

1 EigenFlux: A Stable Multi-Stream Radiative Transfer 2 Method for Strongly Scattering Media

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7 ABSTRACT

Radiative transfer in strongly scattering media remains computationally challenging, particularly for systems with highly asymmetric phase functions and large optical depths, where conventional discrete-ordinate approaches may suffer from numerical instability, slow convergence, or loss of accuracy. We present EigenFlux, a multistream radiative transfer framework based on eigenmode decomposition, natural-reflectance stabilization, and flexible mesh-based angular discretization. Unlike previous approaches relying primarily on global polynomial expansions, EigenFlux permits localized resolution of phase functions with strongly forward scattering or significant backward-scattering components while preserving flux conservation and numerical stability. The method is evaluated across 957 test cases spanning asymmetry factors from $g = -0.998867$ to $g = 0.998867$ and single-scattering albedos from $\omega = 0.0014660$ to $\omega = 0.9999995$, including extreme multiple-scattering regimes that are difficult for conventional solvers. Comparisons with DISORT show competitive accuracy and computational performance, with EigenFlux exhibiting particularly robust behavior for highly asymmetric scattering systems and large numbers of streams. EigenFlux maintained stable solutions for asymmetry factors exceeding $|g| > 0.99$, where conventional polynomial-based expansions became unstable or required substantially larger stream numbers. Analysis of the eigenspectrum further reveals the emergence of asymptotic diffuse transport regimes in optically thick systems and persistent direct-beam structure in semi-transparent media. These results suggest that EigenFlux provides a stable and flexible framework for radiative transfer calculations for atmospheres, snow and ice, ocean optics, pigments and coatings, remote sensing, and graphics rendering, particularly in cases involving extreme scattering asymmetry.

9 Introduction

10 The fundamental theory of light scattering in the atmosphere was developed by Lord Rayleigh in 1871⁴². Rayleigh scattering
11 remains fundamental to the field of radiative transfer today. The earliest known formulations of the modern radiative transfer
12 equation (RTE) were published by Eugen Von Lommel in 1887³⁰, and an integral version by Orest Chwolson in 1889²⁸.
13 However, neither publication spread to the general academic community⁸.

14 In 1905, Schuster used hemispheric isotropy to develop and publish a two-stream RTE and its analytical solution⁴³ and has

15 been traditionally credited with originating the RTE, although a similar two-stream approximation was presented by Eddington
16 in 1916¹³. Papers by Schwarzschild in 1906 and 1914, and by Milne in 1921 on thermodynamic equilibrium within solar
17 and stellar atmospheres led to the establishment of the RTE in its general form^{29,44,45}. A full analytic theory of the RTE was
18 developed and published by Chandrasekhar in 1950⁷.

19 In the meantime, Kubelka and Munk developed a simplified two-constant RTE and its solution in 1931^{23,24} that is equivalent
20 mathematically to Schuster's. Whereas Eddington and Shuster's two-stream RTE and subsequent modifications^{26,34} have been
21 considered only appropriate for mediums without strong asymmetry, Kubelka-Munk's method is accurate for determining the
22 reflectance of materials ~~which are very strongly backscattering or forward-scattering~~ with strong scattering. As a result, K-M
23 and its modifications have become the standard for use by the painting and coating industries²² where the most useful pigments
24 tend to be those with extreme scattering properties.

25 A modern study which uses the K-M method is the Lawrence Berkeley National ~~Laboratories~~ Laboratory's study into
26 pigments for roofing materials that show color in the visible spectrum, but are transparent in IR and UV so as to remain cool
27 even under solar heating². The study determined the scattering and absorbing properties of 85 candidate materials. Most
28 materials were found to be strongly forward scattering²⁵.

29 Researchers in atmospheric physics/chemistry and oceanographers were interested in solutions to the full RTE in order to
30 understand the effects of solar flux. The 1970's saw the development of a variety of RTE solvers starting from Chandrasekhar's
31 work⁷, including the popular DISORT solver⁴⁸ available as public source code, both as a stand-alone distribution²⁷ and as part
32 of the RadTranLib distribution¹⁶.

33 Astronomers and meteorologists were interested in the RTE as a way of understanding the albedo of the Earth and other
34 planets⁹. ~~Of interest here is the formulation and standardization of the~~ The Bi-Directional Radiation Field (BDRF) was
35 formulated and standardized as part of this understanding^{32,54}.

36 Snow and ice are particularly important materials in determining the albedo of the Earth, and microwave sensing has
37 become an important tool for characterizing current snow and ice levels. Current algorithms such as the Discrete Ordinates
38 method are being supplied as part of RTE solvers with custom models for the highly forward scattering and low absorptency
39 features of snow and ice^{14,19,38}.

40 Application of the RTE to oceans began as early as 1922 when Raman⁴¹ invoked scattering to explain the color of the
41 sea. Scattering due to suspended particles (turbidity) and water turbulence play a major role in determining the color⁴⁶. The
42 resulting mathematics was given a firm footing by Preisendorfer's 6 Volume *Hydrologic Optics*^{31,39,40}. The work of Akkaynak
43 and Treibitz provides an interesting example of the use of the RTE in color-correction of underwater imaging³.

44 The Hydrolight tool²⁰ is a commercial tool that can directly solve hydrological depth-varying absorbance and scattering
45 problems, including systems with strong forward scattering. It is coded specifically for hydrological problems and does not
46 address atmospheric radiative transfer or radiative transfer within pigmented materials.

47 A simplified form of the RTE was introduced by Blinn⁵ as an approach to rendering realistic scenes in computer graphics in

1984. This was expanded into a generalized Light Transport Theory a few years later by Kajiya²¹. ~~Although the initial impetus was primarily for models that are easy to compute and appear to be realistic, but are not necessarily accurate to the physics, advances~~ This initial work focused on models that were computationally light and resulted in realistic appearances, but were not necessarily physically accurate. Advances in computer technology have allowed the increased use of physically-realistic models³⁷.

The complexity of rendering lighting and atmospheric effects in computer-generated imagery means that the leading rendering technology is based on ray-tracing and Monte-Carlo sampling. These methods provide an important alternative to the more traditional RTE solvers. An example is the MYSTIC code¹⁵ which is included in the RadTranLib public source distribution¹⁶ along with the more traditional DISORT solver⁴⁸.

In addition to the pigments mentioned above, many key components of meteorological concern show strong forward scattering as well. Stamnes, et. al.⁴⁹ reports $g = 0.85, 0.89, 0.997$ for air bubbles, snow grains, and brine pockets respectively. Clouds are strongly forward scattering and have strong backward scattering components¹⁷ (see Figure 11 below). Materials with true negative asymmetry are less common, but $g < -0.4$ have appeared in theoretical studies of engineered materials⁵⁶.

The *Mesh Approximates* (MA) method presented here is closely related to the classic *Discrete Ordinates* (DO) method presented by Chandrasekhar⁷ (we follow the presentation by Goody and Yung¹⁸ and the paper by Stamnes, et al.⁴⁸).

Both methods start with the integro-differential equation (3) with boundary conditions for the diffuse intensity with an explicit expression for the direct intensity.

~~DO~~ The DO method first defines a finite-dimensional approximation of the integral terms to generate a linear ODE. That finite-approximation uses a Legendre approximation of the phase matrix, and Gaussian quadrature for the integrals, which is mathematically equivalent to using Legendre polynomial expansions for the integrands.

To solve the resulting Ordinary Differential Equations (ODE), DO performs an eigenvalue decomposition to get a system of uncoupled ODEs, half of which are unstable. DO expresses the uncoupled ODEs and boundary conditions as an algebraic eigenvalue problem, which is then solved using specialized algorithms for that purpose.

~~MA~~ The MA method first approximates the original equation using a piecewise-linear approximation of the phase matrix and the diffuse intensity, with exact evaluation of the resulting integrands over each piece-wise linear interval.

As in the previous method, the next step is to use an eigenvalue decomposition to solve the resulting Ordinary Differential Equations (ODE) to get a system of uncoupled ODEs, half of which are unstable. It stabilizes the uncoupled ODEs by ~~initializing solving them using initializing their solution with~~ a natural reflectance assumption (equivalent to the assumption of an ~~infinitely thick~~ semi-infinite atmosphere). Piecewise-linear mesh approximations are again applied along the depth axis and to the boundary conditions in order to generate a well-conditioned linear system which can be solved with standard algorithms. ~~The downward solution is then linearly combined with the reflected upward solution to generate the original~~

The natural reflectance solution can then be used to generate the general two-boundary solution through a linear combination of the downward natural reflectance solution starting at the top boundary, and with an upward natural reflectance

solution starting at the bottom boundary. The linear combination is chosen to satisfy the original boundary conditions at the two boundaries.

The DO method is limited by the fit in the fit that can be obtained using Legendre polynomials, which in particular converge slowly at the endpoints require high orders of approximation to avoid oscillations at the end-points, making it difficult to use for approximating those scattering phase functions which are strongly forward or backward-scattering problems have a significant backward scattering component. The MA approach allows for the use of any subdivision of the ordinates, both for the multi-streams and along the depth axis. This allows for the use of meshes designed to resolve the underlying dynamics, for forward and backward scattering problems of up to ± 0.99 scattering problems with more than 0.99 asymmetry for the existing implementations.

Like the DO, the MA-base MA algorithm is limited to application to plane-parallel homogeneous media. In both cases, the method is used to generate solutions to multiple Inhomogeneous media are managed by introducing multiple homogeneous layers with varying characteristics which can be combined into an overall heterogeneous solution. In its existing implementation, MA is limited to plane-parallel geometry, although the use of pseudo-spherical geometry is contemplated for the future.

Theoretical Development

Based on these considerations, there remains a need for a radiative transfer method that is numerically stable, computationally efficient, and accurate for both optically thick and semi-transparent systems, including systems with highly asymmetric scattering phase functions while also promoting transparency and reproducibility through open implementation. In this paper we introduce EigenFlux, a multistream radiative transfer model based on an eigenmode formulation combined with flexible angular discretization and localized basis functions. The method is designed to preserve stability and conserve flux, while accurately describing strongly forward-scattering systems. We evaluate the approach across a broad range of asymmetry factors and single-scattering albedos, including extreme scattering regimes that are challenging for conventional discrete-ordinate methods. Finally, we compare the behavior of EigenFlux with established approaches and discuss potential applications in atmospheric science, ocean optics, remote sensing, material science, and graphics rendering.

Methods

Radiative Transfer

We investigate radiative transfer through a homogeneous media, illuminated from its surface at its top boundary and a reflective/absorptive surface at its bottom boundary, that contains a uniform mix of pigment-particles which may absorb or scatter light particles that are traveling through the media. We will limit this investigation to the plane-parallel situation, in which we only model the depth to which a light particle has penetrated, (increasing as the light particle descends deeper), the relative velocity (the cosine of the angle of travel relative to the vertical), and its azimuth (the angle of travel relative to the medium's "north").

112 The ~~intensity~~intensity function is interpreted here as the probability that a light particle is at depth y in the media, and has
113 a relative velocity of v , where $v = \cos(\theta)$. So $(v = 1, \theta = \pi/2)$ is straight down, $(v = 0, \theta = 0)$ is horizontal, and $(v = -1,$
114 $\theta = -\pi/2)$ is straight up.

115 The ~~pigment~~ particles may either absorb or scatter a light particle encountering them. ~~This paper~~ The process of a light
116 particle travelling until it is absorbed or scattered out of the beam is known as *extinction*, and the expected path length of the
117 process is the *extinction path length*. Upon extinction, the fraction of the particles that are then scattered is the *single scatter*
118 *albedo* also known as the *scatter fraction*. EigenFlux currently does not consider ~~emission from the pigment, since that is a~~
119 ~~minor issue in determining color and transparency;~~ thermal emission or inelastic scattering from the particles.

120 The scatter distribution is determined by a rotationally symmetric phase function ~~with asymmetry parameter $k(v)$ with~~
121 ~~asymmetry g governing the degree of local backscatter versus forward scatter.~~ measuring the expected amount of back scatter
122 vs. forward scatter, $g = \int 2\pi v k(v) dv$ where $g = 1$ is full forward scatter and $g = -1$ is full backward scatter. The scattering
123 matrix is given by

$$\underline{K(v, w; a - b) = k(vw + \cos(a - b) \sqrt{1 - v^2} \sqrt{1 - w^2})}$$

124 Further information and background can be found in the references^{18,22,34}.

125 Key Variables and Functions

θ	Angle of travel of light particle, relative to vertical
ϕ	Azimuth of travel of light particle, relative to north
$v = \cos \theta$	Velocity relative to vertical
(v, a)	<u>Polar cosine and azimuth angle of incident vector</u>
(w, b)	<u>Polar cosine and azimuth angle of scattered vector</u>
126 x	Depth
y	Depth scaled to extinction path length
z	Exponential depth $z = e^{-y/\hat{y}_{05}}$
$\tilde{I}(v, x, \phi)$	Intensity distribution
$I(v, y, \phi)$	Intensity distribution with respect to y
$s(v, \phi) = I(v, 0, \phi)$	Intensity distribution at top boundary (source illumination)
$K(v, w)$ $K(v, w, a - b)$	Normalized phase scattering distribution (parametrized by g)

127 Model Parameters

σ	Extinction path length for light to interact with pigment particles
ω κ	<u>Probability that upon interaction, light is absorbed by particle</u>
α	<u>Probability that upon interaction, light is scattered by particle</u>
128 $\omega = \alpha / (\alpha + \kappa)$	Single scatter albedo(=scattered/(scattered+absorbed))
g	phase-function-scattering-asymmetry-parameter <u>Average scattering asymmetry where $g = 1$ is full forward scatter and $g = -1$ is full backward scatter</u>
$\hat{y}_{05}$	two-stream <u>Two-stream</u> estimate of the depth at which the transmission was reduced to 5% of the original source

129
130
131 Under stationary conditions, the intensity distribution (1) satisfies Schwartzchild's Equation³⁴. (Note that the emission
132 term is not ~~included as its contribution is not important for our intended application to pigment color.~~

$$\underline{\sigma \tilde{I}(v, x, \phi) + v \partial_x \tilde{I}(v, x, \phi) = \omega \sigma \int K(v, w) \tilde{I}(w, x, \phi) dw}$$

considered here.)

$$I(v, a; y) + v \partial_y I(v, a; y) = \omega \iint K(v, w; a - b) I(w, b; y) dw db$$
$$I(v, a; 0) = s(v, a)$$

(1)

~~After normalization with σ , this becomes-~~

$$I(v, y, \phi) + v \partial_y I(v, y, \phi) = \omega \int K(v, w) I(w, y, \phi) dw$$

Note that $K(v, w; a)$ is cyclic in a , i.e. $K(v, w; a + 2\pi) = K(v, w; a)$ and is the only term in the equation that is azimuth-dependent other than the boundary condition. As a result, the fundamental solution will be cyclically symmetric in azimuth

$$\Phi(v, a; w, b; y) = \Phi(v, a - b; w, 0; y) = \Phi(v, a - b + 2\pi; w, 0; y)$$

~~For pedagogical purposes we now choose to simplify the derivation by considering only azimuth-averaged input sources, allowing us to use an azimuth-independent formulation. See Subsection *Incorporating Azimuth-dependence* below for further details on the additional considerations for azimuth-dependent sources~~

Natural Reflectance

Given a boundary at the top of the medium, the left-hand side of equation (1) is a first-order linear differential equation in y that is unstable for $v < 0$, i.e. for light traveling up to the top of the medium. To fix this problem, we use the notion of natural reflectance to find canonical solutions that can be combined to find two-boundary solutions to (1).

The *natural reflectance* of a material is the reflectance of a sample that is thick enough to be effectively opaque, e.g. adding additional depth does not change the reflectance significantly. In considering atmospheric radiation, this would be the assumption of an infinitely deep atmosphere (without the attendant pressure increase). In considering paints and coatings, this would be the assumption of a completely opaque film.

~~The left-hand side of equation (??) is a first-order linear differential equation in y , so we can~~ Let us write down the standard textbook solution for a first order differential equation with the initial boundary condition $s(v)$.

$$I(v, y, a) = s(v, a) e^{-y/v} + \omega(1/v) e^{-y/v} \int_{\eta=0}^y \int_{w=-1}^1 \int_{b=0}^{2\pi} e^{\eta/v} K(v, w, a - b) I(w, \eta, b) db dw d\eta$$

However, for $v < 0$, $e^{-y/v}$ is unbounded in y , and under the natural reflectance condition, the light exiting the surface,

151 $I(v, 0)$ for $v \leq 0$ should be solely determined by the light entering the surface, $I(v, 0)$ for $v \geq 0$.

152 By using a second boundary at depth M for the light moving up, we can get a two-sided equation in which we specify
 153 $I(v, 0)$ for $v \geq 0$ for light moving from the surface to the bottom, and $I(v, M)$ for $v \leq 0$ for light moving from the bottom to the
 154 surface.

For $v \geq 0$,

$$I(v, y, a) = s(v, a) e^{-y/v} + (\omega/v) e^{-y/v} \int_{\eta=0}^y \int_{w=-1}^1 \int_{b=0}^{2\pi} e^{\eta/v} K(v, w, a-b) I(w, \eta, b) db dw d\eta$$

For $v < 0$,

$$I(v, y, a) = s_M(v, a) e^{(M-y)/v} - (\omega/v) e^{-y/v} \int_{\eta=y}^M \int_{w=-1}^1 \int_{b=0}^{2\pi} e^{\eta/v} K(v, w, a-b) I(w, \eta, b) db dw d\eta$$

155 Now apply the natural reflectance condition by letting $M \rightarrow \infty$ under the condition that ~~$s_M(v)$~~ $s_M(v, a)$ is bounded, and split
 156 out the specific solution for $v = 0$, which results in

For $v \geq 0$,

$$I(v, y) = s(v) e^{-y/v} + (\omega/v) e^{-y/v} \int_{\eta=0}^y \int_{w=-1}^1 e^{\eta/v} K(v, w) I(w, \eta) dw d\eta$$

For $v < 0$,

$$I(v, y) = -(\omega/v) e^{-y/v} \int_{\eta=y}^{\infty} \int_{w=-1}^1 e^{\eta/v} K(v, w) I(w, \eta) dw d\eta$$

For $v > 0$,

$$I(v, y, a) = s(v, a) e^{-y/v} + (\omega/v) e^{-y/v} \int_{\eta=0}^y \int_{w=-1}^1 \int_{b=0}^{2\pi} e^{\eta/v} K(v, w, a-b) I(w, \eta, b) db dw d\eta$$

For $v = 0$,

$$I(v, y, a) = \omega \int_{w=-1}^1 \int_{b=0}^{2\pi} K(v, w, a-b) I(w, y, b) db dw \quad (2)$$

For $v < 0$,

$$I(v, y, a) = -(\omega/v) e^{-y/v} \int_{\eta=y}^{\infty} \int_{w=-1}^1 \int_{b=0}^{2\pi} e^{\eta/v} K(v, w, a-b) I(w, \eta, b) db dw d\eta$$

As desired, the light exiting the surface, $I(v, 0)$ for $v \leq 0$, is determined by the light entering the surface, $I(v, 0)$ for $v \geq 0$.

Figure 1 shows the intensities for a strongly backscattering media.

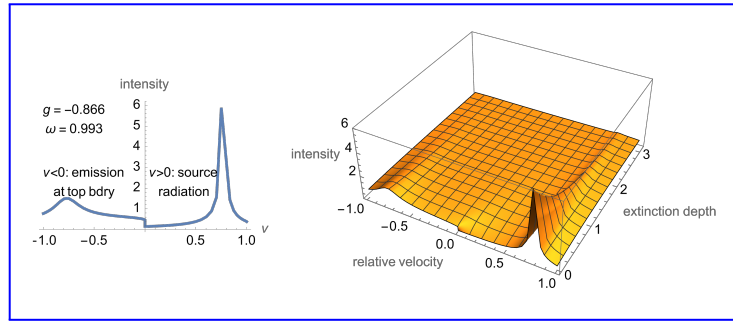


Figure 1. Intensities for strongly backscattering media: (left) ~~source and~~ emission ($v \leq 0$) and source ($v \geq 0$) radiation. (right) internal intensities. $v > 0$ shows particles descending deeper into the media at the different relative velocities, $v < 0$ are light particles ascending towards the top.

Now we turn our attention back to the basic equation ~~(??)(1)~~. This can be decomposed into two parts, the *direct* component which is the intensity of the source rays before they are scattered or absorbed (also known as *extinction*), and the *diffuse* component which is the intensity of the scattered light. Furthermore, we can immediately determine the form of the direct component:

We accordingly define the two components of the intensity as

$$R(v, y) = e^{-y/v} s(v) \quad R(v, y, a) = e^{-y/v} s(v, a) \quad \text{Direct intensity distribution}$$

$$M(v, y) = I(v, y) - R(v, y) \quad M(v, y, a) = I(v, y, a) - R(v, y, a) \quad \text{Diffuse intensity distribution}$$

with the boundary condition $\forall v > 0, I(v, 0) = s(v)$. Substituting these definitions into equation ~~(??)(1)~~, we obtain the basic

equation and side condition for the diffuse component.

$$M(v, y, a) + v \partial_y M(v, y, a) - \omega \int K(v, w, a - b) M(w, y, b) db dw = \omega \int K(v, w, a - b) R(w, y, b) db dw$$

$$\forall v > 0, M(v, 0, a) = 0$$
(3)

Figure 2 shows the internal direct intensity, which depends solely on the distribution of the source radiation and the extinction depth.

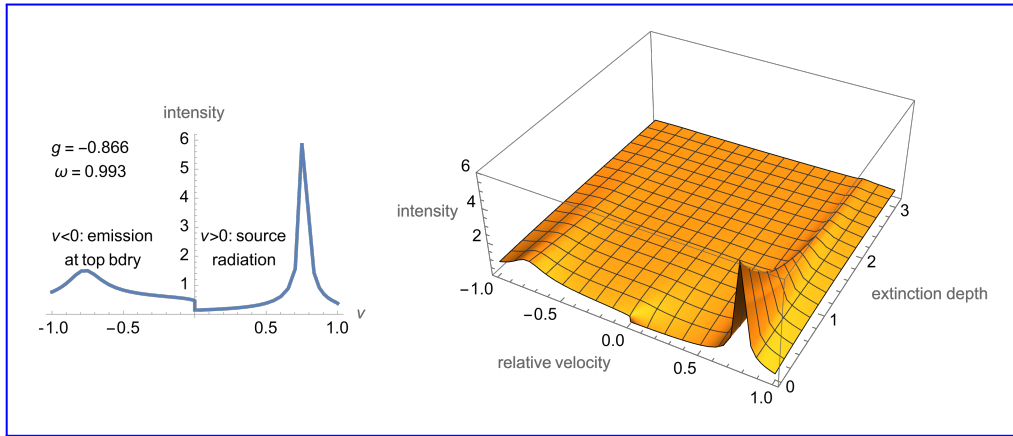


Figure 2. Internal direct intensity, ~~depending on~~ showing the ~~distribution~~ exponential decay of the source radiation ~~and based on~~ the extinction depth.

Figure 3 shows the intensities for strongly ~~fore~~ scattering ~~forward~~ scattering and backscattering media.

~~We will be using~~

Mesh Approximation and Galerkin's Method

We use Galerkin's method¹ to solve the integral equations in (3). This method uses a framework of basis functions and test functions to generate a *mesh approximation* for the various functions that ~~consistute~~ constitute the full solution.

v_i	Approximation points for relative velocity
$\phi_i(v)$	Basis functions
$\psi_i(v)$	Test functions
$\tilde{f}(v) = \sum f_i \phi_i(v)$	Approximating form for 1D functions
$\tilde{h}(v, w) = \sum h_{ij} \phi_i(v) \phi_j(w)$	Approximating form for 2D functions

When seeking an approximate $\tilde{f}(v) \approx g(v)$, we expand $\tilde{f}$ and integrate both against the test functions to get the linear equation which can be solved for the approximating coefficients.

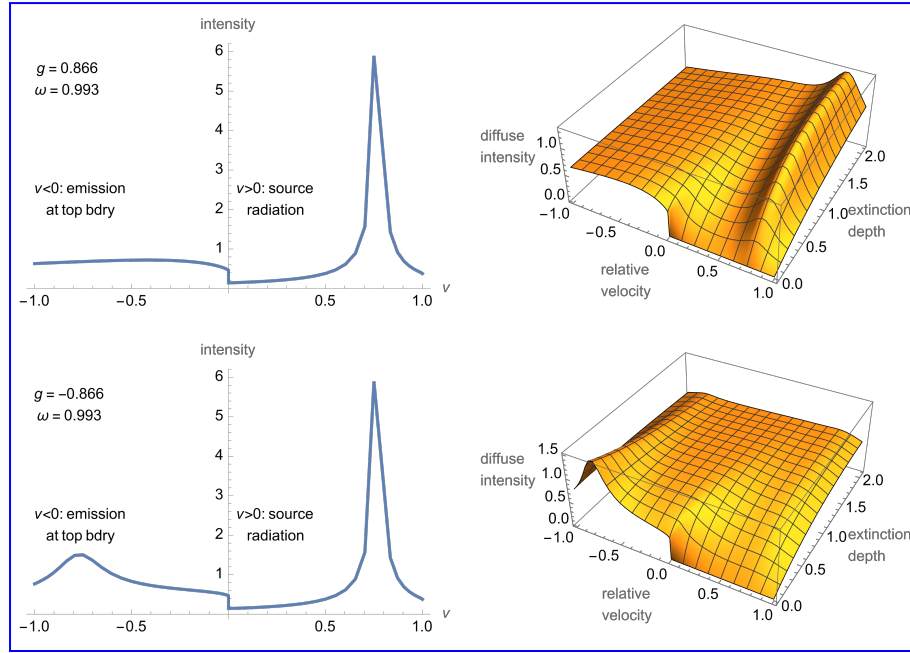


Figure 3. Internal diffuse intensities: (top) strongly forward scattering media, showing a diffuse emission and a strong internal peak at the source peak; (bottom) strongly backscattering media, showing a emission peak at the source peak, and a diffuse internal intensity.

$$\forall i, \sum_j f_j \int \psi_i(v) \phi_j(v) dv = \int \psi_i(v) g(v) dv$$

Any reasonable subdivision of points can be used for v_i . In this paper, v_i will be the concatenation of two sets of the Chebyshev points, one "arc" for $-1 \leq v \leq 0$ and another for $0 \leq v \leq 1$. The basis functions ϕ_i are triangular bump functions centered on the mesh points. We will be using two different choices for test functions in the course of this paper, either

1. $\psi_i = \phi_i$ where the test functions are the basis functions themselves, or
2. $\psi_i(v) = \delta[v = v_i]$ where the test functions are the Dirac delta functions, basically defining $\tilde{f}$ by sampling it at the mesh points.

Using (2) results in equations that are simpler, faster, and frequently better conditioned than (1), but using (1) preserves more of the symmetric properties of the original problem. So we use a mixture, (1) when additional symmetry is required, and (2) when expediency ~~requires it~~ dictates the use of the simpler alternative.

~~To solve~~ Applying the Galerkin approximations to the basic equations (3), we use an eigenvalue expansion of the linear system generated by the mesh approximation. get the mesh approximations:

$$K(v, w) = \sum_{jp} K_{jp} \phi_j(v) \phi_p(w) \quad K(v, w; a-b) = \sum_{jpc} K_{ipc} \phi_i(v) \phi_p(w) \tilde{\phi}_c(a-b)$$

Phase function kernel

$$s(v) = \sum_j s_j \phi_j(v) \quad p_0(v; a) = \sum_{jc} p_{0jc} \phi_j(v) \tilde{\phi}_c(a)$$

Intensity at surface

$$M(v, y) = \sum_j M_j(y) \phi_j(v) \quad M(v, y; a-b) = \sum_{jc} M_{jc}(y) \phi_j(v) \tilde{\phi}_c(a-b)$$

Diffuse intensity distribution

$$\tilde{\phi}_c(a-b) = \tilde{\phi}_c(a-b+2\pi)$$

Azimuth test function

From this we derive the finite dimensional linear ODE.

$$\begin{aligned} \sum_{jc} A_{ij} \tilde{B}_{fc} \partial_y M_{jc}(y) + \sum_{jc} B_{ij} \tilde{B}_{fc} M_{jc}(y) - \omega \sum_{jpc} B_{ij} \tilde{C}_{fdc} K_{jpd} B_{pq} M_{qc}(y) \\ = \omega \sum_{jpc} B_{ij} \tilde{C}_{fdc} K_{jpd} D_{pq}(y) p_{0qc} \end{aligned} \quad (4)$$

where

$$A_{ij} = \int v \psi_j(v) \phi_j(v) dv = \int v \psi_i(v) \phi_j(v) dv$$

$$B_{ij} = \int \psi_j(v) \phi_j(v) dv = \int \psi_i(v) \phi_j(v) dv$$

$$C_{ij} = \int \phi_j(v) \phi_j(v) dv = \int \phi_i(v) \phi_j(v) dv$$

$$\tilde{B}_{fc} = \int \tilde{\psi}_f(a) \tilde{\phi}_c(a) da$$

$$\tilde{C}_{fdc} = \int \tilde{\psi}_f(a) \tilde{\phi}_d(a-b) \tilde{\phi}_c(b) da db$$

$$D_{ij}(y) = \int e^{-y/v} \phi_j(v) \phi_j(v) dv = \int e^{-y/v} \phi_i(v) \phi_j(v) dv$$

More succinctly,

$$A \partial_y M(y) + (B - \omega B K C) M(y) = \omega B K D(y) s$$

$$\tilde{A} \tilde{B} \partial_y M(y) + (\tilde{B} \tilde{B} - \omega \tilde{B} \tilde{C} K C) M(y) = \omega \tilde{B} \tilde{C} K D(y) p_0 \quad (5)$$

Conservation Properties

One of the benefits of using Galerkin's Method is that it is an orthogonal projection from the Hilbert space of differentiable functions to the Hilbert space of piece-wise continuous functions, so it preserves conservation properties under the correct weights. This includes energy conservation, reciprocity, and flux closure.

The original scattering matrix $K(v, w, a-b)$, besides being symmetric in v, w and cyclic in $a-b$, also satisfies the conservation condition that $\forall w, b, \iint K(v, w, a-b) da dv = 1$. In order for the mesh approximate scattering matrix, $K(v, w; a, b) = \sum_{jpc} K_{ipc} \phi_i(v) \phi_p(w) \tilde{\phi}_c(a-b)$ to satisfy that condition, we must have

$$\forall p, b, \sum_{ja} K_{jpc} \int_{v=-1}^{v=1} \phi_j(v) dv \int_{a=0}^{a=2\pi} \tilde{\phi}_c(a-b) da = 1 \quad (6)$$

So the tensor $\tilde{K}_{jpcb} = K_{jp(a-b)}$ must be doubly stochastic with respect to weights $\int \phi_j(v) dv$ and $\int \tilde{\phi}_c(a) da$. In general, the simple choice of defining the approximation by simply sampling the kernel at the mesh points, $\tilde{K}_{jpc} = K(v_j, w_p, a_c)$ will yield a matrix that is close to, but is not doubly stochastic.

To ensure the problem remains conservative, EigenFlux uses a variation on the Sinkhorn-Knopp algorithm⁴⁷ suitable for re-scaling a matrix so that it is doubly stochastic with respect to the weights. The result is that the modified scattering matrix remains conservative in the mesh approximation.

Eigenvalue Decomposition

To solve (3), we use an *eigenvalue expansion* of the linear system generated by the mesh approximation.

The matrix A is near-singular at $v = 0$, but $(B - \omega BKC)$ is diagonally dominant and non-singular. So we can look at the eigenvalue decomposition of $(B - \omega BKC)^{-1}A$ for eigenvalues v_k and eigenvectors Q_k . The resulting ODE becomes

$$v_k \partial_y \mu_k(y) + \mu_k(y) = \rho_k(y) \quad (7)$$

where

v_k, Q_k	Eigenvalues and eigenvectors
$\mu_k(y)$	Eigenvector decomposition of $M_j(y)$
$\rho_k(y)$	Transformed boundary data

and

$$AQ_k = v_k(B - \omega BKC)Q_k$$

$$M_j(y) = \sum_k Q_k \mu_k(y)$$

$$\rho_k(y) = \omega Q^{-1}(B - \omega BKC)^{-1}BKD(y)s$$

Figure 4 shows the computed eigenvalues for varying parameter values. In general, there will be a continuous spectrum from -1 to 1, plus isolated eigenvalues at $v = \pm 1$ that correspond to the limiting distribution at increasing depths. As depth increases, the slowest decay of the intensity is governed by the isolated eigenvalues which can be quantitatively much slower than the extinction decay. See below for further details.

Figure 5 shows the computed eigenvectors for a single set of the parameter values. Note in particular the all-positive eigenvectors for $v = \pm 1.8$. These are the limiting distributions of the intensity.

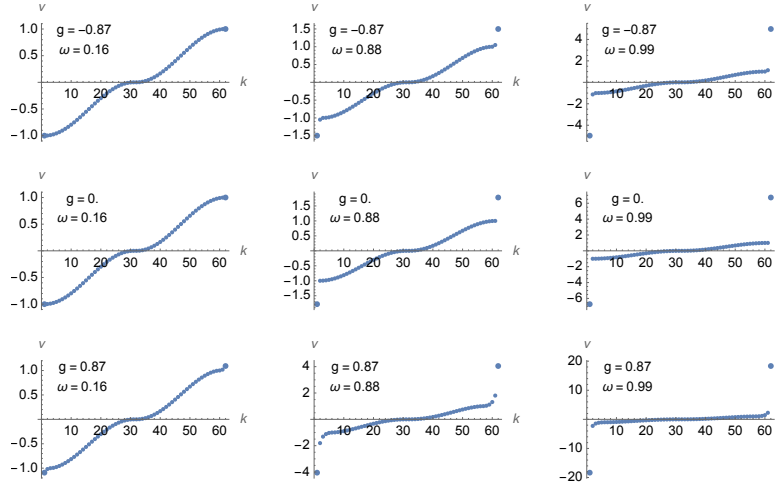


Figure 4. Eigenvalues for varying values of asymmetry g and scatter fraction ω , showing a continuous spectrum from $v = -1$ to $v = 1$ and two isolated eigenvalues. Note that the absolute value of the isolated eigenvalues increases with increasing ω .

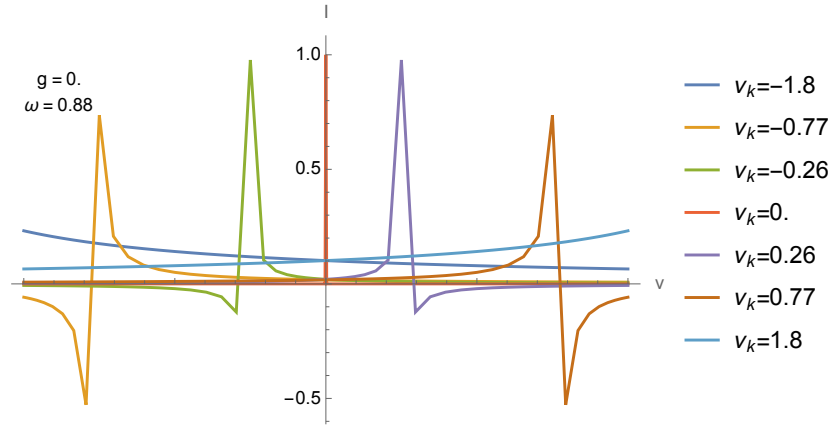


Figure 5. Sampling of eigenvectors for a single run. The eigenvectors for the isolated eigenvalues show as all-positive. The eigenvectors for the continuous spectrum are delta functions around the various relative velocities of the mesh.

We now have decomposed the problem into a set of one-dimensional first-order linear ODEs which have the general solution form

$$\mu_k(y) = \mu_k(0)e^{-y/v_k} + (1/v_k) \int_0^y e^{(\eta-y)/v_k} \rho_k(\eta) d\eta \quad (8)$$

~~However, there are two additional considerations—~~There are additional considerations when the eigenvalue is zero or negative.

One of the eigenvalues will be close to (or equal to) 0. In this case, we can go back to (7) and use the resulting condition

$$\mu_k(y) = \rho_k(y).$$

~~If~~ When the eigenvalue is negative, (8) would imply that we have an unstable mode. In this case, we can apply the natural reflectance assumption discussed previously to the new ODEs.

With these considerations, we have the following more specific solution.

$$\begin{aligned}
 v_k > 0 &\implies \mu_k(y) = \mu_k(0)e^{-y/v_k} + (1/v_k) \int_0^y e^{(\eta-y)/v_k} \rho_k(\eta) d\eta \\
 v_k = 0 &\implies \mu_k(y) = \rho_k(y) \\
 v_k < 0 &\implies \mu_k(y) = -(1/v_k) \int_y^\infty e^{(\eta-y)/v_k} \rho_k(\eta) d\eta
 \end{aligned} \tag{9}$$

Note in particular that this implies that for $k : v_k \leq 0$, $\mu_k(0)$ is determined by ρ_k (and hence by p_0). For positive v_k , determination of $\mu_k(0)$ is slightly more complicated– they are determined by taking the side-condition $v > 0 \implies M[v, 0] = 0$ (see (3)), expanding by $M_j(0) = \sum_k Q_k \mu_k(0)$ (see (7)) and solving for the unknown $\mu_k(0)$.

Block Circulant

The eigenvalue decomposition of $\tilde{R} = (B\tilde{B} - \omega B\tilde{C}KB)^{-1}A\tilde{B}$ can be computationally costly. However, note that all functions/matrices (other than $p_0(v, a)$) that are dependent on azimuth are cyclically symmetric. As a result, the matrix $\tilde{R}$ has some special properties:

- $\tilde{R}$ is block circulant, thus $\tilde{R}_{fcij} = \tilde{R}_{(f-c)0ij}$
- The individual blocks of $\tilde{R}$ are symmetric, implying that $\tilde{R}_{ij}^{fc} = \tilde{R}_{ji}^{fc}$

Because of (1) and (2), the eigenvalues and eigenvectors of $\tilde{R}$ are real and can be determined from the number of blocks and the eigenvalue decomposition of a geometric sum of the individual blocks^{10,50,51}. Let the blocks of $\tilde{R}$ be given by R_{i-j} . Let ρ be a primitive root of unity of order n , the number of blocks, so that $\rho^n = 1$ and such that $\{\rho^j\}$ generates a complete list of all roots of unity of order n . Let $H(\rho) = \sum_j \rho^j R_{n-j}$. Then the eigenvalues of $\tilde{R}$ are the $n * m$ eigenvalues of $H(\rho^k)$, and the eigenvectors are the outer product of $\{\rho_k^j\}$ and the eigenvectors of $H(\rho^k)$.

Depth Model Approximation

Finding a good approximation mesh for the depth y can be problematic because y has an unbounded range, so instead we use the exponential depth z where $z = e^{-y}$, $0 \leq z \leq 1$. ~~With an abuse of notation, so that~~ for a function of depth $f(y)$ we define ~~$f(z) = f(-\ln(z))$~~ have $f(y) = f(-\ln(z))$.

Similar to Step One, we apply the following mesh approximation to (9), (but now with $\tilde{\psi}_i(z) = \delta[z = z_i]$, option (2) in the discussion on Mesh Approximation).

Depth Model Approximations

$$\begin{aligned}
\mu_k(y) &= \sum_m \mu_{km} \phi_m(z) \text{ Transformed intensity} \\
\rho_k(z) &= \sum_m \rho_{km} \phi_m(z) \text{ Transformed boundary conditions} \\
\rho_{kh} &= \omega Q^{-1} (B - \omega B K B)^{-1} B K D(z_h) p_0 \\
D_{ij}(z) &= \int z^{1/v} \phi_j(v) \phi_j(v) dv = \int z^{1/v} \phi_i(v) \phi_j(v) dv
\end{aligned}$$

From this and (9) we get the following.

$$\begin{aligned}
v_k > 0 &\implies \mu_{km} = \mu_{0k} z_m^{1/v_k} + (1/v_k) \rho_{km} \int_{z_m}^1 z_m^{-1} (z_m/\zeta)^{1+1/v_k} \phi_m(\zeta) d\zeta \\
v_k = 0 &\implies \mu_{km} = \rho_{km} \\
v_k < 0 &\implies \mu_{km} = -(1/v_k) \rho_{km} \int_0^{z_m} z_m^{-1} (z_m/\zeta)^{1+1/v_k} \phi_m(\zeta) d\zeta
\end{aligned}$$

where

$$F_{km} = z_m^{1/v_k}$$

and

$$\begin{aligned}
v_k > 0 &\implies G_{khm} = \int_0^1 \int_0^z z_h^{-1} (z_h/\zeta)^{1+1/v_k} d\zeta dz \\
v_k = 0 &\implies G_{khm} = \chi[h = m] \\
v_k < 0 &\implies G_{khm} = \int_0^1 \int_z^1 z_h^{-1} (z_h/\zeta)^{1+1/v_k} d\zeta dz
\end{aligned}$$

Taken together, these This allow us to compute μ and M under the natural reflectance assumption.

~~Figure ?? shows the mesh points for relative velocity and exponential depth.~~

Satisfying the General Two-Boundary Problem

~~Mesh points for relative velocity and exponential depth.~~ The boundary conditions for a two-boundary radiative transfer problem specify the downward sources of illumination at the top boundary, and the upward sources of illumination at the bottom boundary.

$$\begin{aligned}
I(w_j, 0) &= r_0(w_j) && \text{downwards illumination at top boundary } (w_j \geq 0) \\
I(w_j, b) &= r_b(w_j) && \text{upwards illumination at bottom boundary } (w_j \leq 0)
\end{aligned}$$

The general solution for radiative transfer for a medium with finite depth and two boundaries, top and bottom, can be expressed as the linear combination of a the one-boundary solution for the top boundary and a with the one-boundary solution for the bottom boundary.

Let Φ be the fundamental solution for (3), so that $M = \Phi p_0$. ~~With an abuse of~~ To simplify the notation, we replace the indices by the underlying mesh points e.g. $M_i(y) = M(v_i, y)$. We then note that the solution for a system with a bottom boundary

is given by a depth-reversal of the solution for a system with a top boundary.

$$\begin{aligned}
 \tilde{M}^{top}(v_i, y) &= \sum_{w_j \geq 0} \Phi(v_i, w_j, y) r_0(w_j) \\
 \tilde{M}^{bot}(v_i, y) &= \sum_{w_j \leq 0} {}^R\Phi(v_i, w_j, b-y) r_b(w_j) \\
 {}^R\Phi(v_i, w_j, b-y) &= \Phi(-v_i, -w_j, b-y)
 \end{aligned} \tag{10}$$

where

b	depth of bottom boundary
$\tilde{M}^{top}$	solution to top-boundary problem under natural reflectance assumption
$\tilde{M}^{bot}$	solution to bottom boundary problem under natural reflectance assumption
${}^R\Phi$	depth-reversed fundamental solution
$r_0(w_j)$	downwards illumination of top boundary ($w_j \geq 0$)
$r_b(w_j)$	upwards illumination of bottom boundary ($w_j \leq 0$)

Both the downwards and upwards system satisfy the base integro-differential equation (3). A linear combination of the two sets of solutions has the same degrees of freedom as the two-boundary problem, so a linear combination of the two will span the solution set.

To find that solution, first let's decompose the systems into downwards flows ($v \geq 0$) and upwards flows ($v \leq 0$).

$$\begin{bmatrix} I_y^\uparrow \\ I_y^\downarrow \end{bmatrix} = \begin{bmatrix} {}^R\Phi_{b-y}^\downarrow & \Phi_y^\uparrow \\ {}^R\Phi_{b-y}^\uparrow & \Phi_y^\downarrow \end{bmatrix} \begin{bmatrix} r_b^\uparrow \\ r_0^\downarrow \end{bmatrix}$$

where

$I_y^\uparrow$ Intensity of flow upwards at depth y

$I_y^\downarrow$ Intensity of flow downwards at depth y

From this we can determine the boundary input and output flows.

$$\begin{bmatrix} S_b^\uparrow \\ S_0^\downarrow \end{bmatrix} = \begin{bmatrix} {}^R\Phi_0^\downarrow & \Phi_b^\uparrow \\ {}^R\Phi_b^\uparrow & \Phi_0^\downarrow \end{bmatrix} \cdot \begin{bmatrix} r_b^\uparrow \\ r_0^\downarrow \end{bmatrix}$$

$$\begin{bmatrix} O_0^\uparrow \\ O_b^\downarrow \end{bmatrix} = \begin{bmatrix} R\Phi_b^\downarrow & \Phi_0^\uparrow \\ R\Phi_0^\uparrow & \Phi_b^\downarrow \end{bmatrix} \cdot \begin{bmatrix} r_b^\uparrow \\ r_0^\downarrow \end{bmatrix}$$

And so we have found the two-boundary system.

$$\begin{bmatrix} O_0^\uparrow \\ O_b^\downarrow \end{bmatrix} = \begin{bmatrix} R\Phi_b^\downarrow & \Phi_0^\uparrow \\ R\Phi_0^\uparrow & \Phi_b^\downarrow \end{bmatrix} \cdot \begin{bmatrix} R\Phi_0^\downarrow & \Phi_b^\uparrow \\ R\Phi_b^\uparrow & \Phi_0^\downarrow \end{bmatrix}^{-1} \cdot \begin{bmatrix} S_b^\uparrow \\ S_0^\downarrow \end{bmatrix}$$

where

$S_b^\uparrow$ Intensity of source input at bottom boundary at depth b

$S_0^\downarrow$ Intensity of source input at top boundary at depth 0

$O_0^\uparrow$ Intensity of output from top boundary

$O_b^\downarrow$ Intensity of output from bottom boundary boundary

Numerical Investigations Results

Numerical Investigations of Analytical Features

For the numerical experiments, the mesh approximation for the relative velocity v had have 61 sample points distributed as two arcs of shifted Chebyshev points, to give additional resolution around the three points $v = -1, 0, 1$. The mesh approximation for the scaled depths $0 \leq z \leq 1$ had has 67 sample points distributed as shifted Chebyshev points to provide additional resolution at $y = 0, \infty$. The actual scaling used was is $z = e^{-y/\hat{y}_{05}}$ where $\hat{y}_{05}$ is the two-stream estimate of the depth at which the transmission was reduced to 5% of the original source. This provides better resolution by the numerical algorithm when the test case has strong backscattering or forward scattering or a significant backward scattering component.

The test cases used the Henyey-Greenstein phase function parametrized by g the asymmetry parameter, which in this case is equal to the average scatter angle relative to an incident angle.

The 957 test cases used all pairs of 33 values of the asymmetry parameter (g = average scattering angle) g and 29 values of the scattering fraction ω (ω = probability of scatter given extinction $\tau = 1$ - probability of absorption given extinction).

Table 1 shows the parameters for the test cases. Figure ?? shows the mesh points for the asymmetry parameter and the single scatter albedo.

Mesh points for asymmetry parameter and the single scatter albedo

The The Fortran implementation took 0.0742 seconds per test case, the Python implementation took 2.45 seconds per test case. The, and the Mathematica implementation took 23.5 seconds per test case. More detailed timing studies are presented

Table 1. Test case parameters. v_i and z_i were fixed for all runs, so there were 33*29 test cases. These test cases span the range of asymmetry and single stage albedo, with increased density at the extremes of strong scattering and low absorbency, in order to cover the analytic features of the RTM. For v and z Chebyshev point of the second kind were used, remapped to the interval [0,1] (and [-1,0] for v .) Then g is distributed as the remapped Chebyshev points of the first kind. The strong sensitivity to low absorbency dictated that ω be the squares of the Chebyshev points of the first kind.

heightvar	definition	distribution	number	min	max
v_i	relative velocity	double chebyshev <u>Chebyshev</u>	61	-1	1
z_i	scaled exponential depth	chebyshev <u>Chebyshev</u>	67	0	1
g_i	asymmetry factor	chebyshev <u>Chebyshev</u>	33	-0.998867	0.998867
ω_i	single scatter albedo	modified chebyshev <u>Chebyshev</u>	29	0.0014660	0.9999995

Test case parameters

Figure ??

below.

The Asymptotic Diffusion Radiance Distribution

Observations of radiance in deep waters show that the shape of the radiance distribution approaches a limiting distribution, and the rate of decay of that distribution is exponential³¹. This is known as the asymptotic radiance distribution. The maximum eigenvalue of Equation (5) characterizes this behavior.

As shown in Figures 4 and 5 and discussed below, each material has a unique largest eigenvalue ν greater than 1 and a unique positive eigenvector. The rate of attenuation for each eigenvector is the reciprocal of the corresponding eigenvalue, so this positive eigenvector has the slowest rate of attenuation among the diffuse components of the intensity. Since the largest eigenvalue is also greater than 1, it's rate of attenuation is also slower than the rate of attenuation of the direct component.

Hence the positive eigenvector represents the asymptotic limit of the distribution of the intensity as the optical depth increases, and the rate of convergence is given by the reciprocal of the eigenvalue. As shown in Figure 6, the maximum eigenvalue (and hence the slowest decay in diffusion) increases as absorption decreases ($\omega \rightarrow 1$) and as the forward scattering ratio increases ($g \rightarrow 1$). So the diffuse intensity decays slower than the extinction decay of the direct intensity.

Of interest here is that However the maximum eigenvalue is close to one on the majority of the region. Such materials have a very slow decay rate of diffusion relative to the very close to the extinction decay rate of the direct component, so are in a strong sense effectively transparent in that there is always a component of direct intensity relative to the diffuse intensity. Given the direct component source in Figure ??, Figure 7 shows the convergence of an opaque material to the diffusion limit, while Figure 8 shows a semi-transparent material where the direct component remains larger than the limiting diffuse component.

A material is semi-transparent if it allows the direct component to come through the material. If we define the semi-transparency depth to be the maximum depth at which the direct component is at least 10% of the diffuse component, then any material which is thinner than its transparency depth will be semi-transparent. Figures ?? and ?? show the transparency depths and

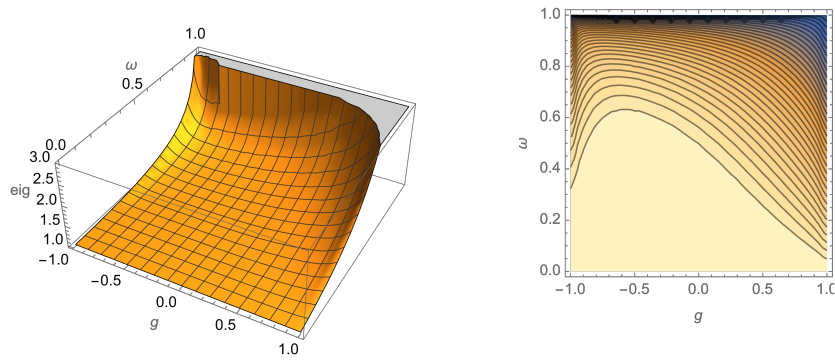


Figure 6. Maximum eigenvalues for varying asymmetry g and scatter fraction ω . The maximums increase as ω increases, i.e. as absorptency decreases.

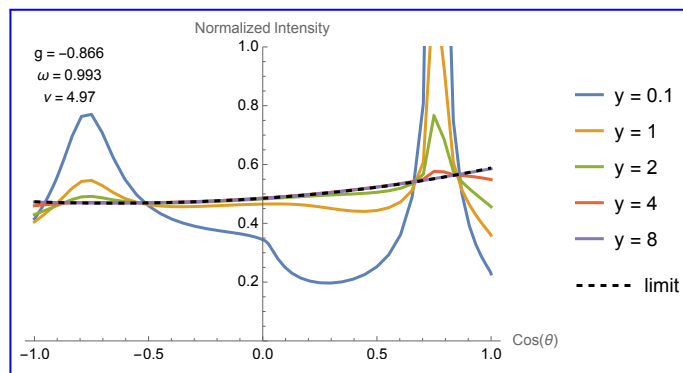


Figure 7. ~~Direct~~-Asymptotic radiance distribution for a low absorptency media ($\omega = 0.993$). As depth y increases, the diffuse internal intensity dominates the direct source intensity (the peak at the right) and the normalized intensity converges to the diffusion limit.

Convergence of normalized intensity distribution for $\nu \gg 1$.

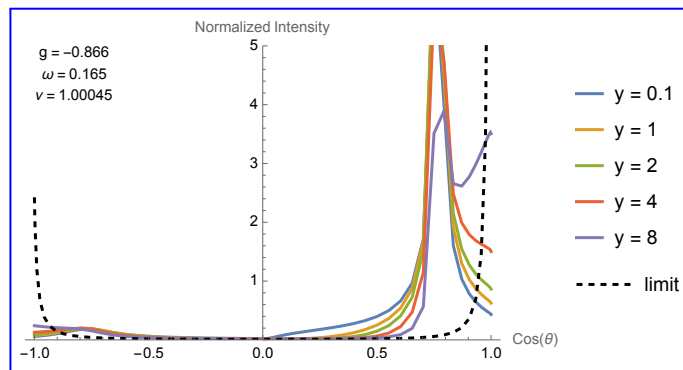


Figure 8. ~~Convergence of normalized intensity~~-Asymptotic radiance distribution as depth y increases for $\nu \approx 1$ a high absorptency media ($\omega = 0.165$). The direct source intensity (the peak at the right) dominates the diffuse limit.

regions.

Comparing Apparent Optical Properties to Two-Stream Approximation

Semi-transparency depths for varying asymmetry g and scatter fraction ω . The two-stream/Kubelka-Munk formulation

provides simple analytic formulas for computing g and ω from R and η , and for the inverse. We use it as a reference baseline in our investigation of the properties of the multi-stream formulation.

Regions for semi-transparency depths 10, 100, 1,000.

Inherent and Apparent Optical Properties and the Difficulty of the Inverse Problem

The *Inherent Optical Properties*³¹ (IOP) are the properties of a media that characterize its optical properties behavior. For equation (1) they consist of the optical depth, the scatter fraction, and the scattering matrix. Since the numerical investigations here assume normalization by the optical depth and use the Henyey-Green one-parameter phase function, it is sufficient to use the scattering asymmetry g and the scattering fraction ω . This is also the common practice when using a two-stream approximation such as the Kubelka-Munk formulation.

The *Apparent Optical Properties*³¹ (AOP) are the observed optical properties of a media. Following common practice, we look at the measured reflectance R and attenuation coefficient η .

The two-stream/Kubelka-Munk formulation provides simple analytic formulas for computing g and ω from R and η , and for the inverse. Unfortunately, numerical investigations show that the situation for the original problem is more complicated.

Figure 9 shows the reflectance and the attenuation coefficient at an optical depth of 0.75 for (a) $g = 0.85$ as ω varies from -1 to 1 , (b) $\omega = -1$ as g varies from 0 to 1 , and (c) $\omega = 1$ as g varies from 0 to 1 , for the numerical simulation and for the two-stream analytic approximation.

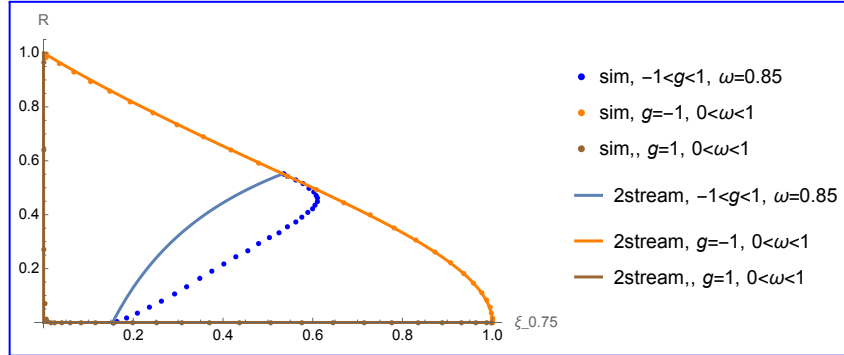


Figure 9. Reflectance and Attenuation Coefficient at 0.75 loss depth as computed by simulation and as given by the Kubelka-Munk two-stream approximation. There is agreement on the borders when $g = \pm 1$, but significant deviation in the interior when $-1 < g < 1$.

This illustrates the following:

1. The two-stream approximation agrees with the full system on the four boundaries, $\omega = \pm 1$ and $g = 0, 1$,
2. Overall error of the two-stream approximation is ≈ 0.15 as shown in Figure ??.
3. The reflectance R and attenuation coefficient $\xi_{0.75}$ (at transmission loss 0.75) do not uniquely determine the asymmetry g and scatter fraction ω for $g < -0.7$.

Absolute error in reflectance and attenuation coefficient for two-stream approximation, for $-1 < g < 1$ and $0 < \omega < 1$.

365 If we look at the scatter diagrams at differing transmission loss depths in Figure 10, we see that the difficulty of invertibility
 366 increases as the loss increases, the data lets us determine g and ω at loss depths of less than 50%, or for $g > -0.5$, while
 367 inversion becomes problematic for loss depths greater than 75% and $g < -0.7$.

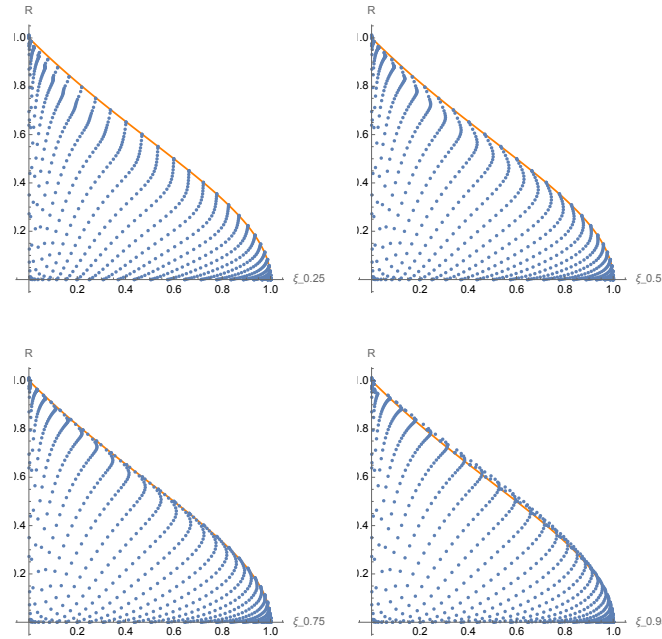


Figure 10. Scatter diagrams for reflectance R and attenuation coefficient ξ at loss depths of 25%, 50%, 75%, and 95%.

368 **Methods**

369 **AII**

370 **Comparison to Other Models**

371 A choice was made as to which Radiative Transfer Models (RTMs) to compare against EigenFlux (See Table 2). The major
 372 concerns were: (1) Does the RTM explicitly model scattering? (2) Does the RTM include the visible spectrum? (3) Is it
 373 available as open access, and is it still supported? (4) Does it use its own solver or does it use a different RTM? (5) Is it mature
 374 (version > 1.0)?

375 These considerations led to two RTMs: DISORT and MYSTIC, of which DISORT was chosen due to its long maturity
 376 and the multiple RTMs that use it as their solver.

Table 2. Radiative Transfer Models (RTMs) considered in course of study. Does not include RTMs intended for spectrographic use which only implicitly consider multiple scattering. Does not include models that use other models as their solver. Includes only mature (version > 1.0) models under active support. List of models obtained from Wikipedia⁵⁷ and independent literature search.

Beginning of Table 2				
Name	UV/Visible?	Open Access?	Other Limitations?	Solver?
6S/6SV1¹	No	Yes		Successive Orders
ARTS²	No	Yes		Discrete Ordinates Iterative, Monte Carlo
CRTM³	Yes	Not publicly released		Doubling-Adding, Successive Orders
DART⁴	No	Yes for research with public funding		Transfer function iteration
DISORT⁵	Yes	Yes		Discrete Ordinates
HydroLight⁶	Yes	Commercial		Invariant Imbedding
libRadtran⁷	Yes	Yes		Uses DISORT, MYSTIC
LinePak⁸	Yes	On-line only		
MODTRAN⁹	Yes	No		

¹6S/6SV1, "Second Simulation of a Satellite Signal in the Solar Spectrum vector code", <https://salsa.umd.edu/6spage.html>, (accessed May 6, 2026)

²ARTS, "The Atmospheric Radiative Transfer Simulator", <https://www.radiativetransfer.org/about/>, (accessed May 6, 2026)

³CRTM, "Community Radiative Transfer Model (CRTM)", <https://www.jcsda.org/jcsda-project-community-radiative-transfer-model>, (accessed May 6, 2026)

⁴DART, "The Discrete Anisotropic Radiative Transfer Model", <https://dart.omp.eu/#/>, (accessed May 6, 2026)

⁵DISORT, "LLLab DISORT Website", <http://www.rtatmocn.com/disort/>, (accessed May 6, 2026)

⁶HydroLight, "HydroLight", <https://www.numericaloptics.com/hydrolight.html>, (accessed May 6, 2026)

⁷libRadtran, "libradtran", <https://www.libradtran.org/doku.php>, (accessed May 6, 2026)

⁸LinePak, "Spectral Calc.com: High-resolution spectral modeling", <https://www.spectralcalc.com/info/about>, (accessed May 6, 2026)

⁹MODTRAN, "MODTRAN®", <http://modtran.spectral.com>, (accessed May 6, 2026)

Continuation of Table 2				
<u>Name</u>	<u>UV/Visible?</u>	<u>Open Access?</u>	<u>Other Limitations?</u>	<u>Solver?</u>
<u>MYSTIC¹⁰</u>	<u>Yes</u>	<u>Yes (See libRadtran)</u>		<u>Monte Carlo</u>
<u>RTMOM¹¹</u>	<u>Yes</u>	<u>No</u>		<u>Method Of Moments</u>
<u>SCIATRAN¹²</u>	<u>Yes</u>	<u>Yes</u>	<u>Scattering limited to built-in functionality</u>	<u>Discrete Ordinates</u>
<u>SMART-G¹³</u>	<u>Yes</u>	<u>Yes</u>	<u>Requires GPU</u>	<u>Monte Carlo</u>
<u>SMRT¹⁴</u>	<u>No</u>	<u>Yes</u>	<u>Specialized for snow pack</u>	<u>Discrete Ordinates</u>
<u>VLIDORT/ LIDORT¹⁵</u>	<u>Yes</u>	<u>By request</u>		<u>Discrete Ordinates</u>
End of Table 2				

Phase Function Limitations

Of the tools considered, the snow and ice models are intended for highly forward scattering problems. Since they use the DO method, they are subject to the instability problems that come with using Legendre expansions of a highly asymmetric phase function. They ameliorate those problems with the use of variations on a specific technique, called the $\delta - N$ method in Thomas-Stamnes⁵³, which treat forward scattering light particles as non-scattering transmitted particles. But as we see below in Figure 16, the effect of this is limited.

Stamnes, et. al.⁴⁹ report that while snow grains have asymmetry $g = 0.9$, brine pockets have asymmetry $g = 0.997$. Their tool, CASIO-DISORT, derived from DISORT, was limited practically to $g_{max} \leq 0.9$. The higher g required for brine-snow mixtures was achieved through a reduced forward scattering transformation⁴⁹ which is essentially an externally applied $\delta - N$

¹⁰MYSTIC, "MYSTIC - the Monte Carlo code for the physically correct tracing of photons in cloudy atmospheres", <http://www.bmayer.de/index.html?mystic.html&l>, (accessed May 6, 2026)

¹¹RTMOM, "Radiative Transfer Model Comparison with Satellite Observations over CEOS Calibration Site Libya-4", <https://www.mdpi.com/2073-4433/13/11/1759>, (accessed May 6, 2026)

¹²SCIATRAN, "SCIATRAN: RADIATIVE TRANSFER MODEL AND RETRIEVAL ALGORITHM ", <https://www.iup.uni-bremen.de/sciattran/>, (accessed May 6, 2026)

¹³SMART-G, "SMART-G: a Monte Carlo GPU Radiative Transfer code", <https://hygeos.com/en/smart-g/>, (accessed May 6, 2026)

¹⁴SMRT, "SMRT: Snow Microwave Radiative Transfer model", <https://smrt-model.science>, (accessed May 6, 2026)

¹⁵VLIDORT/LIDORT, "Overview of RT Solutions Products", http://www.rtslidort.com/about_overview.html, (accessed May 6, 2026)

method.

We can see some of the advantages of the mesh approximation over the Legendre decomposition in managing phase functions. Figure 11 shows the approximations of the cloud phase function defined in Garcia-Stewart¹⁷, which contains both strongly forward scattering and a significant backward scattering component. Approximations used are those used in the case of $n = 56$ streams.

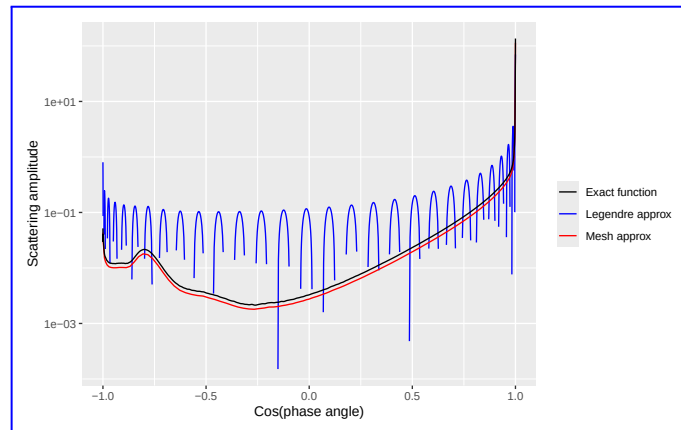


Figure 11. Cloud phase function and approximations for $n=56$. The original phase function is strongly forward scattering, but also has a significant backward scattering component. The gaps in the Legendre approximation are due to the approximation going negative.

Accuracy and Timing Studies

To establish the accuracy of the EigenFlux algorithm, numerical investigations were conducted on a MacBook Pro with 2.6 GHz 6-Core Intel Core i7, 16Gb of RAM, running Ventura V13.5 Tahoe V26.3.1. The final algorithm was coded in GNU Fortran 2008 as well as an implementation in Mathematica V13.1 using a Mathematica notebook, and Python V3.10.12 and numpy V1.25.1 using a Jupyter notebook V6.5.4. Six test cases were developed based on the DISORT distribution, and timing studies were run compared to DISORT using the same compiler.

2008. The accuracy of the implementation was established first verified by hand comparisons to published tables /eitevanhulst1980 (see Table 35 in Van Hulst⁵⁵).

Test Case EigenFlux (ms) DISORT (ms) 1 340 404 2 200 310 3 104 33 4 9083 5188 5 4089 680 6 131 127 Test run results:

In order to validate the fortran implementation of EigenFlux, the distribution for the standard radiative transfer package The distribution for DISORT4 was downloaded from the *Light and Life Lab* website²⁷, along with its standard test suite. The Eigenflux software was then run on the DISORT test suite excluding the unimplemented features thermal emissions and generalized BDRF functions, and compared to the reference DISORT results. See Figure ??, and compiled with the same compiler and options as Eigenflux was.

Additional Topics

Hilbert Space Spectrum and Eigenvalues

In this section we show that much of the structure of the eigenvalues seen in figure 4 for ω is due to the structure of the spectrum of the original equation (??) considered as a Hilbert space operator. conditions, the error between the computed solution and the reference equation was computed. The integral formulation (2) was used as the reference equation because it is a numerically stable exact form. A Schlick phase distribution^{4, 36} was used as the reference phase kernel for two reasons 1) it is very similar to the popular Henyey-Greenstein phase distribution, 2) its kernel function $K(v, w)$ has a simple analytic form, allowing the exact kernel to be used in the reference equation.

We exploit the spectral theory for Hilbert operators¹¹. We rewrite the RHS of equation (??) as linear operators over the Hilbert space $L^2[-1, 1]$ of measurable functions over the interval $[-1, 1]$ with finite L^2 norm where

$$A[v]f(v) = v f(v)$$

$$Kf(v) = \int K(v, w) f(w) dw$$

$$\rho = \omega B K D(y) p_0$$

Runs were performed for varying number of streams ranging from 14 to 280, for a Chebyshev mesh and for a linear mesh. Each run included 49 test cases varying scattering fractions from 0.012 to 0.987 and for asymmetry varying from -0.975 to +0.975.

and then (??) becomes

$$A[v] \partial_y M + (I - \omega K) M = \rho$$

and now we can look at the spectrum of the operator

$$H = (I - \omega K)^{-1} A[v].$$

In general, the spectrum of a bounded operator B is comprised of three parts: spec_d : A discrete spectrum of "true" eigenvalues (e.g. for $v \in \text{spec}(B)$, $\exists f \in L^2, Bf = vf$). The set will be finite or countable, with the only allowed limit point at $v = 0$. essSpec : A continuous spectrum of approximate eigenvalues (e.g. for $v \in \text{essSpec}(B)$, there exists a sequence of unit functions $f_n \in L^2, \|Bf_n - vf_n\| \rightarrow 0$). $\text{spec}_r = \text{spec}_d(B^*) \setminus (\text{spec}_d(B) \cup \text{essSpec}(B))$: A residual spectrum of eigenvalues of the adjoint operator which were not in spec_d or essSpec . Figure 12 shows the accuracy as a function of the number of streams for the Chebyshev mesh and for the linear mesh. It shows that the Chebyshev mesh is more accurate by a range of 1 to 2 orders of magnitude. There is significant variance in the accuracy across the 49 test cases of differing scattering

fraction and asymmetry. Roughly speaking, an increase in number of streams of 30 times results in one order of magnitude increase in accuracy.

enumerate

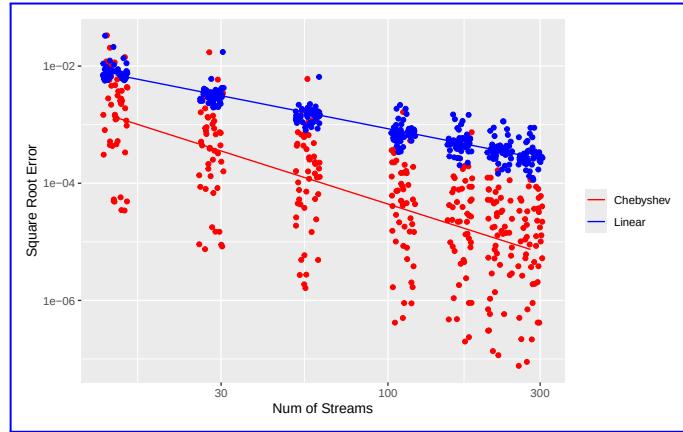


Figure 12. Error as a function of number of streams for Chebyshev spacing and for linear spacing, plotted on a log-log scale. The error is the integration of the squared error between the intensity and its reference equation. Fitting was made to a log-log curve. Jitter added to number of streams to expand point cloud.

By its definition in (??), the operator K is an integral operator with doubly stochastic kernel, so is a compact operator. Since $H = A[v] + \omega KH$, this means that H is a compact perturbation of $A[v]$. $A[v]$ is also a Fredholm operator (provable through direct calculation of its kernel and co-kernels¹¹). It then follows that $\text{essSpec}(A[v]) = \text{essSpec}(H)$, that the residual spectrum spec_r is empty, and that the remaining spectrum $\text{spec}_d(H) \setminus \text{essSpec}(H)$ consists of a finite number of isolated eigenvalues. The change in accuracy for EigenFlux across the test cases is largely a function of the scatter fraction- the lower the absorption, the lower the accuracy as in Figure 13. Figure 14 shows that there is a weak tendency for the method to be more accurate at the extreme points of asymmetry.

Furthermore, H is a positive operator on the positive half $[0, 1]$ of

The change in accuracy for DISORT across the test cases is similar in behavior to EigenFlux as a function of the interval $[-1, 1]$, so it has a maximum eigenvalue $v_{\max} = ||H||$, and the corresponding eigenvector is positive and is the only positive eigenvector for that subspace. scatter fraction- the lower the absorption, the lower the accuracy as in Figure 15. However unlike EigenFlux, Figure 16 shows DISORT is less accurate at the extreme points of asymmetry.

Similar considerations for the negative half lead to the overall conclusion that H will have two isolated eigenvalues $\pm ||H||$, each with a positive eigenvector, and those two eigenvectors are the only two positive members of the overall eigenspace.

From direct calculation on $A[v]$, we find that $A[v]$ (and therefore H) has a continuous distribution of spectrum values from -1 to 1 , each value associated with a spectral function with a jump discontinuity at the spectrum value.

These conclusions are consistent with what we see in Figure 4 and Figure 5. The same input parameters were used to

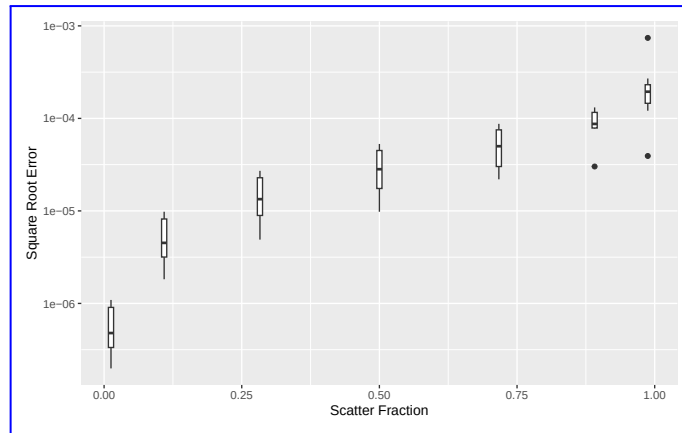


Figure 13. Boxplot of EigenFlux error as a function of scatter fraction. Data is for the run with $nm = 168$. Lower absorbency increases error.

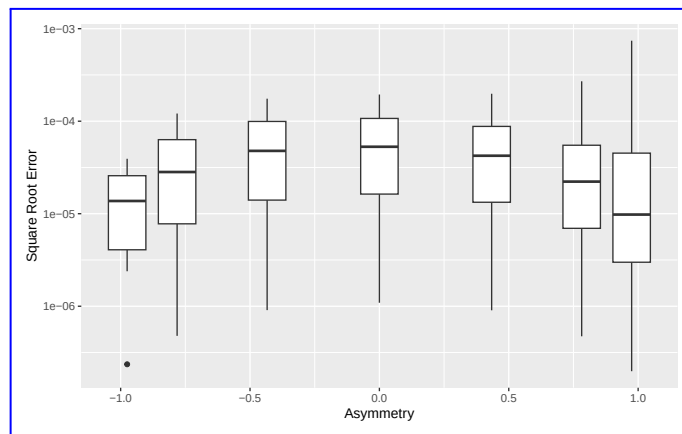


Figure 14. Boxplot of EigenFlux error as a function of asymmetry. Data is for the run with $nm = 168$. Strong asymmetry decreases error slightly.

448 directly compare the EigenFlux and DISORT results and timing. The Henyey-Greenstein phase function was used for these
 449 comparisons.

450 Incorporating Azimuth-dependence

451 Given a non-vertical collimated beam source, the intensity will be azimuth-dependent. While any azimuth dependence in
 452 the individual particles may be azimuth-dependent, it disappears when averaging over the random orientations of the particles.
 453 Nonetheless, even with a rotationally symmetric phase function, the scattering matrix will in general be azimuth-dependent. One
 454 issue in the comparison is that EigenFlux always computes across a spanning set of source functions, whereas DISORT
 455 computes a single source function at a time. As a result, the DISORT times in Figure 17 were scaled to correspond to the
 456 EigenFlux usage. The result shows that DISORT is faster for lower numbers of streams, whereas EigenFlux is faster for higher
 457 numbers of streams, with a crossover at $n = 56$.

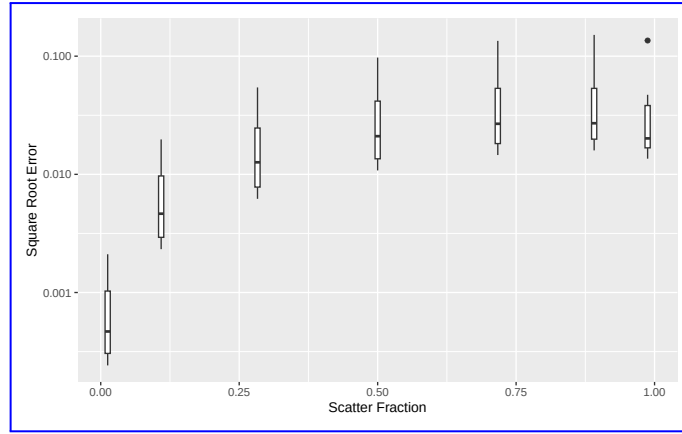


Figure 15. Boxplot of DISORT error as a function of scatter fraction. Data is for the run with $nm = 168$. Lower absorptency increases error.

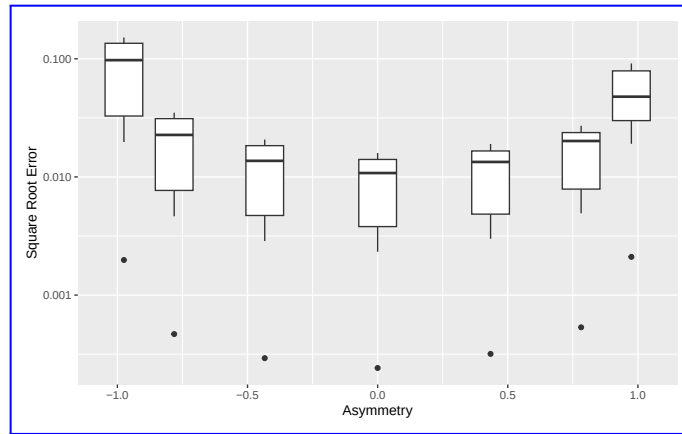


Figure 16. Boxplot of DISORT error as a function of asymmetry. Data is for the run with $nm = 168$. Strong asymmetry increases error.

Let the (averaged) phase function be given by $k(v)$. Then the scattering matrix will be given by-

$$K(v_s, v_i; a_s - a_i) = k(v_s v_i + \cos(a_s - a_i) \sqrt{1 - v_s^2} \sqrt{1 - v_i^2})$$

where-

(v_i, a_i) Polar cosine and azimuth angle of incident vector-

(v_s, a_s) Polar cosine and azimuth angle of scattered vector-

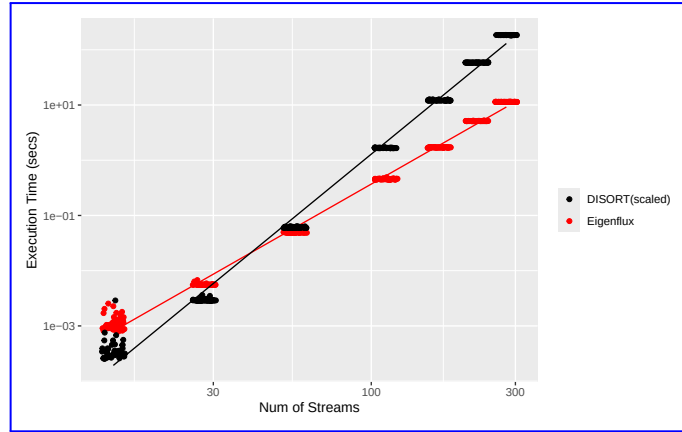


Figure 17. Execution time for EigenFlux vs. DISORT, plotted on a log-log scale. DISORT times have been scaled by number of source beams since EigenFlux always computes multiple source beams.

Under these conditions the main equation (??) becomes-

$$I(v, a; y) + v \partial_y I(v, a; y) = \omega \int K(v, w; a - b) P(w, b; y) dw$$

$$I(v, a; 0) = s(v, a)$$

The difference between the intensities computed by EigenFlux versus DISORT show considerable variance, as seen in Figure 18. Figures 20 and 19 show that this is largely a factor of the asymmetry- DISORT's solution deteriorates for strongly scattering problems.

Note that $K(v, w; a)$ is cyclic in a , i.e. $K(v, w; a + 2\pi) = K(v, w; a)$ and is the only term in the equation that is azimuth-dependent other than the boundary condition. As a result, the fundamental solution will be cyclically symmetric in azimuth-

$$\Phi(v, a; w, b; y) = \Phi(v, a - b; w, 0; y) = \Phi(v, a - b + 2\pi; w, 0; y)$$

We apply the mesh approximation process to the azimuth-dependent diffuse intensity equation (??) above) with the additional property that all functions/matrices (other than $p_0(v, a)$) that are dependent on azimuth are cyclically symmetric-

Model Approximations $K(v, w; a - b) = \sum_{jpc} K_{jpc} \phi_j(v) \phi_p(w) \tilde{\phi}_c(a - b)$ Phase function kernel $p_0(v; a) = \sum_{jc} p_{0jc} \phi_j(v) \tilde{\phi}_c(a)$
Intensity at surface $M(v, y; a - b) = \sum_{jc} M_{jc}(y) \phi_j(v) \tilde{\phi}_c(a - b)$ Diffuse intensity distribution $\tilde{\phi}_c(a - b) = \tilde{\phi}_c(a - b + 2\pi)$ Azimuth test function

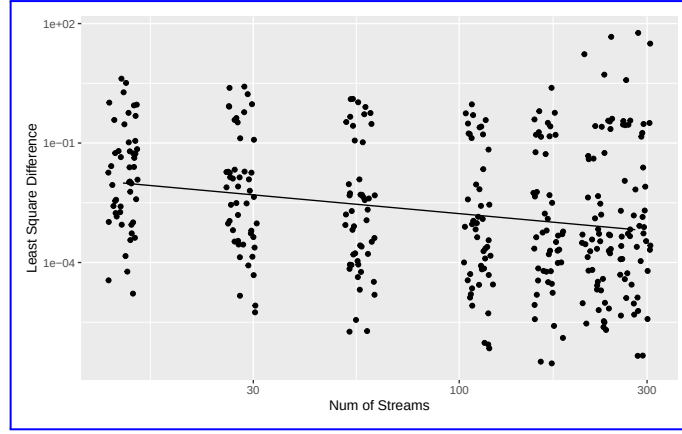


Figure 18. Least square difference between intensities computed by EigenFlux vs. DISORT, plotted on a log-log scale. Jitter added to number of streams to expand point cloud.

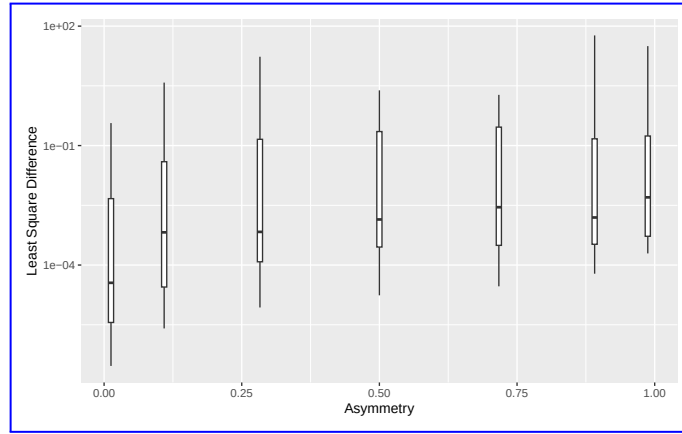


Figure 19. Least square difference between EigenFlux and DISORT as a function of scattering fraction, plotted on a log scale. Lower absorptency increases the difference, very slightly.

From this we derive the finite dimensional linear ODE:-

$$\begin{aligned} \sum_{jc} A_{ij} \tilde{B}_{fc} \partial_y M_{jc}(y) + \sum_{jc} B_{ij} \tilde{B}_{fc} M_{jc}(y) - \omega \sum_{jpc} B_{ij} \tilde{C}_{fdc} K_{jpd} B_{pq} M_{qc}(y) \\ = \omega \sum_{jpc} B_{ij} \tilde{C}_{fdc} K_{jpd} D_{pq}(y) p_{0qc} \end{aligned}$$

Conclusions

where EigenFlux is a novel approach to solving the radiative transfer problem, using a natural reflectance structure and a Galerkin mesh approximation scheme that allows for flexible calculation points including linear and Chebyshev point distributions, which in turn allows for accurate reduced order approximations of phase functions which are strongly forward scattering or have significant backward scattering components. The resulting implementation shows strong numerical accuracy

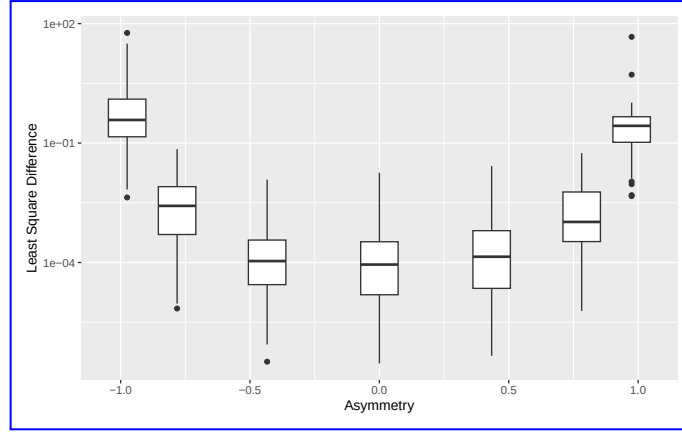


Figure 20. Least square difference between EigenFlux and DISORT as a function of asymmetry, plotted on a log scale. Strong asymmetry increases the difference.

and favorable computational scaling in comparison with DISORT across the test cases investigated here. The primary novelty of EigenFlux lies in its combination of natural-reflectance stabilization, localized mesh approximations, and eigenmode decomposition, allowing stable treatment of highly asymmetric scattering regimes that are difficult for conventional discrete-ordinate methods.

$$A_{IJ} = \int v \psi_J(v) \phi_J(v) dv$$

$$B_{IJ} = \int \psi_J(v) \phi_J(v) dv$$

$$C_{IJ} = \int \phi_J(v) \phi_J(v) dv$$

$$= \int \tilde{\psi}_f(a) \tilde{\phi}_c(a) da$$

$$= \int \tilde{\psi}_f(a) \tilde{\phi}_d(a-b) \tilde{\phi}_c(b) da db$$

$$D_{IJ}(y) = \int e^{-y/v} \phi_J(v) \phi_J(v) dv$$

Figure 14 shows that EigenFlux is more accurate for extreme scattering points while Figure 20 shows that the comparable DISORT code is less accurate for the same extreme points. Figure 17 shows that EigenFlux is faster than DISORT for larger numbers of streams, with an even breakpoint at about 56 streams.

More succinctly,

$$\underline{A\tilde{B} \partial_y M(y) + (B\tilde{B} - \omega B\tilde{C}K C)M(y) = \omega B\tilde{C}K D(y)p_0}$$

Analytically, EigenFlux provides insights into the nature of RTM intensity structures. The max eigenvalue determines the limiting distribution of the intensity, and dictates the degree to which the diffuse intensity overcomes (or fails to overcome) the

direct intensity as depths increase. In a low-absorbency system such as snow and ice, the diffuse intensity quickly overcomes the direct intensity.

As before, the next step is to look at the eigenvalue decomposition of $\tilde{R} = (B\tilde{B} - \omega B\tilde{C}KB)^{-1}A\tilde{B}$. The matrix $\tilde{R}$ has some special properties:-

1. $\tilde{R}$ is block circulant. I.e. $\tilde{R}_{fciJ} = \tilde{R}_{(f-c)0iJ}$
2. The individual blocks of $\tilde{R}$ are symmetric. I.e. $\tilde{R}_{ij}^{fc} = \tilde{R}_{ji}^{fc}$

The analysis here also shows that for low absorbance systems such as snow and ice, the use of two-stream approximation as an inverse model to infer the asymmetry and single stage albedo from measured transmission and reflectance can be problematic as multiple values for the asymmetry and single stage albedo can result in identical transmission and reflectance measures.

Because of (1) and (2), the eigenvalues and eigenvectors of $\tilde{R}$ are real and can be determined from the number of blocks and the eigenvalue decomposition of a geometric sum of the individual blocks^{10,50,51}. Let the blocks of $\tilde{R}$ be given by R_{i-j} . Let ρ be a primitive root of unity of order n , the number of blocks, so that $\rho^n = 1$ and such that $\{\rho^i\}$ generates a complete list of all roots of unity of order n . Let $H(\rho) = \sum_j \rho^j R_{n-j}$. Then the eigenvalues of $\tilde{R}$ are the $n * m$ eigenvalues of $H(\rho^k)$, and the eigenvectors are the outer product of $\{\rho_k^j\}$ and the eigenvectors of $H(\rho^k)$. The present implementation of EigenFlux is limited to scalar radiative transfer in plane-parallel geometries and does not currently include polarization, thermal emission, or fully spherical atmospheric effects. Inhomogeneous systems are represented through combinations of homogeneous layers rather than through fully continuous spatial variation. These limitations define important directions for future development. Future work could include extension to pseudo-spherical geometries, inclusion of polarization and thermal emission, adaptive mesh refinement, and optimization for specific architectures. The flexible mesh and eigenmode structure of EigenFlux may also prove useful for inverse retrieval applications and coupling to atmospheric, oceanographic, and graphics-rendering systems.

Data Availability

Data Availability

Materials described in the manuscript are available from the University of Copenhagen Electronic Research Data Archive at address: <https://doi.org/10.17894/ueph.cfd8e267-97f3-496d-b434-e29984b931e8> <https://doi.org/10.17894/ucph.cfd8e267-97f3-496d-b434-e29984b931e8>

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