Review of the Ocean Science manuscript egusphere-2025-3359

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Summary. This manuscript is based on an incorrect expression for the buoyant restoring force on a displaced fluid parcel. I describe what I think is the correct approach, relying heavily on Archimedes (personal communication 213BCE), and then suggest how this energy required to displace a fluid parcel can be calculated using the existing numerical algorithms in the TEOS-10 Gibbs Seawater Toolbox (McDougall and Barker, 2011).

1. Archimedes and the force on a displaced fluid parcel

Like the authors of the above manuscript, consider a vertical water column of the ocean, and concentrate on a water parcel at a certain depth whose Absolute Salinity [for convenience we omit the usual subscript A of S_{A}], Conservative Temperature and Absolute Pressure (in Pa) are $\left(\tilde{S}, \widetilde{\Theta}, \tilde{P}\right)$. The Absolute Salinity and Conservative Temperature of the ocean water column can be regarded as a function of pressure down the cast, that is, as $S(P)$ and $\Theta(P)$.

The parcel $\left(\tilde{S}, \widetilde{\Theta}, \tilde{P}\right)$ is now enclosed in an insulating plastic bag and is moved slowly until it arrives at a different depth in the water column. Let's call this new depth "location 2" where the pressure is P_2, and the parcel's properties there are $\left(\tilde{S}, \widetilde{\Theta}, P_2\right)$, while the surrounding ocean's properties at this location are $(S(P_2), \Theta(P_2), P_2)$. Along the journey to get from the original location to "location 2" the general pressure is labelled P', and the parcel has properties $\left(\tilde{S}, \widetilde{\Theta}, P'\right)$, while the surrounding ocean at this general pressure has properties $(S(P'), \Theta(P'), P')$.

We want to find the expression for the net buoyant force experienced by our fluid parcel $\left(\tilde{S}, \widetilde{\Theta}, P'\right)$ at the general pressure P' due to it being surrounded by the ocean environment fluid $(S(P'), \Theta(P'), P')$ which has different salinity and temperature to that of our plastic-bag-enclosed parcel at this pressure. The key result that is needed here was discovered by Archimedes of Syracuse and published by him in 213BCE; yes, published two thousand two hundred and thirty-eight (2038) years ago! Archimedes (213BCE) stated that a floating body in equilibrium displaces a certain volume of the fluid in which it floats whose weight is that of the floating body. [If I could talk to Archimedes, I would say to him how very impressive it is that he was able to prove this important result eighteen centuries before Isaac Newton developed the calculus.]

To be specific, let the mass of the fluid in our insulting plastic bag be 1 kilogram. At its original location, the volume of this one kilogram of seawater is $\hat{v}(\tilde{S},\widetilde{\Theta},\tilde{P})$, and its density is the reciprocal of this, namely $\hat{\rho}(\tilde{S},\widetilde{\Theta},\tilde{P})$. These quantities can be calculated from the GSW Oceanographic Toolbox as gsw_specvol(SA_tilda,CT_tilda,P_tilda) and gsw_rho(SA_tilda,CT_tilda,P_tilda) respectively. Note that the over-hat symbol of $\hat{v}$ and $\hat{\rho}$ simply draws our attention to the fact that these algorithms expect to have Conservative Temperature as its temperature input.

Having moved our insulated seawater parcel adiabatically and without exchange of matter from $\tilde{P}$ to P', the volume of this 1kg of seawater has changed from $\hat{v}(\tilde{S},\widetilde{\Theta},\tilde{P})$ to $\hat{v}(\tilde{S},\widetilde{\Theta},P')$. This is the volume of seawater at P' that our special bag displaces there. Invoking the ground-breaking result of Archimedes (213BCE), the surrounding seawater provides an upwards force on our specially insulated fluid parcel of exactly the weight of seawater displaced. What is this upwards force? It is

$$\text{Upwards force on the plastic bag} \;=\; g\;\hat{v}(\tilde{S},\widetilde{\Theta},P')\;\hat{\rho}(S(P'),\Theta(P'),P'). \qquad (1)$$

This is understood as follows. $\hat{\rho}(S(P'),\Theta(P'),P')$ is the mass per unit volume of the oceanic environment water at this pressure. The mass of displaced oceanic water is the product of this with the volume of the displaced water, $\hat{v}(\tilde{S},\widetilde{\Theta},P')$, and to convert this mass of displaced ocean water to a force, we multiply by the gravitational acceleration, g.

OK, we have the force that the ocean exerts on our bag-enclosed special parcel. Now we need the weight of our parcel itself. This is g. That is, the gravitational acceleration times 1 kg, making 1 Newton. If you like, this can also be found by a similar calculation to that of Eq. (1), namely as $g\;\hat{v}(\tilde{S},\widetilde{\Theta},P')\;\hat{\rho}(\tilde{S},\widetilde{\Theta},P')$ which is simply g since density and specific volume are reciprocals of each other. So the net upwards vertical force acting on the parcel is

$$\text{Net upwards force on the plastic bag} \;=\; g\;\left\{\hat{v}[\tilde{S},\widetilde{\Theta},P']/\hat{v}[S(P'),\Theta(P'),P'] - 1\right\}. \qquad (2)$$

Thank you Mr Archimedes of Syracuse; we couldn't have done this without your insight.

2. The work done to displace the fluid parcel

Eq. (2) is the net force on the insulated bag containing our 1 kg of reference seawater. In order to keep the insulated bag at this location we have to supply an equal and opposite external force to it [hence the minus sign at the beginning of the right-hand side of Eq. (3) below]. This force grows from the reference location, and we wish to quantify the total energy required to slowly move the insulated bag of fluid from its location at $\tilde{P}$ to the final location at P_2. This energy is the integral of the force with respect to height,

$$\text{Energy required per kg } = -\int_{\tilde{z}}^{z_2} g \left\{ \frac{\hat{v}[\tilde{S},\tilde{\Theta},P']}{\hat{v}[S(P'),\Theta(P'),P']} - 1 \right\} dz'. \qquad (3)$$

Now we use the hydrostatic relationship, $P_z = -g\rho$, or $g\, z_P = -\hat{v}[S(P'),\Theta(P'),P']$, to find

$$\text{Energy required per kg } = \int_{\tilde{P}}^{P_2} \left\{ \hat{v}\left[\tilde{S},\tilde{\Theta},P'\right] - \hat{v}[S(P'),\Theta(P'),P'] \right\} dP'. \qquad (4)$$

This is the accurate version of equation (3) of the submitted manuscript 2025-3359.

3. Evaluating the work done using GSW software

Eq. (4) above is almost identical to the Cunningham geostrophic streamfunction (see section 3.29 of the TEOS-10 Manual, IOC et al. 2010). This means that the present submitted manuscript egusphere-2025-3359 provides a nice physical explanation of the Cunnigham geostrophic streamfunction; an explanation/justification that is new to this reviewer.

The difference between Eq. (4) above and Eqs. (3.19.1) and (3.19.2) of the TEOS-10 Manual is that there the Cunningham geostrophic streamfunction is set up with the reference pressure being the sea surface pressure P_0 rather than being a general starting pressure $\tilde{P}$ as we do here. Hence, I will use the gsw software, `gsw_geo_strf_dyn_height(SA,CT,p,p_ref)` as the basis for writing Eq. (4) above in terms of GSW algorithms. This `gsw_geo_strf_dyn_height` algorithm delivers Eq. (3.27.1) of the TEOS-10 Manual. In terms of this function, we can evaluate Eq. (4) above by

$$\text{Energy required per kg} \;=\; \int_{\tilde{P}}^{P_2} \left\{ \hat{v}\left[\tilde{S}, \widetilde{\Theta}, P'\right] - \hat{v}\left[S(P'), \Theta(P'), P'\right] \right\} dP'$$

$$\begin{aligned} &= \text{gsw\_enthalpy(SA\_tilda, CT\_tilda, p\_2)} \\ &\quad - \text{gsw\_enthalpy(SA\_tilda, CT\_tilda, p\_tilda)} \\ &\quad + \text{gsw\_geo\_strf\_dyn\_height(SA,CT, p, p\_tilda)} \qquad (5) \\ &\quad - \text{gsw\_enthalpy(SSO, 0°C, p\_2)} \\ &\quad + \text{gsw\_enthalpy(SSO, 0°C, p\_tilda).} \end{aligned}$$

4. Remarks on the approximate approach based on potential density

Equation (1) of the submitted manuscript 2025-3359 is an approximate expression for the net force on our insulating impervious bag. This approximate expression ignores an important nonlinearity of the equation of state: the thermobaric nonlinearity. To see this, note that the manuscript's approximate expression for the force <u>per unit mass</u> is

$$\text{Approximate net force} \;=\; g\left\{ \hat{v}\left[\tilde{S}, \widetilde{\Theta}, P_2\right] / \hat{v}\left[S(P'), \Theta(P'), P_2\right] - 1 \right\}, \qquad (6)$$

[where the two specific volumes are the potential specific volumes refereed to the reference pressure P_2] which is to be compared with the accurate net force of Eq. (2) above. In the approximate case the force is proportional to $\left\{\hat{\rho}\left[S(P'), \Theta(P'), P_2\right] - \hat{\rho}\left[\tilde{S}, \widetilde{\Theta}, P_2\right]\right\}$ while in the accurate case it is proportional to $\left\{\hat{\rho}\left[S(P'), \Theta(P'), P'\right] - \hat{\rho}\left[\tilde{S}, \widetilde{\Theta}, P'\right]\right\}$. The (linearized) difference between these two expressions is

$$\text{Error in net force per kg} \;\approx\; \{P_2 - P'\}\left\{ \hat{\rho}_P\left[S(P'), \Theta(P'), \tfrac{(P'+P_2)}{2}\right] - \hat{\rho}_P\left[\tilde{S}, \widetilde{\Theta}, \tfrac{(P'+P_2)}{2}\right] \right\}. \quad (7)$$

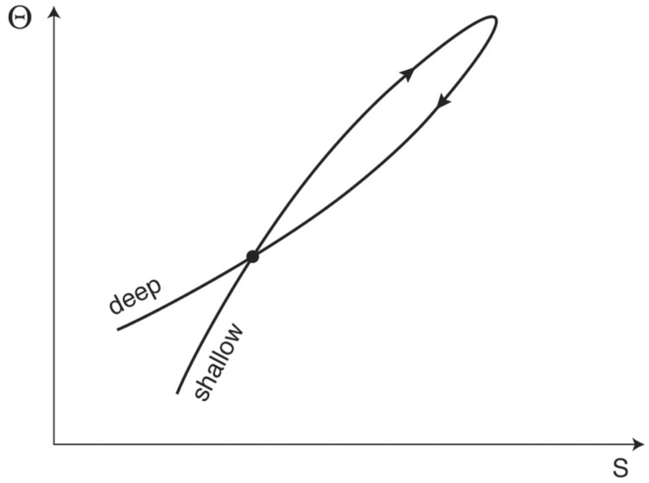
Concentrating on the dependence of the adiabatic compressibility, on temperature (and ignoring its dependence on salinity) this Eq. (7) scales as

$$\text{Error in net force per kg} \;\approx\; \{P_2 - P'\}\left\{\Theta(P') - \widetilde{\Theta}\right\}\hat{\rho}_{P\Theta}. \qquad (8)$$

Eq. (8) contains the product of a pressure difference and a temperature difference, multiplied by $\hat{\rho}_{P\Theta}$ which represents the rate at which the adiabatic compressibility (square of sound speed) varies with temperature, or equivalently, the rate at which the thermal expansion coefficient varies with pressure. This is the essence of the thermobaric non-linearity of the equation of state.

As with all uses of potential density, what is ignored is related to the thermobaric non-linearity of the equation of state of seawater. That is, the approach of Geoff Vallis' textbook is approximate, and it ignores this thermobaric nonlinearity. For large vertical excursions, this approximation can result in huge errors. As an example, consider that potential density referenced to the sea surface actually **decreases** with pressure in the deep North Atlantic, even though the water column there is statically stable. Deploying the method of this draft manuscript 2025-3359 would be greatly in error in such a region.

The thermobaric nonlinearity of the equation of state of seawater is actually sufficiently large that a salinity-temperature diagram as below is theoretically possible.



This SA-CT diagram is of a single vertical CTD cast which doubles back on itself but is statically stable at all heights. An analysis based on potential density would be hopelessly inaccurate when applied to such an extreme SA_CT vertical cast.

5. Concluding remarks

When the manuscript is corrected so that the expression for the net buoyant force is correct at finite amplitude, is it publishable as a new contribution to oceanographic knowledge? Perhaps it is. I, for one, did not know the connection between the Cunningham geostrophic streamfunction and the energy required to move an insulated parcel through the range of pressures. This connection between gravitational potential energy and a geostrophic streamfunction was new to me and only revealed itself when carefully deriving the above results, using the key insightful result of Archemedes of Syracuse.

REFERENCES

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IOC, SCOR and IAPSO, 2010: *The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties*. Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English), 196 pp. Available from www.TEOS-10.org [IOC et al. (2010) is colloquially called “the TEOS-10 manual”]

McDougall T. J. and P. M. Barker, 2011: Getting started with TEOS-10 and the Gibbs Seawater (GSW) Oceanographic Toolbox, 28pp., SCOR/IAPSO WG127, ISBN 978-0-646-55621-5, available from www.TEOS-10.org