Supplement to Quantifying the minimum ensemble size for asymptotic accuracy of the ensemble Kalman filter using the degrees of instability

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1 Comparison of error evaluation criteria

Mathematical studies for filters often focus on the long-term behavior of the analysis error (Kelly et al., 2014; Kelly and Stuart, 2019; Takeda and Sakajo, 2024; Biswas and Branicki, 2024; Sanz-Alonso and Waniorek, 2025) and aim to establish the bound known as time-asymptotic filter accuracy, which is defined as Eq. (17) in the manuscript. It aims to bound the expectation of the squared error

$$\mathrm{SE}_n = \mathbb{E}[\|\boldsymbol{x}_n - \overline{\boldsymbol{x}}_n^a\|^2].$$

Here, we compare this criterion with the commonly used RMSE. The expectation of RMSE at time t_n is defined as

$$\mathrm{RMSE}_n = \mathbb{E}\left[\frac{\|\boldsymbol{x}_n - \overline{\boldsymbol{x}}_n^a\|}{\sqrt{N_x}}\right]. \tag{1}$$

From Jensen's convex inequality, we have

$$N_x \mathrm{RMSE}_n^2 = \mathbb{E}\left[\|\boldsymbol{x}_n - \overline{\boldsymbol{x}}_n^a\|\right]^2 \leq \mathbb{E}[\|\boldsymbol{x}_n - \overline{\boldsymbol{x}}_n^a\|^2] = \mathrm{SE}_n. \tag{2}$$

This implies that small values of the expectation of the squared error lead to small RMSE values in expectation, but not vice versa. In addition, supremum over time in the asymptotic limit is larger that or equal to the time-averaged value in general. Hence, the criterion Eq. (17) is stronger than the time-averaged RMSE criterion.

References

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