised them accordingly (see L103, L150, L220).

L106: Consistency for v_j(t) or v_j.

Response: We revised the equation so that the notations are consistent (see Eq. (7)).

L112: At least one exponent is zero

Response: We revised the phrase accordingly (see L145).

L124: For Nx ¥in N number of variables, ...

Response: We revised the phrase **appropriately** (see L156).

L128: used in data assimilation algorithms.

Response: We will change the preposition accordingly (see L160).

L165: What does this mean for "taking over the observation noises and initial ensemble"? Is the expectation computed over different observation noises?

Response: We will revise the phrase to specify the underlying probability distribution (see L200-201). The expectation is defined theoretically as an expectation with respect to the underlying probability distributions of the observation noise and the initial ensemble. In numerical experiments, this expectation is approximated by averaging over different realizations of observation noise and different independently generated initial ensembles.

Figure 2 and Figure 6: in both x- and y-axis only 10^{-} is shown. caption: black line indicates the order of r^2

Response: We revised the caption in Figure 2 and Figure 6.

Figure 3, 4: if spin-up time is 0, it is perhaps better to avoid 41 -> 14 in the legend.

Response: Thank you for proposing an improvement. We avoided the redundant legends as you indicated (Figure 3, 4).

Response to RC2

We thank Reviewer 2 for their careful reading of our manuscript and their constructive comments. These suggestions have significantly improved the clarity and presentation of the paper. As a result, the title of the paper is revised.

Below, we provide point-by-point responses to all reviewer comments. Reviewer comments are presented in italics, followed by our responses.

Overall, this is an interesting study. But it seems to lack novelty. The novelties that I could see are (i) the downsizing of the ensemble in the spin-up of the EnKF, (ii) using Eq. (17) to define asymptotic filter accuracy. But in both cases, I believe their introduction should be better justified, and their added value demonstrated. Concerning (i), in my experience whatever divergence the downsizing is meant to prevent can be addressed flawlessly with an adaptive inflation ensemble Kalman filter (EnKF) such as the EnKF-N (see below) in the Lorenz 96 context. Concerning (ii), the large majority of authors on similar topics use the mean root mean square error (RMSE) which is known to also be a measure of asymptotic filter accuracy in the Lorenz 96 context. I am not sure of the added value of the criterion based on Eq. (17) in the Lorenz 96 context. In any quick numerical tests that I performed, one implied the other (i.e., mean RMSE vs Eq. (17)). A configuration where the mean RMSE is proper but would violate your definition of filter accuracy Eq. (17) would be weird, and I was not able exhibit any. Can the authors prove me wrong?

Response: We thank the reviewer for the careful assessment of the manuscript and for giving us the opportunity to clarify the novelties of the paper, specifically regarding points (i) and (ii). We deeply compared our approaches with existing and standard approaches and clarified their differences through theoretical explanation and numerical evidence. We addressed these points in detail in our responses to the specific comments below.

Major remarks

The title is referring to a subject that has already been addressed by several other papers in the data assimilation and dynamical systems literature. Instead, the paper should point to its novelty, unless the paper is meant to be a topic overview (see below for a selection of references).

Response: We revised the title to highlight the scope and the novelty of the current study within the existing literature.

Similarly, in the abstract the only sentence that may point to some novelty is Therefore, we propose an ensemble spin-up and downsizing method within data assimilation cycles (see below for a selection of references).

Response: We revised the abstract to emphasize the scope and the novelty of the current study, focusing on the reformulation of the filter accuracy and the ensemble downsizing method.

I believe that you have to make a disclaimer on the use of localisation. Most of the issues addressed in the manuscript are fixed by localisation in high-dimensional geophysical systems. I understand and support why one wants to carry on such an investigation without localisation. But this should quickly be mentioned once, as a disclaimer.

Response: Thank you for your suggestion. We aim to clarify the relationship between the degrees of freedom in dynamical systems and the required ensemble size, isolated from localization effects. Incorporating localization is the next step. We included a disclaimer explaining why we do not use localization (see L41-44).

Since using Eq. (17) as a filter accuracy criterion seems (to me) like the real novelty of the paper, you should better explain why you introduce it, how different it is from the criteria of accuracy used so far in the literature.

Response: We thank the reviewer for pointing this out. We clarified the motivation introducing Eq. (19) (Eq. (17) in the previous version) and its relation to existing accuracy criteria more explicitly in the revised manuscript (see L51-58).

You have to better explain the added value of the ensemble downsizing method, because spinning-up an ensemble with the Lorenz 96 model rarely poses any problem. Especially if one is using an adaptive inflation technique, such as Bocquet (2011); Bocquet et al. (2015); Raanes et al. (2019).

Response: We thank the reviewer for this comment. We agree that for the Lorenz–96 model with sufficiently large ensembles or with adaptive inflation, ensemble spin-up rarely poses a practical difficulty. However, the current study focuses on a borderline regime, where the ensemble size is close to the minimal dimension required to capture the unstable subspace ($m \approx N_+ + 1$). In such cases, increasing the ensemble size is not possible for computational reasons. In this regime, although the ensemble mean may converge quickly, the alignment of ensemble perturbations with the unstable subspace can take a much longer time. We clarified this point in additional experiments (see Figure 4 and L259-274).

There are merits in considering Eq. (17) rather that say the mean RMSE. But it might be highly dependent on the chosen multiplicative scheme. If this the case, how to deal with such dependence on the inflation scheme in this context? This should be discussed.

Response: Thank you for the comment. The primary focus of the current study is to identify a lower bound on the ensemble size required to achieve filter accuracy under favorable conditions, rather than to optimize or analyze the inflation strategy itself. In this context, inflation is treated as a supporting mechanism that maintains sufficient ensemble spread, while the central object of interest is the geometric requirement that the ensemble captures the unstable subspace of the dynamics. The dependence on inflation tuning and the use of adaptive inflation schemes (such as EnKF-N) are therefore complementary to, but beyond the scope of, the current study. We clarified this scope and the dependence of inflation more explicitly in the revised manuscript (see L298-299).

Related remarks, suggestions, and typos

1. l.17-18: The degree of instability is quantified by the Lyapunov exponents (LE's), which are defined as the exponential growth or decay rates of infinitesimal perturbations in the tangent space (Beckman and Ruler, 1985): This is not an accurate definition of the Lyapunov exponents.

Response: We revised the sentence to avoid incorrect definition (see L19-22).

2. l.19-20: For continuous-time dynamical systems, such as ordinary differential equations, one of the Le's is zero, corresponding to perturbations parallel to the vector field.: The existence of a zero Lyapunov exponent comes from the fact that such continuous in time dynamical system is autonomous, i.e. it does not explicitly depends on time (Haken, 1983). It stems from the time-translation invariance of the dynamics. It is critical to emphasise this point since many geophysical systems are not autonomous. Your current statement is wrong in general, and especially for most dynamics in the geosciences.

Response: Thank you for your detailed comment. In the manuscript, we focus on autonomous systems. Therefore, we rephrased the sentence to specify autonomous dynamical systems only (see L22).

3. l.50: Bocquet et al. (2017); Bocquet and Carrassi (2017) are not papers about AUS but about the (ensemble) Kalman filters and smoothers, even though they are connected.

Response: We rearrange the literature review properly (see L69-71).

4. l.53-55: Theoretical analyses for linear systems in (Bocquet et al., 2017; Bocquet and Carrassi, 2017) suggest that the rank of the ETKF covariance is asymptotically bounded by N_+ due to the exponential decay of uncertainty in the stable subspace and the slower decay in the neutral subspace under some conditions.: You seem to *downplay the results obtained in Gurumoorthy et al. (2017); Bocquet et al. (2017); Bocquet and Carrassi (2017). In particular, Bocquet et al. (2017) proved mathematically in the linear case that the ensemble should be of size $N_0 + 1$ or greater. They went much farther than just suggestions or conjectures.*

Response: We agree that Bocquet et al. (2017) provides a rigorous mathematical result for linear systems, proving that an ensemble size of at least N_0 + 1 is required to properly approximate the error covariance. This theoretical result motivates the ensemble downsizing method considered in our study to maintain filter accuracy. We revised the manuscript to clearly acknowledge this theoretical result, together with related numerical evidence (see L73-80).

5. l.62: However, these results do not address the time-asymptotic accuracy of the ETKF : On the contrary, Bocquet et al. (2017); Bocquet and Carrassi (2017); Grudzien et al. (2018a,b) are mainly focused on the time-asymptotic accuracy of the EnKF (and smoother as well). In particular, they use the mean RMSE which is a measure of asymptotic accuracy in an ergodic dynamics context. Bocquet et al. (2017) even demonstrate an analytic formula for the asymptotic (forecast or analysis) error covariance matrix of rank N0 in the linear case. Your statement seems to be wrong. What am I missing?

Response: Thank you for the accurate comment. We agree that our statement was incorrect. In an ergodic setting, the long-time mean RMSE (after an initial transient) is indeed a standard measure of time-asymptotic performance. Moreover, Bocquet et al. (2017) derive an analytic expression for the asymptotic error covariance (of rank N_0) in the linear case. We revised the sentence accordingly. In the revised manuscript, we will also clarify that Eq. (19) is introduced not to contrast “asymptotic vs time-averaged”, but for motivations explained above (see L82-87).

6. Hence, we conjecture that the minimum ensemble size for asymptotic accuracy of the ETKF is $m^* = N_+ + 1$, where only the unstable directions are tracked by the forecast ensemble covariance. That has already been conjectured and in some cases proven, see e.g., Gurumoorthy et al. (2017); Bocquet et al. (2017); Bocquet and

Carrassi (2017); Grudzien et al. (2018a,b). Your statement implies that you are the first to make such conjecture.

Response: We revised the statement with appropriate citations (see L87-90).

7. l.78: Although our target model is the same as that in (Bocquet and Carrassi, 2017), our objective differs in that we focus on asymptotic accuracy and its dependence on the order of the observation noise: Bocquet and Carrassi (2017) also focus on asymptotic accuracy and they also report on the dependence onto the observation noise. So at this stage, I do not see any difference in both their objective and their focus.

Response: We agree that the sentence makes no difference. We revised the sentence to state that we focus on the reformulated minimum ensemble size based on the reformulated accuracy criterion (see L101-105).

8. l.93: f should be bold.

Response: We revised the equation (see L125).

9. l.97: Eq.(3) should be Equation (3).

Response: We revised the sentence accordingly (see L129).

10. l.112: . . . regardless of x0 in an invariant subset of RNx : is a verb missing here? I believe that I understand the sentence, but I am not so sure.

Response: Thank you for your suggestion. We revised the sentence to clarify the mathematical statement (see L144).

11. l.117-118: See (Kuptsov and Parlitz, 2012; Carrassi et al., 2022) for a more comprehensive introduction to LEs and their associated vectors. A very solid *reference, especially because it was written by theoreticians of the geofluids, is Legras and Vautard (1996).*

Response: Thank you for the valuable information. We cited the reference appropriately (see L150).

12. l.128: . . . used to . . . : used for?

Response: We revised the preposition (see L160).

13. p.6: Please use bold symbol for matrices (as you did for vectors) since this is the widely adopted convention for data assimilation papers, especially with journals connected to the geosciences.

Response: Thank you for the suggestion. We revised the manuscript to use bold symbol for matrices in the revised manuscript.

14. Algorithm 2: To be rigorous, you need to tell that the SVD algorithm returns singular values and vectors by decreasing order.

Response: Thank you for the accurate comment. We revised the explanation accordingly (see L215-L216).

15. l.195: In this section, we assimilate observations every $n_{obs} = 5$ integration steps.: This is a bit awkward. Why don't you just write that the time interval is 0.05 which is the standard value used in such experiments.

Response: Thank you for the comment. We revised the representation accordingly (see L239).

16. We set $r = 10^{-1}$: The very often used value is $r = 1$. Why this specific choice? I tested numerically the EnKF-N with $r = 1$ and $m = 15$, and I found it to achieve filter *accuracy as defined by Eq. (17).*

Response: Thank you for confirming the experiment by yourself. We respond to this at comment#20 below.

17. p.11, Fig. 2: You would obtain very similar results for the mean RMSE, questioning the relevance (or added value) of the criterion Eq. (17).

Response: Thank you for the comment. We respond to this at comment#25 below.

18. This property guarantees that the squared analysis error remains of order r2: This seems like a tautology.

Response: We removed the sentence since it provides no additional information.

19. The dashed line indicates the order of r2: I don't see the dashed line in the figure.

Response: Thank you for pointing out. The correct word is 'black line'. We revised it accordingly (see captions of Figs. 2, 6).

20. l.12, Figure 3: I used the EnKF-N and have no issues spinning-up the ETKF with $r = 0.1$ and $m = 15$. So what would be the point of Fig. 3? What is the corresponding mean RMSE which is very common in such study? I obtain with the EnKF-N 0.017 for the mean RMSE (one single run, not tuning at all).

Response: We agree that the purpose of Fig. 3 is unclear. Therefore, we removed it from the revised manuscript.

21. p.10: I am not sure to see the point of the experiments of Fig. 4, i.e with $r = 10^{-4}$. With the EnKF-N, I obtain in the same setup a mean RMSE of 0.46×10^{-4} which is on par with your best run ($\alpha = 1.5$).

Response: Since the point of Fig. 4 is unclear, we removed it in the revised manuscript.

22. p.11: What is the point of the experiments with $F = 16$? You could justify them.

Response: This experiment aims to confirm that the proposed approach remains effective even when N+ varies as a result of changes in the external forcing.

23. Figures 2 and 6: some of the y-axis labels are missing.

Response: Thank you for the comment. We defined notations used in labels of Figures 2 and 6 appropriately.

24. l.226: We proposed an ensemble downsizing method for the EnKF to generate an ensemble aligned with the unstable subspace of the dynamics: Fine, but what is the added value of such scheme? Using an adaptive multiplicative inflation EnKF such as the EnKF-N does the job even without tuning the inflation.

Response: As shown in Fig. 3(b) of the revised manuscript, when the ensemble size is set to the borderline value m=14, the convergence of the ETKF remains very slow even when the inflation factor is optimally tuned, if the ensemble spin-up and downsizing procedure is not applied. This slow convergence is attributable to the gradual reduction of uncertainty in the neutral direction as discussed in Bocquet et al. 2017, which is inherent to borderline ensemble sizes that barely span the unstable subspace. Such slow convergence associated with neutral directions cannot be eliminated solely by adaptive inflation schemes.

In contrast, Fig. 3(a) demonstrates that introducing an ensemble spin-up period leads to rapid convergence, after which the ensemble downsizing method successfully selects a smaller ensemble while maintaining accuracy. As a result, the proposed ensemble spin-up and downsizing strategy substantially accelerates convergence compared with approaches relying only on adaptive inflation.

25. l.227-228: . . . we verified our conjecture that the minimum ensemble size for asymptotic accuracy Is $m^* = N_+ + 1$, where N_+ is the number of positive LEs: This conjecture has been already formulated and verified in the literature, and in some cases mathematically proven. If you meant that your statement specifically applies to the criterion Eq. (17), you have to be clearer and additionally explained why Eq. (17) would be different from using the mean RMSE as a filter accuracy indicator. I even believe that in the context of the Lorenz 96 model, one implies the other. And the few numerical tests that I performed in support of this review concur.

Response: We agree with the reviewer that the conjecture $m^* = N_+ + 1$ has already been formulated and, in some cases, rigorously established in the literature under accuracy criteria based on the mean RMSE or related quantities. We did not intend to claim originality of the conjecture itself.

Instead, our contribution is to revisit this conjecture specifically under the filter accuracy criterion defined by Eq. (19). Owing to this criterion, we can distinguish between accurate and divergent filter behavior based on the scaling with respect to the observation noise level, which is an added value of our approach. We clarified these points explicitly in the revised manuscript (see L288-289, L293-294).

References

Bocquet, M., Gurumoorthy, K. S., Apte, A., Carrassi, A., Grudzien, C., and Jones, C. K. R. T.: Degenerate Kalman Filter Error Covariances and Their Convergence onto the Unstable Subspace, SIAM/ASA J. Uncertainty Quantification, 5, 304–333,

https://doi.org/10.1137/16M1068712, 2017