Reply to Referee 3

Referee *I appreciate the author's efforts in addressing the reviewers' comments. However, after carefully evaluating the responses, I find that several major concerns remain insufficiently resolved.*

Response I thank the referee for their additional comments and for engaging with the revised arguments. The local APE framework is still at an early stage of development. Over the past decade I have worked extensively on this topic, with 16 publications on local APE and closely related issues in both the ocean and the atmosphere [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. Inevitably, this means that many colleagues who kindly agree to review my work bring a broad physical perspective rather than detailed familiarity with this specific theory.

In what follows, I will clarify where I believe the referee's concerns arise from differing interpretations, and I will strengthen the manuscript where the current presentation may have contributed to such differences. My aim is to make the underlying assumptions and logical steps as explicit as possible so that readers can fairly assess the framework and its implications.

Referee: On the use of gravitational acceleration as an analogy *Even when restricting the discussion to Newtonian mechanics, gravity does not seem to constitute an appropriate analogy for the Lorenz reference state (LRS) ... Determining the LRS corresponding to a given spatial distribution of S and θ requires a global rearrangement of fluid parcels, which is inherently difficult to express in a local form. This is precisely why the physical interpretation of the LRS has remained a challenging issue.*

Response I agree that the LRS is more intricate than the gravitational potential field, and that its physical interpretation has long been non-trivial. The analogy with gravity is not meant to suggest that the LRS can be written as a simple Poisson problem, but to highlight a conceptual parallel: both are fields whose local values are constrained by global distributions (mass for gravity, water-mass properties for the LRS) and yet are used locally in the dynamics.

In fact, there exist explicit formulations that link the reference density profile to the water-mass distribution in a way that is directly analogous, in spirit, to the Poisson equation. For a Boussinesq fluid in a simple vertical-walled basin, [17] derived the following explicit expression for the reference position of a parcel, $z_\star(\mathbf{x}, t)$:

$$z_\star(\mathbf{x}, t) = \frac{1}{A} \int H\big(\rho(\mathbf{x}', t) - \rho(\mathbf{x}, t)\big)\, dV', \qquad (1)$$

where H is the Heaviside function and A is the (constant) horizontal area; see their Eq. (11). This can be inverted to yield $\rho_0(z, t)$.

For a realistic ocean with nonlinear equation of state and variable basin geometry, [12] generalised this construction. Their approach defines $z_r(\rho)$, the inverse of $\rho_0(z)$ such that $\rho_0\big(z_r(\rho)\big) = \rho$, as the solution of

$$\int_{z_r(\rho)}^{0} A(z)\,dz = V_T \int_{\theta_{\min}}^{\theta_{\max}} \int_{\hat{S}(\theta,\rho,p_0(z_r(\rho)))}^{\hat{S}(\theta,\rho_{\min},p_0(0))} \Pi(S,\theta)\,dS\,d\theta, \tag{2}$$

where $A(z)$ is the ocean area at depth z, $V_T = \int_{-H_{\max}}^{0} A(z)\,dz$ is the total volume, and $\Pi(S,\theta)$ is the normalised volume distribution function (see their Eq. (29) and discussion).

In both cases, the determination of the reference density profile can be written abstractly as

$$\mathcal{L}\big(z_r(\rho)\big) = 0, \tag{3}$$

where $\mathcal{L}$ is an operator that explicitly links the local value of $z_r(\rho)$ to the global water-mass distribution. While (3) is not identical to the Poisson equation (**??**), the structure is similar in that a local reference field is determined by an integral or differential relation involving the global state.

I will revise the manuscript to: - Cite [17] and [12] explicitly and explain how their constructions provide a mathematically well-defined route from the global distribution to a locally defined reference profile. - Clarify that the gravity analogy is conceptual and pedagogical: it aims to illustrate a dual "global–local" character, not to suggest an exact mapping of operators.

Referee: On the observability of the Lorenz reference state *The theoretical argument presented through equations (2)–(5) is not convincing. While the mathematical manipulations themselves are correct, the reference profiles $\rho_0(z)$ and $p_0(z)$ introduced here may be chosen arbitrarily; there is no inherent reason to identify them with the LRS. Furthermore, the statement that "while $p_0(z)$ enters as a passive reference, its choice determines N_0 and is thus constrained by observations" is not meaningful. The authors may be implicitly assuming that N_0 corresponds to the in-situ Brunt–Väisälä frequency, but this is not the case ... In contrast, the quantity N_0 defined in the author's reply depends solely on the arbitrarily chosen reference state and is therefore not observable.*

Response The exact vertical momentum balance,

$$\frac{D^2\zeta}{Dt^2} + \frac{1}{\rho_\star}\frac{\partial \delta p}{\partial z} + \int_0^{\zeta} N_0^2\big(S,\theta,z_r+\zeta'\big)\,d\zeta' = 0, \tag{4}$$

and its small-amplitude approximation,

$$\frac{D^2\zeta}{Dt^2} + N_0^2(S,\theta,z_r)\,\zeta \approx 0, \tag{5}$$

provide a first-principles derivation of the buoyancy frequency in terms of displacements about a neutral reference level $z_r(S,\theta)$ (defined by $b(S,\theta,z_r) = 0$). This construction is deductive and rests only on the governing equations and the definition of neutral buoyancy.

To relate N_0^2 to observed salinity and temperature profiles, the classical environmental approach sets

$$\rho_0(z) = \rho\bigl(S_0(z), \theta_0(z), p_0(z)\bigr),$$

with $S_0(z)$ and $\theta_0(z)$ describing locally defined environmental profiles. Under the assumption that $\rho_0(z)$ is obtained from the actual state by adiabatic and isohaline rearrangement, and that $S = S_0(z_r)$, $\theta = \theta_0(z_r)$, one recovers

$$N_0^2(S, \theta, z_r) = g\left(\alpha\,\frac{\partial\theta_0}{\partial z}(z_r) - \beta\,\frac{\partial S_0}{\partial z}(z_r)\right), \tag{6}$$

which is the familiar expression for the (environmental) squared buoyancy frequency used in oceanography. In this setting:

- N_0^2 is not arbitrary; it is fixed by the choice of $\rho_0(z)$.
- $\rho_0(z)$ is itself constrained by the requirement that it correspond to the Lorenz reference state obtained by adiabatic, isohaline rearrangement.
- The observable quantities are the buoyancy oscillation frequencies (from, e.g., internal waves) and the environmental gradients ($\partial_z S_0, \partial_z \theta_0$); these jointly constrain the admissible $\rho_0(z)$ and hence the LRS.

In contrast, the quantity introduced by the referee,

$$N_{\text{inst}}^2 = -\frac{g}{\rho_\star}\left(\rho_\theta(S, \theta, p_0(z))\,\frac{\partial\theta}{\partial z} + \rho_S(S, \theta, p_0(z))\,\frac{\partial S}{\partial z}\right), \tag{7}$$

is based on instantaneous vertical gradients and therefore differs conceptually from the environmental buoyancy frequency in (6). In a turbulent ocean, N_{inst}^2 can be highly variable and may take positive or negative values locally, reflecting transient overturning and small-scale variability, and represents a fundamentally meaningless approach to defining the buoyancy frequency. By contrast, N_0^2 derived from $\rho_0(z)$ represents a smoothed, underlying reference stratification associated with the LRS.

In the manuscript, "observability" is used in the sense of determining $\rho_0(z)$ (and thus the LRS) from measurements of buoyancy oscillations (direct observability) and from the consistency of various dynamical constraints (indirect observability). I will revise Section 3 to:

- Clearly distinguish between instantaneous, small-scale estimates of N^2 and the reference N_0^2 associated with $\rho_0(z)$.
- Emphasise that $\rho_0(z)$ is not arbitrary: it is selected by the Lorenz rearrangement and constrained by the requirement that the resulting N_0^2 be consistent with observed buoyancy behaviour and environmental structure.
- Clarify the precise sense in which N_0^2 is "constrained by observations" and how this links to the LRS.

Referee: On the formulation using the static energy function *As I noted in my previous comments, the decomposition $\Sigma = \Sigma_{\rm heat} + \Sigma_{\rm dyn}$ and the use of the LRS to define the latter is interesting. However, I remain unconvinced by the strengthened arguments. The author's example involving surface diabatic processes shows that choosing $\Sigma_{\rm heat}$ based on the LRS makes expression (6) represent the APE production rate, but this is not unexpected since APE is defined relative to the LRS. It reiterates a well-known result and does not substantively support the claim that "the Lorenz reference state enters the equations in the way an external constraint would."*

Response I agree that, at first sight, it may appear unsurprising that a decomposition built around the LRS recovers standard APE production forms. The purpose of the example, however, is to show that physically meaningful behaviour is obtained only for a very restricted class of choices for $\Sigma_{\rm heat}$, and that this class is tied to the LRS. In other words, the decomposition is not arbitrary.

To illustrate this, consider the surface-forced production/destruction of $\Sigma_{\rm dyn}$ by heat and freshwater fluxes. Denoting by $Q_{\rm net}$ the net surface heat flux, by $\rho_f = \rho(0, T, p)$ the surface freshwater density, and by $E - P$ the net evaporation minus precipitation $(\mathrm{m\,s^{-1}})$, one finds

$$F_{\rm dyn} = \left(\frac{T - T_r}{T} \right) Q_{\rm net} + \left[\mu - \mu_r - (T - T_r) \frac{\partial \mu}{\partial T} \right] \rho_f S (E - P), \qquad (8)$$

with

$$T_r = \frac{\partial \Sigma_{\rm heat}}{\partial \eta}, \qquad \mu_r = \frac{\partial \Sigma_{\rm heat}}{\partial S}. \qquad (9)$$

For the LRS-based, APE-consistent choice of $\Sigma_{\rm heat}$, $F_{\rm dyn}$ reduces to the exact APE production form [e.g., 18, 12, 19] and is positive when surface fluxes destabilise the stratification, in line with empirical evidence and established energetics.

By contrast, if $\Sigma_{\rm heat}$ is defined in terms of potential enthalpy [20], for which $T_r = \theta$ and $\mu_r = \mu$, one obtains $F_{\rm dyn} = 0$, implying that surface fluxes do not contribute to APE production. This is at odds with both observations and standard theoretical understanding. This simple comparison demonstrates that:

- The choice of $\Sigma_{\rm heat}$ is strongly constrained by dynamical consistency; not all mathematically admissible decompositions yield physically acceptable behaviour. - The LRS-based definition of $\Sigma_{\rm heat}$ is singled out by these constraints, supporting the view that the LRS plays a distinguished role in the energetic decomposition.

Regarding the phrase "enters the equations in the way an external constraint would": I will soften and clarify this wording in the manuscript to avoid any implication that the LRS exerts a causal forcing. The intended meaning is that once $\Sigma_{\rm heat}$ is fixed by the LRS, the form of $\Sigma_{\rm dyn}$ and of the associated source terms (such as $F_{\rm dyn}$) is determined and imposes non-trivial constraints on the dynamics—much as an externally imposed constraint restricts the set of admissible states. I will rephrase this to emphasise the constraining role, rather than suggesting an additional external agent.

In summary, while I recognise that some of these issues involve subtle conceptual distinctions, I believe that with the clarifications and additions outlined above—particularly concerning (i) the mathematical constructions underpinning the LRS, (ii) the precise notion of observability used, and (iii) the physical constraints on Σ_{heat}—the revised manuscript will address the remaining concerns and present a coherent and testable framework for local APE theory.

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