Reply to M-F Loutre (RC1)

> *« The author presents and explains the different formula for the computation of the insolation, and other related values, that formed the basis for the a new version of AnalySeries, called "PyAnalySeries", to be released soon. As the author indicates some of the formulas presented in this paper are classical, while others are new. This paper is an important piece of work, to make all these formula available at one place. Many paleoscientists are using several of them (maybe even without knowing) and it will continue to be the case in the future. Therefore I consider that this paper is important and is worth to be published. Although the topic is difficult and technical, the author tried to be as pedagogical as possible. In that sense, the abstract clearly reflects the content of the paper. »*

I warmly thank M.F. Loutre for these encouraging remarks and for the very useful and detailed comments listed below.

> *« Here are some more specific comments.*
> *1. References*
> *a. As the author recognizes that some formulas are classical, it would be good to offer a reference for that, in particular for section 2.2 »*

Indeed, and this was also a point raised by Michel Crucifix.
I will certainly include the references pointed out by Michel concerning other available softwares, in particular :

- Crucifix, M. (2023). palinsol : a R package to compute Incoming Solar Radiation (insolation) for palaeoclimate studies (v1.0 (CRAN)). Zenodo. https://doi.org/10.5281/zenodo.14893715
- Oliveira, E. D.: Daily INSOLation (DINSOL-v1.0): an intuitive tool for classrooms and specifying solar radiation boundary conditions, Geosci. Model Dev., 16, 2371–2390, https://doi.org/10.5194/gmd-16-2371-2023, 2023.

and I will also look for other such softwares, though an exhaustive review of the subject would be quite a difficult endeavour.

Concerning section 2.2 (daily insolation), I will also explicitly refer to the classical references such as Milankovitch (1941) or Berger (1978).

> *« b. I also suggest to add a reference related to the use of the elliptical integral. I can suggest this one (Berger et al, 2010, https://doi.org/10.1016/j.quascirev.2010 .05.007) although there might be others as well. »*

Thank you for this reference. I will add it in the revised paper.

> *« 2. Line 159: T=1 year. A few words about what is one year (Gregorian year, sidereal year, tropical year, anomalistic year, ...), which one is chosen and why might be welcome. »*

It is the « anomalistic year », but a detailed explanation might be somewhat out of the scope of the paper. In a few words :
Since we are interested in astronomical phenomena, we are not talking about a « Gregorian year » nor about a « Julian year » which are both political approximations of the astronomical tropical year. We are interested only about the Earth's motion on its orbit (point mechanics), without considering its obliquity or precession (solid mechanics), therefore the year of interest is in fact also not the tropical year.

- In the context of Kepler's 2nd law (in order to translate true longitude into mean longitude, or the opposite), it is assumed (by Kepler) that the orbit is a perpetual ellipse. In other words, Kepler's equation neglects the secular changes associated with the precession of the perihelion (the difference between anomalistic and sidereal years, about 4'43" per year), and the same approximation holds true within the classical 2-body problem of celestial mechanics solved by Newton.
- In the context of modern celestial mechanics, the orbit is not a perfect ellipse and Kepler's equation is only an approximation. For instance, the Moon location affects significantly the time interval between two perihelions (from about 363 to about 368 days), and the same is true (to a lesser extent) for the time interval between two identical sidereal location.
- Still, it is useful to introduce long-term averaged trends using an hypothetical elliptic orbit that moves slowly through time : these are the so-called secular variations. In this context it is possible to define the difference between the anomalistic year (365 days 6 hours 13 minutes 53 seconds) and the sidereal year (365 days, 6 hours, 9 minutes and 10 seconds). It is also in this context that we can possibly apply Kepler's equation. This equation relates the mean anomaly with the eccentric anomaly ($M = E - e \sin E$) which are angles measured from the perihelion. It is therefore often simply stated in textbooks that, here, a « year » should be an anomalistic year, ie. a complete turn around the ellipse, from perihelion to perihelion. This explanation appears to me a bit short : Kepler's 2nd law (the basis of Kepler's equation) corresponds to the conservation of angular momentum, and such conservation laws are usually valid only « with respect to the stars » (Galilean frameworks). It seems therefore a bit strange and counter-intuitive to use the (moving) perihelion as a reference, instead of the stars. A deeper justification can only arise from perturbation theory : elliptic orbits can be described by eccentricity vectors (or a Laplace-Runge-Lenz vectors) which interact with each others in a many-body problem. Conservation laws (in a Galilean framework) lead to a slow rotation of these vectors (ie. the precession of the apsids) and angular momentum for each planet (orbit) is not conserved anymore. Still, it looks as if it where conserved, but only in a rotating framework, something which is tightly linked to the averaging description used. We can therefore still apply Kepler's 2nd law (and Kepler's equation) with the addition of the apsidal precession.
In a revised paper, I will try to summarize in just a few words such an explanation for why we need to use T = 1 anomalistic year.

> *« 3. Line 202. I do not fully agree with the author's definition of 'caloric insolation'. The caloric Summer half year is defined such that any day of the Summer half year receives more insolation than any day of the Winter half year. In particular, it means that in the tropics caloric half years may not be continuous »*

Marie-France Loutre is right, and this point was also mentionned by Michel Crucifix. The precise definition given by Milankovitch (1941 ; in §87 of his book) is indeed :
« ... we divide the year into two equally long and consequently real-half years, one of which comprises all those days of the year during which the irradiation of the latitude in question is stronger that on any day of the other half year... » .

It turns out that the computation of this quantity is difficult and Milankovitch used several approximations that are valid only at high latitudes (§88 of his book):

« … with the exception of a narrow belt around the equator… the beginning of the caloric summer half year is adjacent to the vernal equinox while the beginning of the caloric winter half year is adjacent to the automnal equinox ».
Using series expansions and assuming that the beginning of the caloric summer half year is very close to the vernal equinox (and similarly for its end), he finally treats both the beginning and the end in a strict symmetric way (see formulae (165) to (167) in §89). In other words, the assumptions and approximations used by Milankovitch correspond in fact to the computation performed by my software, which is the integrated irradiation over a half-year centered at the solstice.
But I fully agree with Marie-France Loutre and Michel Crucifix, that it does not correspond to Milankovitch's definition, though it corresponds rather closely to Milankovitch's computations. This will be clarified in the revised version of the paper.

« 4. Line 326. $\varphi_{Ext}(-\lambda,e,\varepsilon,-\varpi) = \varphi_{Ext}(\lambda,e,\varepsilon,\varpi)$. Isn't it also $\varphi_{Ext}(-\lambda,e,\varepsilon,-\varpi) = -\varphi_{Ext}(\lambda, e,\varepsilon,\varpi)$? »

Yes, thank you ! This will be corrected.

« 5. Figure 4.
a. The titles of the bottom part read 'insolation %'. Percentage of what? Is it a percentage of the solar constant? This should be explained. »

Yes, it is indeed in percent of the solar constant, since these situations do not correspond to actual planets (at least, not in our solar system). This will be explained in a revised version.

« *b.* Would it be possible to explain in plain language what are 'turning points'? »

I must admit that I am myself not quite satisfied with this denomination « turning points ». The plain language would be the « extrema of the $\varphi_{Ext}(\lambda)$ curve ». This is written in the figure legend (line 333) : « the 'turning points' at λ_M where $\varphi_{Ext}(\lambda_M)$ is extremal. But calling it an extrema is problematic, since φ_{Ext} is already the curve of extrema. The plain language should therefore state the « extrema of the curve of extrema φ_{Ext}, as a function of true longitude λ », which I find quite complicated and not so easy to understand. The vocabulary is even more problematic, since I need to introduce later on the « critical points » which are the « extrema » of the turning points themselves (the extrema of the extrema of the extrema !). Therefore I decided to avoid the word « extrema » and choose something else : turning points and critical points.
The mathematical definition is given line 342 and the points themselves are visible on Figure 4.
In a revised version, I will insist a bit more on the logic of all these points, which is to decide how many roots (if any) there is to the equation $\varphi_{Ext}(\lambda) = \varphi_0$. Indeed, to obtain easily and systematically all roots, we need first to find all the extrema of the function $\varphi_{Ext}(\lambda)$ and look for roots when φ_0 lies between two successive extrema of $\varphi_{Ext}(\lambda)$. Therefore the « turning points » $(\lambda_M, \varphi_{Ext}(\lambda_M))$. obtained by solving $d\varphi_{Ext}/d\lambda = 0$.

« 6. Line 450. Please provide here a full list of all the astronomical solutions that can be used in the library (or none of them). »

The list is actually given later on, lines 454 to 457, so there is no need to repeat it line 450.

« 7. Line 455. 'expressed in kyr AP'. Does that mean that time is expressed in thousands of year positive for the future and negative for the past? »

Yes, time is positive for the future (negative for the past). I will specify this explicitly in a revised version.

« 8. Line 460. '...'Berger 1978' solution... is a trigonometric approximation of some older astronomical computations'. Berger (1978) is based on Bretagnon (1974) astronomical solution, which is a trigonometric solution of simplified equations (first order of the Lagrange equations) »

Indeed, this is more correct. Thank you for the suggestion.

« 9. Line 465. Laskar's solutions are given in years (or thousand years) before/after 2000A.D. while Berger (1978) is given in years (or thousand years) before/after 1950A.D. Does this affect the computation ? »

There is no correction applied to the time scale in my software. In fact, the « astro.py » library does not perform any computation at all beyond what is provided in the original publications (it is just a uniform interface). It is therefore the user's responsability to know what time scale is used as an input.
But it is indeed important that the user is aware of the problem and I will add more information on that point in the revised version.

« 10. Lines 512 and 533. At line 512, refL is the true longitude of the reference point, while on line 533 refL is the mean longitude of the reference point. It is not so clear in the text. »

refL is always the true longitude used as the zero for « mean longitudes ». In other words, mean longitude (time) is computed from this point (usually = 0 for the march equinox).
I understand that this might not be clear, especially at line 533 , and I will clarify this point in the revised version.
Note that it is sometimes useful to compute mean longitude (time) from another reference point, as illustrated in the next comment :

« 11. Line 535. Why is np.pi/2 used here? Isn't it the reference longitude? In that case why isn't it 0? »

The idea is to compute the integrated insolation over half a year centered at the june solstice. It is therefore convenient to use « june solstice » (true longitude = pi/2) as the reference longitude, and then compute mean longitudes from this point (between -pi/2 and +pi/2) to get half a year.

« 12. Line 561. 'eps' should probably be 'obl'. »

Yes. This will be corrected.