Reply to Referee 3

I thank the referee for their careful reading and thoughtful comments, which help clarify and strengthen the manuscript. I agree that revisions are warranted regarding the observability discussion in Section 3 and the formulation of Σ_{heat} in Section 4. I respectfully disagree, however, that the analogy with gravity in Section 2 is problematic within the stated scope of classical Newtonian mechanics, which is the framework relevant for the oceanographic applications considered.

In what follows, I address each point in turn. I believe the revisions outlined below will resolve the referee's concerns and improve the manuscript.

On the use of gravitational acceleration as an analogy *The manuscript treats gravitational acceleration as an example of a field with a global origin that is locally observable. However, this analogy is problematic... [Referee's comment omitted here for brevity in the final manuscript; retained in the editorial correspondence]*
Response The purpose of the analogy is to contrast the global character of the gravity field—set by the large-scale mass distribution—with the local character of the force entering Newton's second law,

$$\frac{d^2 z}{dt^2} = -g(x, y, z, t). \tag{1}$$

This illustrates a duality: a physical quantity can possess both global and local aspects, depending on context. The manuscript argues that the Lorenz reference state (LRS) exhibits a similar duality. Recognising this is important for reframing aspects of APE theory.

I agree that general relativity provides the modern, comprehensive description of gravity. However, the oceanographic setting of this work is Newtonian, where (1) is appropriate and widely used, including in practical determinations of g (e.g., with gravimeters). Within this classical framework, the analogy is scientifically accurate for the intended pedagogical purpose and does not impact the subsequent ocean energetics analysis. I will clarify in the text that the analogy is explicitly Newtonian and is invoked solely to highlight the global–local distinction.

On the observability of the Lorenz reference state (Section 3) *Referee: The manuscript suggests that the LRS can, in principle, be reconstructed from locally measured buoyancy frequency if the system is sufficiently close to rest... Consequently, the practical and theoretical usefulness of the arguments presented in Section 3 is unclear.*
Response I appreciate the referee's emphasis on realism. Establishing observability in a controlled limit (near-rest conditions) is a necessary first step: if the proposition failed in that limit, it would be unlikely to hold more generally. Demonstrating that the LRS is, in principle, locally inferable from buoyancy

frequency in simple cases provides a clear foundation for arguing that the LRS retains an observable character more broadly, even if the inference becomes more involved in energetic regimes.

To make this explicit, I will revise Section 3 to frame the near-rest analysis as a baseline result and then outline a more general formulation based on anomalous forces. Specifically, let $p = p_0(z) + \delta p$, with $p_0(z)$ dynamically passive. For the non-hydrostatic primitive equations, the momentum balance can be written as

$$\frac{D\mathbf{v}}{Dt} + f\,\mathbf{k} \times \mathbf{u} + \frac{1}{\rho_\star}\nabla \delta p = b\,\mathbf{k}, \tag{2}$$

where $b = -g\big(\rho(S, \theta, p_0(z)) - \rho_0(z)\big)/\rho_\star$ and $\rho_0(z) = -p_0'(z)/g$. Define the parcel's reference equilibrium level $z_r = z_r(S, \theta)$ by $b(S, \theta, z_r) = 0$, and the displacement $\zeta = z - z_r$. For adiabatic and isohaline motion, $w = Dz/Dt = D\zeta/Dt$. The vertical momentum balance becomes

$$\frac{D^2\zeta}{Dt^2} + \frac{1}{\rho_\star}\frac{\partial \delta p}{\partial z} + \int_0^\zeta N_0^2(S, \theta, z_r + \zeta')\, d\zeta' = 0, \tag{3}$$

using

$$b(S, \theta, z) = \int_0^\zeta b_z(S, \theta, z_r + \zeta')\, d\zeta', \tag{4}$$

$$b_z(S, \theta, z) = -N_0^2(S, \theta, z) = \frac{g}{\rho_\star}\left(\frac{d\rho_0}{dz}(z) + \frac{g\,\rho_0(z)}{c_s^2(S, \theta, p_0(z))}\right). \tag{5}$$

Equation (3) shows that, for finite-amplitude ζ, the vertical motion comprises forced and free nonlinear buoyancy oscillations across a range of processes (turbulence, internal waves, balanced motions), all of which depend—albeit in complex ways—on N_0^2. Inferring $\rho_0(z)$ and $p_0(z)$ in the general case is therefore a nontrivial inverse problem rather than an impossibility. The key point is that the LRS remains tied to observable quantities, preserving its status as an observable construct even away from rest.

This derivation also clarifies an apparent arbitrariness: while $p_0(z)$ enters as a passive reference, its choice determines N_0 and is thus constrained by observations; there is not an arbitrary family of equally acceptable reference states once observational consistency is imposed. I will incorporate these clarifications in the revised Section 3.

On the formulation using the static energy function (Section 4) *Section 4 presents a reformulation in terms of a static energy function Σ... The author defines Σ_{heat} based on the LRS... Yet, this statement is questionable, as the choice of Σ_{heat} in equation (8) is not unique... This arbitrariness weakens the claim that the LRS directly shapes the dynamics...*

Response I agree that, formally, one might contemplate alternative definitions of Σ_{heat}. Physically, however, the distinction between available and non-available energy is meaningful only if the "non-available" part is selected so that the remaining "available" part exhibits the observed dynamical signatures. This provides a concrete criterion that constrains Σ_{heat}.

One informative test is the sign and structure of surface-forced production/destruction of Σ_{dyn} by heat and freshwater fluxes. Denoting the net surface heat flux by Q_{net}, the surface freshwater density by $\rho_f = \rho(0, T, p)$, and the net evaporation minus precipitation by $E - P$ (m s^{-1}), we obtain

$$F_{\mathrm{dyn}} = \left(\frac{T - T_r}{T}\right) Q_{\mathrm{net}} + \left[\mu - \mu_r - (T - T_r)\,\frac{\partial \mu}{\partial T}\right] \rho_f S(E - P), \qquad (6)$$

with

$$T_r = \frac{\partial \Sigma_{\mathrm{heat}}}{\partial \eta}, \qquad \mu_r = \frac{\partial \Sigma_{\mathrm{heat}}}{\partial S}. \qquad (7)$$

For the APE-consistent choice of Σ_{heat} (i.e., based on the LRS), F_{dyn} coincides with the exact APE production form [e.g 1, 2, 3] and is positive when surface fluxes destabilise the water column, in agreement with observational and modeling evidence. In contrast, if Σ_{heat} were taken, for example, as potential enthalpy [4], for which $T_r = \theta$ and $\mu_r = \mu$, one obtains $F_{\mathrm{dyn}} = 0$, implying that surface fluxes do not contribute to APE production—at odds with empirical understanding.

Thus, while multiple mathematical decompositions are possible in principle, the requirement that Σ_{dyn} encode the observed energetics imposes strong physical constraints that single out the LRS-based Σ_{heat} as the relevant choice. I will expand Section 4 to include this argument and additional implications that further reduce any perceived arbitrariness.

Summary statement *In summary, while the manuscript raises important conceptual questions... I would encourage the author to make it publicly available as a non-refereed contribution...*

Response I appreciate the referee's constructive engagement. With the clarifications and additions outlined above—especially the strengthened treatment of observability in Section 3 and the physical constraints on Σ_{heat} in Section 4—I believe the revised manuscript will address the concerns raised and meet the standards for publication. I will implement these revisions and resubmit.

References

[1] G. O. Hughes, A. M. Hogg, and R. W. Griffiths. Available potential energy and irreversible mixing in the meridional overturning circulation. *J. Phys. Oceanogr.*, 39:3130–3146, 2009.

[2] J. A. Saenz, R. Tailleux, E. D. Butler, G. O. Hughes, and K. I. C. Oliver. Estimating lorenz's reference state in an ocean with a nonlinear equation of state for seawater. *J. Phys. Oceanogr.*, 45:1242–1257, 2015.

[3] V. E. Zemskova, B. L. White, and A. Scotti. Available potential energy and the general circulation: partitioning wind, buoyancy forcing, and diapycnal mixing. *J. Phys. Oceanogr.*, 45:1510–1531, 2015.

[4] T. J. McDougall. Potential enthalpy: a conservative oceanic variable for evaluating heat content and heat fluxes. *J. Phys. Oceanogr.*, 33:945–963, 2003.