Response to M. Crucifix

> *« The author proposes a new package written in Python for the computation of incoming solar radiation and integrals thereof, based on standard solutions for planetary motion and precession. The package is open source and licensed under the CeCILL free software licence agreement. Computation of insolation is fairly standard and already done in several other open source packages. Scientifically and computationally, a particularly valuable contribution of this article lies in section 4 where the author proposes numerical and analytical procedures for the computation of the minimum and maximum of the insolation for the year. One particularly interesting outcome is displayed in Figure 5, showing that in some realistic configurations equatorial insolation has only one and not two maxima. This could have some implications for the interpretation of the double-precession signal, which is relevant for cyclostratigraphic interpretations. The idea of computing insolation above a threshold by computing an elliptic integral between true solar longitudes that are first identified is elegant and welcome. »*

I warmly thank M. Crucifix for his kind comments and for the very useful and detailed observations listed below.

 # Lacking references

> *Mention of existence of other, similar packages by other authors is only made once and very indirectly line 549: "while most of the routines available in 'astro.py' and 'insol.py', as described above, are available in several other software packages or languages"*

Indeed, and this was also noted by Marie-France Loutre. I can only confess that I am certainly not aware of all available softwares designed to compute insolation time series, and I was in particular not aware on the ones mentioned by Michel Crucifix.
In a revision of this manuscript, I will include these recent references:

- Crucifix, M. (2023). palinsol : a R package to compute Incoming Solar Radiation (insolation) for palaeoclimate studies (v1.0 (CRAN)). Zenodo. https://doi.org/10.5281/zenodo.14893715
- Oliveira, E. D.: Daily INSOLation (DINSOL-v1.0): an intuitive tool for classrooms and specifying solar radiation boundary conditions, Geosci. Model Dev., 16, 2371–2390, https://doi.org/10.5194/gmd-16-2371-2023, 2023.

and will also look for other ones, like for instance:

- Bryan Lougheed, https://github.com/bryanlougheed/orbital chime/

> *These packages are never explicitly cited and the phrasing be read as casting doubt on their reliability: l. 10: "some people might not be aware of" ; l. 49 : "A frequent mistake" ; l. 204 : "as some people tend to believe".*

It was not in my intention to cast doubt on existing softwares, but only to warn the reader about some "self-made" tricks that I have met several (many) times in the paleoclimate community, as explained below:

> *In addition, the description of the incorrect procedure ("add daily insolation at some different positions") could be made more explicit to avoid ambiguity. Indeed, the time integral of any quantity can be well approximated with the rectangle or trapeze rule, over equally spaced intervals in time. That is: $\int_{\lambda 1}^{\lambda 2} W(\lambda) dt$ is correct; One can also replace the increment by $d\lambda * dt/d\lambda$. By contrast, it would be incorrect to simply substitute $d\lambda$ by dt . This is certainly what the author implies.*

This is indeed what I had in mind. Some people tend to compute the "averaged insolation" between two orbital positions $\lambda 1$ and $\lambda 2$ simply by computing insolation at regularly spaced values λi between $\lambda 1$ and $\lambda 2$ and computing the average. Indeed, this corresponds to replacing dt by $d\lambda$ in the integral.
This mistake is closely associated to the confusion between "true longitude" (λ) and "mean longitude" (basically t), and to the difficulty to understand the difference between astronomical seasons and the present-day calendar. This is why I devoted the whole paragraph 2.3 on this question.
I will clarify this difficulty in a revised manuscript.

> *Note that neither palinsol nor, to my knowledge, DINSOL make this mistake.*

I am sorry that the manuscript could potentially be misinterpreted in this direction: this was not my intention. I will be much more explicit in the revised version. For instance, in §3.1 (around line 204) I will suggest the reader to use "existing softwares (with references)" instead of trying to compute "by hand" an averaged insolation, since the task is not entirely trivial, as shown in the equations detailed in the text. Note that DINSOL does not compute integrated insolations, but only instantaneous and daily insolation.

> *Reference to previous work is also lacking in the introduction of the elliptic integrals. Berger et al., 2010 introduces these equations (also reproduced in palinsol) and also introduces the history of their usage in the context of insolation calculations, citing works in German, e.g. Fempl.*

Indeed, the elliptic integrals have already been described before and I will refer to these previous publications.

> *My recommendation here would be to provide a more extensive review of previous work, outlining the specific needs addressed by this new package and acknowledge what has already been done successfully; and focus on the more specific contribution of identifying minima and maxima of daily insolation.*

I will follow this recommendation in the best possible ways, though an "extensive review" is probably difficult to do and will certainly not be an "exhaustive" one. I will also highlight which routines are available in which software package (at least for the ones I am aware of) in a short table in a new appendix. This would help to better identify and explain the specificities of this new package.

 # Incorrect definition of caloric insolation

> *The definition of caloric insolation provided by Milanković (1941; 2002), in his paragraph 87, "the half year that comprises all the days of stronger radiation", and therefore "experiences the greatest possible irradiation. The boundaries of this half-year are determined by solving a differential equation (his eq. 138), whose solution is used by Berger (1978) in his "Long-term variations of caloric insolation resulting from the earth's orbital elements", and not in general time-centred on the solstices as implied by eq. l. 215.*

This is indeed true and should be explained in a revised version of this paper. This problem was also noted by Marie-France Loutre, and I copy below the corresponding paragraph I wrote on this point:
The precise definition given by Milankovitch (1941 ; in §87 of his book) is indeed :
« … we divide the year into two equally long and consequently real-half years, one of which comprises all those days of the year during which the irradiation of the latitude in question is stronger that on any day of the other half year… » .

It turns out that the computation of this quantity is difficult and Milankovitch used several approximations that are valid only at high latitudes (§88 of his book):
« … with the exception of a narrow belt around the equator… the beginning of the caloric summer half year is adjacent to the vernal equinox while the beginning of the caloric winter half year is adjacent to the automnal equinox ».
Using series expansions and assuming that the beginning of the caloric summer half year is very close to the vernal equinox (and similarly for its end), he finally treats both the beginning and the end in a strict symmetric way (see formulae (165) to (167) in §89). In other words, the assumptions and approximations used by Milankovitch correspond in fact to the computation performed by my software, which is the integrated irradiation over a half-year centered at the solstice.
But I fully agree with Marie-France Loutre and Michel Crucifix, that it does not correspond to Milankovitch's definition, though it corresponds rather closely to Milankovitch's computations. This will be clarified in the revised version of the paper.

> *Admittedly, the definition of Milanković is slightly ambiguous, since he does not explicitly mention that the half-year is meant to be continuous, though his Figure 42 implies this. This is a subtlety that I realised while doing the present review, and lead me to conclude that the brute-force approach in palinsol is not applicable in the tropics (this will be corrected in a next version). Outside the tropics, I would consider that the computations of Berger (1978), and the brute-force approach in palinsol are, in principle, closer to the intended definition than in the current contribution.*

This is indeed likely to be the case for the brute-force approach (palinsol), since there is no reason for the beginning and for the end of the "caloric summer" (according to Milankovitch's definition) to be symmetric around the solstices. I nevertheless suspect that the difference should remain small for all paleoclimatic practical situations (somewhat like the difference shown on Fig. 8 of this manuscript between the maximum insolation and the summer solstice one), while it seems to me that Berger (Quarter. Res. 1978, equation 4) uses the same approximation as Milankovitch 1941, which is more or less equivalent to my definition.
Interestingly, the computation of minima and maxima of insolation could help designing a more systematic way to compute the actual "caloric summer" insolation at all latitudes and for all possible orbital parameters.

 # Minor remark on geocentric vs heliocentric

> *The words "geocentric" and "heliocentric" are swapped ll. 125-126. Indeed, the equation coming just after eq 1,* $\sin\delta = \sin\varepsilon\ \sin\lambda$ *emerges from the resolution of a spherical triangle on the celestial sphere (geocentric), with, thus* δ *positive when* λ *is between 0 and* π*.* λ *is in that case the true longitude of the Sun, and is 0 when the Sun is aligned with the first point of Aries.* ϖ *is, indeed, commonly supplied in e.g., Laskar, in heliocentric coordinates, thus* $\varpi = 0$ *when the perihelion in reach in September.*

Yes, Michel Crucifix is (again) right… I will correct this in a revised version.

Minor typographic / others

> Use English quotes (") rather than guillemets(«).
> l. 282 : for "a" generic planet.
> l. 265 : "sun hour angle" rather than "time-step". ;
> l. 640 : hyperbolic functions: use LaTeX straight characters (as for $\cos$, etc.).

Yes, thank you! This will be corrected.

> l. 252 : The sentence "This is a pity" is probably unnecessarily colloquial; yes analytical solutions need to be preferred to numerical approximations when necessary, however assuming convergence checks are made a numerical approximation is not necessarily inaccurate ;

Maybe I should elaborate a bit on this.
Sometime ago, a student in my lab was comparing results from two different models at different resolutions, for exactly the same orbital configurations. He was surprised to discover that the solar energy inputs were significantly different (in particular at some high latitude locations), the difference between models being much larger than orbitally induced ones within each model. After some (hard) thinking, he found that this was simply due to resolution differences, in the way illustrated on Figure 3, since climate models are almost systematically using mid-grid point values instead of the actual integrals. "Assuming convergence checks" is sometimes assuming a lot. Thus my colloquial sentence: this is a pity.
I a revised version, I will avoid this expression and explain in more details why "it is a pity".

> Note that Eq. A2a -> A3 have not been thoroughly checked.

(I have double-checked them many times over several years… before deciding to publish this paper).

> One point also that I forgot to mention: other state-of-the art solutions now compete with La04 and I would encourage the authors to consider orbital solutions by the group of R. Zeebe. This is specially relevant for studies beyond the Pleistocene.

I will gladly add such additional astronomical solutions, both more state-of-the-art ones, but also historical ones like the ones used by Milankovitch.
Overall, I hope I will have some time in the future to expand this software package by providing not only new astronomical solutions, but also new routines. The inclusion of this library into the new "PyAnalySeries" package could be an incentive to do so.