Reply to Rainer Feistel

Many thanks to Prof. Feistel for useful comments, which we anticipate will help clarify the arguments made in my paper and hopefully help the reader recognise their importance.

Major points

Comment. To assist readers in taking advantage, the author should explicitly demonstrate his new method at a simple analytical tutorial example, perhaps in an appendix of the paper.

Response. We thank Prof. Feistel for the constructive suggestion. Before addressing the tutorial aspect, we clarify a central point where, we believe, a misunderstanding arises.

First, the quantity appearing in our Eq. (4),

$$N_0^2(S,\theta,z) = -\frac{g}{\rho_0(z)}\left(\frac{d\rho_0}{dz} + \frac{g\,\rho_0(z)}{c_s^2}\right), \tag{1}$$

is defined from the *environmental* reference profiles $(\rho_0(z), p_0(z))$, and of the square speed of sound c_s^2 of the fluid parcel. Hence, it is an extrinsic function of state, that is, a joint property of the environment and of the fluid parcel. By contrast, the locally defined squared buoyancy frequency used by Prof. Feistel is based on the *local* parcel derivatives,

$$\left[\frac{d\rho}{dp} - \left.\frac{\partial\rho}{\partial p}\right|_{S,\eta}\right] = \left[\left.\frac{\partial\rho}{\partial S}\right|_{\eta,p}\frac{dS}{dp} + \left.\frac{\partial\rho}{\partial\theta}\right|_{S,p}\frac{d\theta}{dp}\right], \tag{2}$$

where $d\rho/dp$ represents the vertical gradient of the *local* in-situ density profile. His expression is therefore exclusively a property of the water colum and hence of the fluid only. When the actual state is close to its mechanically balanced resting state, (1) and (2) can be approximately consistent. However, in the Baltic Sea toy example invoked by Prof. Feistel, the flow is far from rest, so N_0^2 from (1) generally differs substantially from the local buoyancy frequency computed from CTD profiles via (2). This limitation is explicitly acknowledged in the paper (lines 76–78): *"For larger departures, buoyancy oscillations are modified by background flows and nonlinear effects, so the simple relation (4) no longer suffices. Still, we expect the LRS to remain tied to observable dynamical behaviour, although extracting properties from more complex motions might be impractical."*

Context and scope. The construction of Lorenz-type global APE from a local principle—hence a non-negative APE density computable for each fluid parcel from CTD or climatology—has been established for decades [1, 2, 3]. Its practical utility has been demonstrated across diverse problems: the ocean energy cycle [4], atmospheric storm tracks [5, 6], tropical cyclone intensification [7, 8], turbulent stratified mixing [9, 10, 11], double-diffusive instabilities [12], and even magneto-thermal turbulence in astrophysics [13].

Applications to semi-enclosed or marginal basins (e.g. the Baltic) are of high interest but are *outside the scope* of the present paper. A fundamental open question is whether such basins "feel" the same environment as the global ocean; that is, whether the same $(\rho_0(z), p_0(z))$ can meaningfully serve both the World Ocean and the Baltic Sea in defining APE. This issue has been touched on [14, 11] but remains unresolved. The present paper focuses on interpretative aspects of local APE where the use of *single* reference profiles is appropriate (implicitly, sufficiently simple domains).

Main contributions. (i) We provide an argument that, in local APE theory, the Lorenz reference state (LRS) is best interpreted as a property of the *environment*, not of the fluid parcel. (ii) We give a new proof that the APE/BPE partition, locally and globally, is a *structural* property of the Navier–Stokes equations.

Comment. If it turns out that the proposed method of estimating APE from local CTD profiling is not applicable to estuarine cases like the Baltic, the related validity limits should be explicitly addressed.

Response. We agree that validity limits should be clearly stated. Importantly, the structural result established in Section 4 (building on [15]) shows that the partition of potential energy into dynamically active and passive components is an *exact* property of the Navier–Stokes equations. In this sense, local APE theory is *always* applicable, at least in principle. What remains nontrivial in practice is the choice of *environmental* reference state, which determines $\Sigma_{\rm heat}$ and the APE definition itself. Theory predicts that an optimal partition $\Sigma_{\rm heat}/\Sigma_{\rm dyn}$ must exist, but it does not yet uniquely prescribe it; hence heuristics are sometimes needed.

If one accepts that the LRS is an environmental property, then a definitive formulation of APE in complex settings awaits a satisfactory theory of the "environment." Any perceived limitation in practice thus lies not with APE as a concept (which has rigorous foundations) but with our current, incomplete understanding of how to define the environment and the equations governing it.

Illustration. This sensitivity is well illustrated by [8] in the context of tropical cyclone (TC) intensification. The volume-integrated APE budget can be written

$$\frac{d\,APE}{dt} \;=\; G_A \;-\; C(APE, KE), \tag{3}$$

where G_A is the APE production by surface enthalpy fluxes and $C(APE, KE)$ is the APE-to-KE conversion. In (3), APE, $d\,APE/dt$, and G_A all depend on the reference state, while $C(APE, KE)$ does not. Figure 1 shows that one can select a reference state for which G_A closely tracks the reference-state-independent

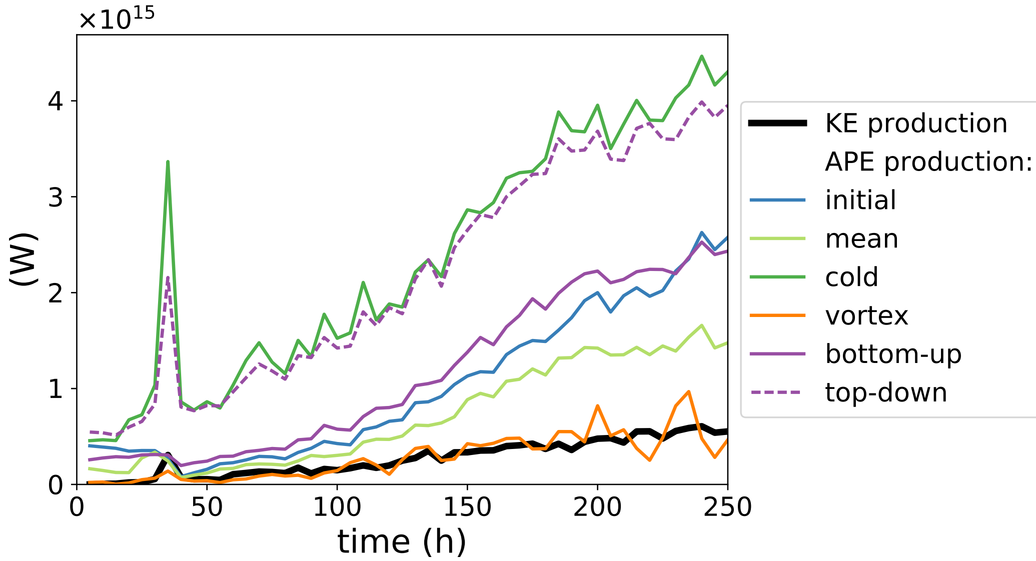

Figure 1: Comparison of APE production rates G_A for different reference states (colours) versus the APE-to-KE conversion $C(APE, KE)$ (black). See [8] for details and assumptions.

conversion $C(APE, KE)$, thereby identifying a "best" environment for that problem.

Implications for the Baltic. By the same logic, there must exist an appropriate environmental specification for computing APE density in the Baltic. The fact that a universally accepted choice is not yet established reflects an open theoretical question about the definition of environment in semi-enclosed basins, not a failure of APE theory. This challenge is not unique to APE: the notion of environment is also central in thermodynamics (e.g. practical exergy calculations). Its resolution will likely have implications beyond APE.

On a tutorial example. We are open to the idea of adding some kind of illustrative examples that could help the reader better understand our arguments, and will try to do so in the revision provided that this is feasible within space constraints.

Minor points

The term "equilibrium" in the abstract and elsewhere should be more specific: is it thermodynamic, hydrodynamic or mechanic equilibrium, for example. The term "equilibrium" corresponds to that underlying theory of available potential energy, that is, a notional rest state obtainable from the actual state by means of an adiabatic and isohaline re-arrangement of mass. This corresponds to a mechanical equilibrium. We'll make sure to be more specific when revising the paper.

The "static energy function" Σ should be specified in terms of proper conventional thermodynamic energies of TEOS-10, such as enthalpy h or Helmoltz energy f, etc. How is Σ to be computed from the TEOS-10 software library?

Static energy Σ is defined as the sum of specific enthalpy and geopotential,

$$\Sigma(\eta, S, p, \Phi) = h(\eta, S, p) + \Phi = h(\eta, S, p) + gz, \tag{4}$$

so that its 'natural' canonical dependent variables are (η, S, p, Φ). If the pressure can be approximated as hydrostatic so that $dp/d\Phi \approx -\rho$, its evolution equation can be approximated as

$$\begin{aligned} \frac{D\Sigma}{Dt} =& T\frac{D\eta}{Dt} + \mu\frac{DS}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} + \frac{D\Phi}{Dt} \\ \approx& -\frac{1}{\rho}\nabla\cdot\mathbf{J}_h + \varepsilon_K + \frac{1}{\rho}\frac{D_h p}{Dt}, \end{aligned} \tag{5}$$

where $D_h/Dt = \partial/\partial t + \mathbf{u}\cdot\nabla_h$ only contains horizontal advection by the ageostrophic component of the velocity field. This is very similar to the evolution equation for specific enthalpy

$$\rho\frac{Dh}{Dt} = -\nabla\cdot\mathbf{J}_h + \rho\varepsilon_K + \frac{Dp}{Dt} \tag{6}$$

As a result, Σ is much less affected by compressibility effects than specific enthalpy. Here, non-conservative effects are the sum of viscous dissipation rate ε_K and of the $D_h p/Dt$. Based on observations [16], ε_K tends to like within $10^{-10} Wkg^{-1}$ and $10^{-8} Wkg^{-1}$. As regards to $D_h p/Dt$, [17] estimated the vertical integral of $\mathbf{u}\cdot\nabla_h p$ in two different coupled climate models. They found that it rarely exceed $10 mWm^{-2}$, which averaged over the mean depth of the ocean $1000m$ corresponds to an energy conversion rate of about $10^2 mWm^3$ or $10^{-8} Wkg^{-1}$, similar to the upper range of observed viscous dissipation rate [16].

For this reason, static energy is very accurately conservative, which explains why it has formed the basis for the study of poleward energy transports in the atmosphere, even though it is not a quasi-material quantity (that is, a function of just (η, S)). Its boundary fluxes exactly coincide with the enthalpy fluxes both at the surface as well as at the bottom, whereas this is only approximately true for potential enthalpy at the bottom.

How is Σ_{heat} related to potential enthalpy (McDougall et al. 2021)?

The APE-based definition of Σ_{heat} consists in defining it as the value of static energy in Lorenz reference state, that is, as

$$\Sigma_{\text{heat}}(\eta, S) = h(\eta, S, p_r) + \Phi_r \tag{7}$$

where $p_r = p_0(\Phi_r)$ and $\Phi_r = \Phi_r(\eta, S)$ are the reference pressure and geopotential of a fluid parcel in Lorenz reference state. Mathematically, Φ_r minimises the function $F(\Phi) = \Sigma(\eta, S, p_0(\Phi)) + \Phi$ at fixed value of (η, S) and is therefore solution of

$$F_0'(\Phi_r) = -\upsilon(\eta, S, p_0(\Phi_r))\rho_0(\Phi_r) + 1 = 0, \tag{8}$$

referred to as the level of neutral buoyancy (LNB) equation in the literature, e.g., [18], in which which $dp_0/d\Phi = -\rho_0(\Phi)$. As a result, the evolution equation for $\Sigma_{\rm heat}$ can be shown to be

$$\begin{aligned} \frac{D\Sigma_{\rm heat}}{Dt} =& T_r \frac{D\eta}{Dt} + \mu_r \frac{DS}{Dt} \\ =& -\frac{1}{\rho}\nabla\cdot\mathbf{J}_r + \varepsilon_K + \varepsilon_p, \end{aligned} \tag{9}$$

in which $T_r = T(\eta, S, p_r)$ and $\mu_r = \mu(\eta, S, p_r)$, while ε_p represents the APE dissipation, which is generally about 20% of the viscous dissipation rate ε_K [16]. As a result, $\Sigma_{\rm heat}$ is very accurately conservative.

The same equation is also satisfied by potential enthalpy, except that the latter uses a reference pressure $p_r = p_a$ corresponding to the mean surface atmospheric pressure.

The boundary fluxes of $\Sigma_{\rm heat}$ are equal to the boundary fluxes of enthalpy minus the APE production rate G_A by surface buoyancy fluxes. According to [4], G_A rarely exceeds $30mWm^{-2}$. The boundary fluxes of $\Sigma_{\rm heat}$ are therefore very close to the boundary fluxes of enthalpy. The APE-based $\Sigma_{\rm heat}$ therefore corresponds to the definition of heat that most accurately isolate the passive component of potential energy, while being both accurately conservative and having boundary fluxes accurately matching boundary fluxes of enthalpy.

In eq. (5) and below, the partial derivatives should be written in thermodynamic convention indicating which variables are kept constant. Standard mathematical convention is that if one explicitly declares the dependent variables to be (η, S, p, Φ), then a derivative such as $\partial\Sigma/\partial p$ assumes by default that (η, S, Φ) are held constant. In any case, whether to use mathematical or thermodynamic convention is generally determined by the field of study and personal preferences. I do not know many papers or textbooks in oceanography or atmospheric sciences that still follow thermodynamic convention. Mathematical convention yields simpler formula uncluttered by needless symbols, and is preferred here. I only follow thermodynamic convention when there is a risk of ambiguity or confusion, which is not the case here.

Eqs. (10) and (11) are typical thermodynamic Legendre transforms (Alberty 2002). Presenting exact differentials of Σ, $\Sigma_{\rm dyn}$, E_p, E_a, E_b along with explaining the physical quantities of the implied partial derivatives in terms of standard thermodynamics would greatly help the reader to follow the given arguments.

There is nothing special or surprising in the use of Legendre transform in Eqs. (10) and (11), as it simply corresponds to obtaining internal energy from enthalpy as $u = h - p\partial h/\partial p = h - p/\rho$. This being said, I agree that it is useful to clarify how each of the quantities listed can be written in terms of standard thermodynamic quantities. The relevant formula, which I propose to add in an appendix, are:

$$\Sigma = h(\eta, S, p) + \Phi \tag{10}$$

$$\Sigma_{\rm dyn} = \Sigma - \Sigma_{\rm heat} = h(\eta, S, p) - h(\eta, S, p_0(\Phi_r)) + \Phi - \Phi_r \tag{11}$$

$$E_p = \Sigma - p\frac{\partial \Sigma}{\partial p} = h(\eta, S, p) - \frac{p}{\rho} + \Phi \tag{12}$$

$$E_a = \Sigma_{\rm dyn} - p\frac{\partial \Sigma_{\rm dyn}}{\partial p} = h(\eta, S, p) - h(\eta, S, p_0(\Phi_r)) + \Phi - \Phi_r - \frac{p}{\rho} \tag{13}$$

$$E_b = \Sigma_{\rm heat} - p\frac{\partial \Sigma_{\rm heat}}{\partial p} = \Sigma_{\rm heat} = h(\eta, S, p_0(\Phi_r)) + \Phi_r \tag{14}$$

Notes:

- The quantity E_p corresponds to the traditional potential energy, that is, the sum of internal energy $h - p/\rho$ and gravitational potential energy Φ;
- The dynamical component E_a may be rewritten in the following form

$$E_a = \Pi_1 + \Pi_2 - \frac{p_0(\Phi)}{\rho} \tag{15}$$

  where

$$\Pi_1 = h(\eta, S, p) - h(\eta, S, p_0(\Phi)) - \frac{p - p_0(\Phi)}{\rho} \tag{16}$$

$$\Pi_2 = h(\eta, S, p_0(\Phi)) - h(\eta, S, p_0(\Phi_r)) + \Phi - \Phi_r \tag{17}$$

  Physically, Π_1 and Π_2 are positive definite quantities, generally referred to as available compressible energy (ACE) or available acoustic energy (AAE), and APE density respectively. Π_1 represents the compressible work needed to bring the fluid parcel from the reference pressure $p_0(\Phi)$ to the actual pressure p. Π_2 represents the work against buoyancy forces needed to bring a parcel from its reference position Φ_r at reference pressure $p_r = p_0(\Phi_r)$ to the reference pressure $p_0(\Phi)$ at location Φ. These terms have been first defined in local APE theory [19]. Note that the Lagrangian derivative may be written as

$$\rho\frac{DE_a}{Dt} = \rho\frac{D(\Pi_1 + \Pi_2)}{Dt} - \nabla\cdot[p_0(\Phi)\mathbf{v}] \tag{18}$$

  using the continuity equation $D\rho/Dt + \rho\nabla\cdot\mathbf{v} = 0$.

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