https://doi.org/10.5194/egusphere-2025-2874
Preprint. Discussion started: 14 July 2025

Collisional Transport Coefficients In Kappa-Distributed Plasmas

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Abstract

In this paper, we present a set of transport equations (continuity, momentum, and energy) using the Kappa velocity distribution as our zeroth-order function within the framework of the five-moment approximation. Then, we derive the corresponding transport momentum and energy coefficients using the Boltzmann collision integral. The results are expressed in terms of hypergeometric functions. These calculations have been done for three types of collisions: Coulomb collisions, hard-sphere interactions, and Maxwell molecules collisions. Furthermore, we explore the transport coefficients in two special cases: (1) the limiting case when kappa index approaches infinity, where the results converge to those of the Maxwellian distribution, and (2) the case of a non-drifting Kappa distribution. Finally, we discuss the behaviour of the transport coefficients in the case of Coulomb collisions for the Kappa distribution and compare it with the result of Maxwellian distribution.

keywords Transport equations, Five-moment approximation, Transport coefficients, Kappa distribution, Coulomb collisions.

1 Introduction

The transport equations based on the Maxwellian velocity distribution function were first derived by Tanenbaum [1967] and Burgers [1969], with a subsequent review by Schunk [1977]. These studies also obtained the transport coefficients (i.e. the collision integrals) using the Boltzmann collision integral approach and expressed them in terms of Chapman-Cowling collision integrals Chapman and Cowling [1970]. These coefficients are valid for arbitrary temperature differences between the interacting gases but are restricted to small drift velocities. In Jubeh and Barghouthi [2018], the transport coefficients were calculated, for the five-moment approximation, without restriction on the drift velocity and presented in closed form using hypergeometric functions. The transport equations and the transport coefficients for a bi-Maxwellian velocity distribution function have also been derived in several studies Chew et al. [1956], Demars and Schunk [1979], Hellinger and Trávníček [2009], and Jubeh and Barghouthi [2017] in order to develop a more accurate model. Since, in space plasmas, the particle velocity distribution functions often deviate from the standard Maxwellian distribution, exhibiting non thermal suprathermal tails that decreases following a power law in terms of velocity. Such distributions are well-fitted by the so-called Kappa velocity distribution function, or generalized Lorentzian velocity distribution functions. The Kappa distribution, is a power-law distribution with long tails describing highly energetic particles, appears in a wide variety of settings, from low-density to high-density plasmas. It is commonly found at high altitudes in the solar wind and in various locations, including the terrestrial magnetosphere, radiation belts, and the magnetospheres of planets like Jupiter, Saturn, Uranus, and Neptune, as well as in other space plasma environments like Titan and the Io plasma torus. The prevalence of Kappa distributions in these regions has been extensively corroborated by Observations and satellite data from WIND Collier et al. [1996], Ulysses Meyer-Vernet et al. [1995], Cassini Steffl et al. [2004], and the Hubble Space Telescope Retherford et al.

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[2003]. For a detailed discussion of the Kappa distribution, see Pierrard and Lazar [2010], Livadiotis [2018] and Davis et al. [2023]. Due to the frequent occurrence of the Kappa distribution in many space plasma environments, it's significantly important to see how the transport equations and the transport coefficients will be affected when we expand the velocity distribution function around the Kappa distribution. To address this, the study starts in Section 2 with a discussion of the theoretical formulation of Boltzmann's equation and the relevant collision terms, i.e., the Boltzmann collision integral. Section 3 presents the five-moment approximation of the transport equations. In Section 4, we derive the five-moment approximation for the Kappa velocity distribution function. Then, in Section 5, we calculate the transport coefficients for drifting Kappa plasmas, applicable for arbitrary drift velocity differences as well as temperature differences between the interacting plasma species. Next, we express the resulting coefficients in a hypergeometric representation. Section 6 investigates two special cases; the first, where the kappa index approaches infinity; the second, for the non-drifting Kappa distribution. These calculations cover three types of interactions: Coulomb collisions, hard-sphere interactions, and Maxwell molecules collisions. Finally, in Section 7, we study the behaviour of the transport coefficients in the case of Coulomb collisions and see how collisions affect both the momentum and the energy of the colliding particles, as well as how the effect differs for both Maxwellian and Kappa distributions.

2 Boltzmann Equation

In dealing with plasma, we describe each species in the plasma by a separate velocity distribution function $f_s(\mathbf{r}, \mathbf{v}_s, t)$, defined such that $f_s(\mathbf{r}, \mathbf{v}_s, t)\, d\mathbf{v}_s d\mathbf{r}$ represents the number of particles of specie s, which at time t have velocity between $\mathbf{v}_s$ and $\mathbf{v}_s + d\mathbf{v}_s$ and positions between $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$. The evolution in time of the species velocity distribution function is determined by the net effect of collisions and the flow in phase space of species under the effect of external forces. The mathematical description of this evolution is given by Boltzmann equation, Schunk [1977],

$$\frac{\partial f_s}{\partial t} + \mathbf{v}_s \cdot \nabla f_s + \mathbf{a}_s \cdot \nabla_{v_s} f_s = \frac{\delta f_s}{\delta t}. \tag{1}$$

Here, ∇ represents the gradient in coordinate space, ∇_{v_s} is the gradient in velocity space, and $\mathbf{a}_s$ denotes the particle acceleration due to external forces. In most plasma applications, the main external forces acting on the charged particles are gravitational and Lorentz forces . With allowance for these forces, the acceleration becomes

$$\mathbf{a}_s = \mathbf{G} + \frac{e_s}{m_s}\left(\mathbf{E} + \frac{\mathbf{v}_s \times \mathbf{B}}{c}\right), \tag{2}$$

where $\mathbf{G}$ is the acceleration due to gravity, e_s and m_s are the charge and mass of species s respectively, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and c is the speed of light. The quantity in the right side of the Boltzmann equation, $(\delta f_s/\delta t)$, represents the rate of change of f_s in a given region of phase space as a result of collisions, and its form depends on the type of collision process.

Boltzmann Collision Integral

The appropriate expression for $(\delta f_s/\delta t)$ in case of binary elastic collisions between particles (collisions governed by inverse power potentials, and resonant charge exchange collisions) is the Boltzmann collision integral, given by, Schunk [1977], Schunk and Nagy [2009]

$$\frac{\delta f_s}{\delta t} = \sum_t \int_{\mathbb{R}^3 \times \Omega} \left[f_s' f_t' - f_s f_t\right] \mathrm{g}_{st}\, \sigma_{st}(\mathrm{g}_{st}, \theta)\; d\Omega\, d\mathbf{v}_t, \tag{3}$$

where $d\mathbf{v}_t$ is the velocity space volume element for the target species t, $\mathbf{g}_{st}$ is the relative velocity of the colliding particles s and t, written as

$$\mathbf{g}_{st} = \mathbf{v}_s - \mathbf{v}_t, \qquad \text{and} \qquad \mathrm{g}_{st} = |\mathbf{v}_s - \mathbf{v}_t|, \tag{4}$$

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$d\Omega$ is the element of solid angle in the s particle reference frame, θ is the scattering angle, $\sigma_{st}(\mathrm{g}_{st}, \theta)$ is the differential scattering cross section, defined as the number of particles scattered per solid angle $d\Omega$, per unit time, divided by the incident intensity, and the primes denote quantities evaluated after the collision.

$$f'_s f'_t = f'_s(\mathbf{r}, \mathbf{v}'_s, t)\, f'_t(\mathbf{r}, \mathbf{v}'_t, t)$$

Moments Of The Velocity Distribution Function

If $N_s(\mathbf{r}, \mathbf{v}_s, t)$ represents the number of species s at a given point in phase space $(\mathbf{r}, \mathbf{v}_s, t)$, then, by the definition of the velocity distribution function, we can write, Bittencourt [2004],

$$dN_s = f_s\, d\mathbf{v}_s\, d\mathbf{r}. \tag{5}$$

Dividing both sides of the equation (5) by the volume element $d\mathbf{r}$, and then integrate over the velocity space, gives the zeroth-order moment,

$$n_s(\mathbf{r}, t) = \int_{\mathbb{R}^3} f_s\, d\mathbf{v}_s, \tag{6}$$

which represents the *number density* of particles. Hence, the number density n_s is the number of particles per unit volume. From relation in equation (6), it is clear that the term f_s/n_s represents a normalized probability density function. As a result, the average value of $\xi_s(\mathbf{v}_s)$ at any position $\mathbf{r}$ and time t is given by

$$\langle \xi_s \rangle = \frac{1}{n_s} \int_{\mathbb{R}^3} \xi_s\, f_s\, d\mathbf{v}_s. \tag{7}$$

The first-order moment is the average value of $\mathbf{v}_s$, and it represents the *drift velocity* of a species s,

$$\mathbf{u}_s(\mathbf{r}, t) = \langle \mathbf{v}_s \rangle.$$

For higher moments, it is more convenient to evaluate them with respect to the drift velocity $\mathbf{u}_s$, so we introduce the *random velocity*, Grad [1949], as follows

$$\mathbf{c}_s(\mathbf{r}, \mathbf{v}_s, t) = \mathbf{v}_s - \mathbf{u}_s, \tag{8}$$

clearly,

$$\langle \mathbf{c}_s \rangle = 0. \tag{9}$$

Then, the physically significant velocity moments of the species distribution function are

$$\text{Temperature}: \qquad T_s(\mathbf{r}, t) = \frac{m_s}{3k_B} \left\langle c_s^2 \right\rangle, \tag{10}$$

$$\text{Pressure tensor}: \qquad \mathbf{P}_s(\mathbf{r}, t) = n_s m_s \left\langle \mathbf{c}_s \mathbf{c}_s \right\rangle, \tag{11}$$

$$\text{Stress tensor}: \qquad \boldsymbol{\tau}_s(\mathbf{r}, t) = \mathbf{P}_s - p_s\, \mathbf{I}, \tag{12}$$

$$\text{Heat flow vector}: \qquad \mathbf{q}_s(\mathbf{r}, t) = \frac{n_s m_s}{2} \left\langle c_s^2\, \mathbf{c}_s \right\rangle, \tag{13}$$

$$\text{Heat flow tensor}: \qquad \mathbf{Q}_s(\mathbf{r}, t) = n_s m_s \left\langle \mathbf{c}_s \mathbf{c}_s \mathbf{c}_s \right\rangle. \tag{14}$$

$$\text{Higher-order pressure tensor}: \qquad \boldsymbol{\mu}_s(\mathbf{r}, t) = \frac{n_s m_s}{2} \left\langle c_s^2\, \mathbf{c}_s \mathbf{c}_s \right\rangle, \tag{15}$$

where k_B is the Boltzmann constant, $\mathbf{I}$ is a unit dyadic, p_s is the partial pressure and defined as

$$p_s = n_s k_B\, T_s. \tag{16}$$

The average value, defined in equation (7), can be written using the definition of the random velocity,

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from which the relation $d\mathbf{c}_s = d\mathbf{v}_s$ follows, as

$$\langle \xi_s(\mathbf{c}_s) \rangle = \frac{1}{n_s} \int \xi_s(\mathbf{c}_s)\, f_s(\mathbf{r}, \mathbf{c}_s, t)\, d\mathbf{c}_s. \tag{17}$$

3 The Five - Moment Approximation

The transport equations describe the spatial and temporal evolution of the physically significant velocity moments, mentioned in the previous section. These equations can be obtained by multiplying the Boltzmann equation by an appropriate function of velocity and then integrating over the velocity space. If we multiply equation (1) by the factors 1, $m_s\mathbf{c}_s$, $\frac{1}{2}m_s c_s^2$, $m_s\mathbf{c}_s\mathbf{c}_s$, and $\frac{1}{2}m_s c_s^2\mathbf{c}_s$, and then integrate over the velocity space, we obtain the continuity, momentum, energy, pressure tensor, and heat flow equations, respectively. These are the general transport equations for species s, as given in Schunk [1977], Schunk and Nagy [2009], and Bittencourt [2004]. The general transport equations do not constitute a closed system because the equation governing the moment of order l contains the moment of order $l+1$. That is, the continuity equation describes the evolution of the density, but it also contains the drift velocity, and similar dependencies occur in the higher moment equations. In order to close the system, it is necessary to adopt an approximate expression for f_s. A common mathematical technique can be used to do that, is expanding f_s in a complete orthogonal series of the form, Grad [1949] and Mintzer [1965],

$$f_s(\mathbf{r}, \mathbf{c}_s, t) = f_s^{(0)}(\mathbf{r}, \mathbf{c}_s, t) \sum_p a_p(\mathbf{r}, t) M_p(\mathbf{r}, \mathbf{c}_s, t), \tag{18}$$

where $f_s^{(0)}$ is an appropriate zeroth-order velocity distribution function, M_p represents a complete set of orthogonal polynomials, a_p represents the unknown expansion coefficients, and the subscript p is used to indicate that the summation is generally over more than one coordinate index. The zeroth-order distribution function and the orthogonal set of polynomials are generally chosen so that the series converges rapidly, meaning that only a few terms in the series expansion are needed to describe the distribution function. Different levels of approximation are possible depending on the number of terms retained in the series expansion. The first term in the series expansion in equation (18) is 1, regardless of the choice of the zeroth-order distribution function $f_s^{(0)}$ Mintzer [1965]. Therefore, assuming the species distribution function f_s is represented by the first term of the expansion, we have

$$f_s = f_s^{(0)}. \tag{19}$$

This approximation reduces the general system of transport equations to just the continuity, momentum, and energy equations for each species,

$$\frac{\delta n_s}{\delta t} = \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s), \tag{20}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = n_s m_s \frac{\mathrm{D}_s \mathbf{u}_s}{\mathrm{D}\, t} + \nabla \cdot \mathbf{P}_s - n_s m_s \mathbf{G} - n_s e_s \left(\mathbf{E} + \frac{\mathbf{u}_s \times \mathbf{B}}{c} \right), \tag{21}$$

$$\frac{\delta E_s}{\delta t} = \frac{\mathrm{D}_s}{\mathrm{D}\, t} \left(\frac{3}{2} p_s \right) + \frac{3}{2} p_s (\nabla \cdot \mathbf{u}_s) + \nabla \cdot \mathbf{q}_s + \mathbf{P}_s : \nabla \mathbf{u}_s, \tag{22}$$

where the operation $\mathbf{P}_s : \nabla \mathbf{u}_s$, corresponds to the double product of the two tensors $\mathbf{P}_s$, and $\nabla \mathbf{u}_s$, and

$$\frac{\mathrm{D}_s}{\mathrm{D}\, t} = \frac{\partial}{\partial t} + \mathbf{u}_s \cdot \nabla \,. \tag{23}$$

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The set of equations (20−22) were initially derived with no assumption about the $f_s^{(0)}$, Tanenbaum [1967]. If the choice of zeroth-order distribution function $f_s^{(0)}$ satisfies

$$\mathbf{q}_s = \boldsymbol{\tau}_s = 0, \tag{24}$$

as in the drifting Maxwellian distribution, we can write equations (20−22) as, Schunk [1977],

$$\frac{\delta n_s}{\delta t} = \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \mathbf{u}_s) \tag{25}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = n_s m_s \frac{\mathrm{D}_s \mathbf{u}_s}{\mathrm{D}\, t} + \nabla p_s - n_s m_s \mathbf{G} - n_s e_s \left(\mathbf{E} + \frac{\mathbf{u}_s \times \mathbf{B}}{c} \right) \tag{26}$$

$$\frac{\delta E_s}{\delta t} = \frac{\mathrm{D}_s}{\mathrm{D}\, t} \left(\frac{3}{2} p_s \right) + \frac{5}{2} p_s \left(\nabla \cdot \mathbf{u}_s \right) \tag{27}$$

These equations are known as the five-moment approximation of the transport equations because each species is characterized by five parameters: density, three components of drift velocity, and temperature. At this level of approximation, stress, heat flow, and all higher-order velocity moments are neglected, and the properties of the species are expressed in terms of just the species density, drift velocity, and temperature. The terms on the left-hand side of equations (25−27) are called the collision integrals, or, as we will refer to, the transport coefficients. These coefficients are the moments of the Boltzmann collision integral and describe the rate of change of density, momentum, and energy due to collisions. They are defined as follows:

$$\frac{\delta n_s}{\delta t} = \int_{\mathbb{R}^3} \frac{\delta f_s}{\delta t}\, d\mathbf{c}_s, \tag{28}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = m_s \int_{\mathbb{R}^3} \mathbf{c}_s \frac{\delta f_s}{\delta t}\, d\mathbf{c}_s, \tag{29}$$

$$\frac{\delta E_s}{\delta t} = \frac{m_s}{2} \int_{\mathbb{R}^3} c_s^2 \frac{\delta f_s}{\delta t}\, d\mathbf{c}_s, \tag{30}$$

where the Boltzmann's collision integral, expressed in terms of the random velocities $\mathbf{c}_s$ and $\mathbf{c}_t$, and takes the form:

$$\frac{\delta f_s}{\delta t} = \sum_t \int_{\mathbb{R}^3 \times \Omega} [f_s' f_t' - f_s f_t]\; \mathrm{g}_{st}\, \sigma_{st}(\mathrm{g}_{st}, \theta)\; d\Omega\, d\mathbf{c}_t, \tag{31}$$

with the functions f_s', f_t', f_s, and f_t depends on $(\mathbf{r}, \mathbf{c}_s \text{ or } \mathbf{c}_t, t)$.

4 The Five-Moment Approximation For Drifting Kappa Distribution

In 3-dimensions, the drifting Kappa distribution function of particle velocities is commonly written in the form, Livadiotis [2018], Davis et al. [2023],

$$f_\alpha^{\kappa_\alpha}(\mathbf{r}, \mathbf{c}_\alpha, t) = \frac{n_\alpha \eta(\kappa_\alpha)}{\pi^{3/2}\, w_\alpha^3} \left(1 + \frac{c_\alpha^2}{\kappa_{0_\alpha}\, w_\alpha^2} \right)^{-\kappa_\alpha - 1}, \tag{32}$$

where α denotes the particle type, $\mathbf{c}_\alpha$ is the random velocity of species α, and it is defined in terms of the particle drift velocity $\mathbf{u}_\alpha$, as

$$\mathbf{c}_\alpha = \mathbf{v}_\alpha - \mathbf{u}_\alpha, \tag{33}$$

the thermal velocity of particle α, w_α, is given by

$$w_\alpha = \sqrt{\frac{2 k_B T_\alpha}{m_\alpha}}, \tag{34}$$

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m_α and T_α are the mass and the absolute temperature of particle α, respectively, k_B is the Boltzmann constant, $\eta(\kappa_\alpha)$ is a function of kappa index κ_α, defined as

$$\eta(\kappa_\alpha) = \kappa_{0\alpha}^{-3/2} \frac{\Gamma(\kappa_\alpha + 1)}{\Gamma(\kappa_\alpha - 1/2)}, \quad \kappa_{0_\alpha} = \kappa_\alpha - \frac{3}{2}, \tag{35}$$

κ_{0_α} is the invariant kappa index and κ_α is a shape parameter that controls the power-law tails, sometimes referred to as the spectral index, with the condition $\kappa_\alpha > 3/2$. This condition prevents the Kappa distribution function in equation (32) from collapsing, as the equivalent temperature would not be defined Pierrard and Lazar [2010]. Setting the zeroth-order distribution function $f_s^{(0)}$ to be a drifting Kappa distribution, it is expressed as

$$f_s^{(0)} = f_s^{\kappa_\alpha} = \frac{n_s \eta(\kappa_\alpha)}{\pi^{3/2}\, w_s^3} \left(1 + \frac{c_s^2}{\kappa_{0_\alpha}\, w_s^2}\right)^{-\kappa_\alpha - 1}, \tag{36}$$

leads to the five-moment approximation described in equations (25–27), provided the condition in equation (24) is satisfied. Since if we substitute the following average values of the Kappa distribution, see Appendix A for details,

$$\langle \mathbf{c}_s \mathbf{c}_s \rangle = \frac{k_B T_s}{m_s} \mathbf{I}, \qquad \langle c_s^2\, \mathbf{c}_s \rangle = 0. \tag{37}$$

into equations (11), (12), and (13) yields to

$$\mathbf{P}_s = n_s m_s \langle \mathbf{c}_s \mathbf{c}_s \rangle = n_s k_B T_s\, \mathbf{I} \tag{38}$$

$$\boldsymbol{\tau}_s = \mathbf{P}_s - p_s \mathbf{I} = 0, \tag{39}$$

$$\mathbf{q}_s = \frac{n_s m_s}{2} \langle c_s^2\, \mathbf{c}_s \rangle = 0. \tag{40}$$

5 Transport Coefficients

Calculating the transport coefficients involves solving integrals known as the transfer collision integral, which take the form

$$\frac{\delta \xi_s}{\delta t} = \int_{\mathbb{R}^3} \xi_s \frac{\delta f_s}{\delta t}\, d\mathbf{c}_s, \tag{41}$$

where ξ_s represents a general velocity moment. Specifically, for $\xi_s = 1$, $m_s \mathbf{c}_s$, and $\frac{1}{2} m_s c_s^2$, the integral corresponds to the expressions in equations (28−30). We are interested in collisions described by the Boltzmann collision integral mentioned in equation (31). Therefore, the transfer collision integral is

$$\frac{\delta \xi_s}{\delta t} = \sum_t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times \Omega} \xi_s \left[f_s' f_t' - f_s f_t\right] \mathrm{g}_{st}\, \sigma_{st}(\mathrm{g}_{st}, \theta)\; d\Omega\, d\mathbf{c}_t\, d\mathbf{c}_s. \tag{42}$$

Rewriting equation (42) in an equivalent form

$$\frac{\delta \xi_s}{\delta t} = \sum_t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3 \times \Omega} f_s f_t \left[\xi_s' - \xi_s\right] \mathrm{g}_{st}\, \sigma_{st}(\mathrm{g}_{st}, \theta)\; d\Omega\, d\mathbf{c}_t\, d\mathbf{c}_s, \tag{43}$$

that does not require the distribution functions after the collision, $f_s' f_t'$, which can be reorganized to obtain

$$\frac{\delta \xi_s}{\delta t} = \sum_t \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_s f_t\, \mathrm{g}_{st} \left(\int_\Omega \left[\xi_s' - \xi_s\right] \sigma_{st}(\mathrm{g}_{st}, \theta)\; d\Omega \right) d\mathbf{c}_t\, d\mathbf{c}_s. \tag{44}$$

Since we are interested in the transport coefficients for density, momentum, and energy, we obtain the following by substituting the corresponding value of ξ_s into equation (44), and using the momentum transfer cross-section integral,

$$Q_{st}^{(1)}(\mathrm{g}_{st}) = \int_\Omega (1 - \cos\theta)\; \sigma_{st}(\mathrm{g}_{st}, \theta)\; d\Omega, \tag{45}$$

https://doi.org/10.5194/egusphere-2025-2874
Preprint. Discussion started: 14 July 2025

we can write the transport coefficients as, Schunk and Nagy [2009],

$$\frac{\delta n_s}{\delta t} = 0, \tag{46}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = -\sum_t m_{st} \int_{\mathbb{R}^3\times\mathbb{R}^3} f_s f_t \,\mathrm{g}_{st}\, Q_{st}^{(1)}\, \mathbf{g}_{st}\, d\mathbf{c}_t\, d\mathbf{c}_s, \tag{47}$$

$$\frac{\delta E_s}{\delta t} = -\sum_t m_{st} \int_{\mathbb{R}^3\times\mathbb{R}^3} f_s f_t \,\mathrm{g}_{st}\, Q_{st}^{(1)} \left(\hat{\mathbf{V}}_c \cdot \mathbf{g}_{st}\right) d\mathbf{c}_t\, d\mathbf{c}_s, \tag{48}$$

where the dot product is written as

$$\hat{\mathbf{V}}_c \cdot \mathbf{g}_{st} = \frac{m_s \mathbf{c}_s \cdot \mathbf{g}_{st} + m_t \mathbf{c}_t \cdot \mathbf{g}_{st} + m_t\, \Delta\mathbf{u} \cdot \mathbf{g}_{st}}{m_s + m_t}, \tag{49}$$

and the reduced mass m_{st} defined as:

$$m_{st} = \frac{m_s m_t}{m_s + m_t}. \tag{50}$$

Equations (47) and (48) are expressed in terms of the momentum transfer cross-section integral $Q_{st}^{(1)}$, which is considered in three cases: Coulomb collisions, hard-sphere interactions, and Maxwell molecule collisions.

Coulomb Collision

The momentum transfer cross section for Coulomb collisions is given by

$$Q_{st}^{(1)}\left(\mathrm{g}_{st}\right) = \frac{\mathrm{Q}_{\mathrm{Co}}}{\mathrm{g}_{st}^4}, \tag{51}$$

where

$$\mathrm{Q}_{\mathrm{Co}} = 4\pi \left(\frac{e_s e_t}{4\pi\varepsilon_0 m_{st}}\right)^2 \ln\Lambda. \tag{52}$$

Here, e_s and e_t are the charges of species s and t, respectively, ε_0 is the permittivity of free space, and $\ln\Lambda$ is the Coulomb logarithm.

Hard-Sphere Interactions

For hard-sphere interactions, the momentum transfer cross section,

$$Q_{st}^{(1)} = \mathrm{Q}_{\mathrm{HS}} = \pi\,\sigma^2, \tag{53}$$

is a constant (σ is the sum of the radii of the colliding particles).

Maxwell Molecule Collisions

In the case of Maxwell molecule collisions, the product of the relative velocity and the momentum transfer cross-section,

$$\mathrm{g}_{st} Q_{st}^{(1)} = \mathrm{Q}_{\mathrm{MC}}, \tag{54}$$

is constant and can be removed from the integrals in equations (46–48).

Transport Coefficients For Drifting Kappa Distribution

Up to this point, the evaluation of the transport coefficients was generalized. Now, we will calculate the transport coefficients for the five-moment approximation, under the assumption that the velocity distribution function is a drifting Kappa distribution for both interacting particles. Evaluating the transport coefficients for each type of collision for the drifting Kappa distribution requires rewriting the distribution in an integral representation. So, rather than writing the distributions in the form

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of equation (32), they can be written as

$$f_s^{\kappa_s} = \frac{n_s \eta(\kappa_s)}{\pi^{3/2}\, w_s^3} \frac{1}{\Gamma(\kappa_s+1)} \int_0^\infty \xi_1^{\kappa_s}\, e^{-\xi_1} \exp\left[-\frac{\xi_1\, c_s^2}{\kappa_{0_s}\, w_s^2}\right] d\xi_1, \tag{55}$$

$$f_t^{\kappa_t} = \frac{n_t \eta(\kappa_t)}{\pi^{3/2}\, w_t^3} \frac{1}{\Gamma(\kappa_t+1)} \int_0^\infty \xi_2^{\kappa_t}\, e^{-\xi_2} \exp\left[-\frac{\xi_2\, c_t^2}{\kappa_{0_t}\, w_t^2}\right] d\xi_2, \tag{56}$$

Coulomb Collision

Using equation (51), The transport coefficients for Coulomb collisions, are

$$\frac{\delta n_s}{\delta t} = 0, \tag{57}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = -\sum_t m_{st}\, \mathrm{Q_{Co}} \int_{\mathbb{R}^3\times\mathbb{R}^3} f_s^{\kappa_s} f_t^{\kappa_t} \frac{\mathbf{g}_{st}}{\mathrm{g}_{st}^3}\, d\mathbf{c}_t\, d\mathbf{c}_s, \tag{58}$$

$$\frac{\delta E_s}{\delta t} = -\sum_t m_{st}\, \mathrm{Q_{Co}} \int_{\mathbb{R}^3\times\mathbb{R}^3} f_s^{\kappa_s} f_t^{\kappa_t} \frac{(\hat{\mathbf{V}}_c \cdot \mathbf{g}_{st})}{\mathrm{g}_{st}^3}\, d\mathbf{c}_t\, d\mathbf{c}_s. \tag{59}$$

The Momentum Coefficient

The momentum coefficient for Coulomb collision is given in equation (58). Substituting the distributions given in equations (55) and (56) into the integral

$$I_M = \int_{\mathbb{R}^3\times\mathbb{R}^3} f_s^{\kappa_s} f_t^{\kappa_t} \frac{\mathbf{g}_{st}}{\mathrm{g}_{st}^3}\, d\mathbf{c}_t\, d\mathbf{c}_s, \tag{60}$$

gives

$$\begin{aligned} I_M =& \frac{n_s n_t}{\pi^3\, w_s^3\, w_t^3} \frac{\eta(\kappa_s)\eta(\kappa_t)}{\Gamma(\kappa_s+1)\Gamma(\kappa_t+1)} \int_0^\infty \int_0^\infty \xi_1^{\kappa_s}\, \xi_2^{\kappa_t}\, e^{-\xi_1-\xi_2} \\ & \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \exp\left[-\frac{\xi_1\, c_s^2}{\kappa_{0_s}\, w_s^2} - \frac{\xi_2\, c_t^2}{\kappa_{0_t}\, w_t^2}\right] \frac{\mathbf{g}_{st}}{\mathrm{g}_{st}^3}\, d\mathbf{c}_s d\mathbf{c}_t\right) d\xi_1\, d\xi_2. \end{aligned} \tag{61}$$

To solve the integral in equation (61) we will introduce the following transformation, let

$$\mathbf{c}_s = \left(\frac{\kappa_{0_s}}{\xi_1}\right)^{1/2} (\mathbf{c}_* - \mathrm{A}\, \mathbf{g}_*)\,, \qquad \mathbf{c}_t = \left(\frac{\kappa_{0_t}}{\xi_2}\right)^{1/2} (\mathbf{c}_* + \mathrm{B}\, \mathbf{g}_*)\,. \tag{62}$$

The constant A and B are defined as:

$$\mathrm{A} = \frac{m_t T_s}{m_t T_s + m_s T_t}, \qquad \mathrm{B} = \frac{m_s T_t}{m_t T_s + m_s T_t}. \tag{63}$$

The variables $\mathbf{c}_*$ and $\mathbf{g}_*$ are expressed as:

$$\mathbf{c}_* = \mathbf{V}_c - \mathbf{u}_c + \xi\, \Delta\mathbf{u} + \xi\, \mathbf{g}_{st}, \tag{64}$$

$$\mathbf{g}_* = -\mathbf{g}_{st} - \Delta\mathbf{u}, \tag{65}$$

with

$$\mathbf{u}_c = \frac{m_s \mathbf{u}_s + m_t \mathbf{u}_t}{m_s + m_t}, \qquad \xi = \frac{m_{st}}{m_s + m_t} \frac{T_t - T_s}{T_{st}}. \tag{66}$$

The reduced temperature T_{st} are defined as:

$$T_{st} = \frac{m_s T_t + m_t T_s}{m_s + m_t}, \tag{67}$$

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and the difference between the drift velocities of s and t particles is

$$\Delta\mathbf{u} = \mathbf{u}_t - \mathbf{u}_s. \tag{68}$$

By calculating the determinant of the Jacobian matrix $\mathbf{J}$ for the transformation described in equations (62) and (63), we have

$$d\mathbf{c}_s d\mathbf{c}_t = \det(\mathbf{J})\, d\mathbf{c}_* d\mathbf{g}_* = \left[\frac{\kappa_{0_s}\kappa_{0_t}}{\xi_1\xi_2}\right]^{1/2} d\mathbf{c}_* d\mathbf{g}_*. \tag{69}$$

Applying the transformation make the integrals in equation (61) independent of each other, so by evaluating the integrals on $\mathbf{c}_*$, ξ_1 and ξ_2 , and rewriting the $\mathbf{g}_*$ integral, gives

$$I_M = \frac{n_s n_t}{\pi^{3/2}\, b^3} \mathrm{D}(\kappa_s,\kappa_t) \int_{\mathbb{R}^3} e^{-|\mathbf{g}_{st}+\Delta\mathbf{u}|^2/b^2}\ \frac{\mathbf{g}_{st}}{\mathrm{g}_{st}^3}\, d\mathbf{g}_{st}. \tag{70}$$

where

$$\mathrm{D}(\kappa_s,\kappa_t) = \frac{(\kappa_s - 1/2)}{(\kappa_s - 3/2)}\frac{(\kappa_t - 1/2)}{(\kappa_t - 3/2)} \tag{71}$$

and

$$b = \left(2k_B\,\frac{T_{st}}{m_{st}}\right)^{1/2}. \tag{72}$$

Using the result of the integral I_1^n for $n = -3$ (see Appendix B for details), we have

$$I_M = -\frac{4}{3}\frac{n_s n_t}{\pi^{1/2}\, b^3} \mathrm{D}(\kappa_s,\kappa_t)\, \Delta\mathbf{u}\ \Phi_{\mathrm{Co}}(\varepsilon_{st}). \tag{73}$$

Substituting equation (73) into equation (58), we can write the momentum coefficient for Coulomb collision as

$$\frac{\delta\mathbf{M}_s}{\delta t} = \sum_t n_s m_s \nu_{st}^{\mathrm{Co}}\, \mathrm{D}(\kappa_s,\kappa_t)\, \Delta\mathbf{u}\ \Phi_{\mathrm{Co}}(\varepsilon_{st}), \tag{74}$$

where Φ_{Co} is defined as in equation (153), Appendix B, and

$$\nu_{st}^{\mathrm{Co}} = \frac{4}{3}\frac{m_t}{m_s+m_t}\frac{n_t}{\pi^{1/2}\, b^3}\mathrm{Q}_{\mathrm{Co}}, \qquad \varepsilon_{st} = \frac{|\Delta\mathbf{u}|}{b}. \tag{75}$$

The Energy Coefficient

The energy coefficient for Coulomb collision is given in equation (59). Substitute the dot product from equation (49) into the integral

$$I_E = \int_{\mathbb{R}^3\times\mathbb{R}^3} f_s^{\kappa_s} f_t^{\kappa_t}\, \frac{(\hat{\mathbf{V}}_c \cdot \mathbf{g}_{st})}{\mathrm{g}_{st}^3}\, d\mathbf{c}_t\, d\mathbf{c}_s, \tag{76}$$

leads to three integrals, which can be evaluated by following the same steps as the integral in equation (60), where we substitute the distributions given in equations (55) and (56) into the integrals, and apply the transformation mentioned in equations (62) and (63). This results in two integrals that depend on $\mathbf{g}_{st}$. Using the results of the integrals I_2^n and I_3^n at $n = -3$, as given in Appendix B, and combining all three integrals, we obtain:

$$I_E = -\frac{1}{m_s+m_t}\frac{4}{3}\frac{n_s n_t}{\pi^{1/2}\, b^3}\left[3k_B\,\Delta\mathrm{T}\ \mathrm{W}(\kappa_s,\kappa_t)\,\Psi_{\mathrm{Co}}(\varepsilon_{st}) + m_t\,\mathrm{D}(\kappa_s,\kappa_t)\,|\Delta\mathbf{u}|^2\,\Phi_{\mathrm{Co}}(\varepsilon_{st})\right], \tag{77}$$

where $\Delta\mathrm{T}$ is the difference between the drift velocities of s and t particles,

$$\Delta\mathrm{T} = T_t - T_s, \tag{78}$$

and $\mathrm{W}(\kappa_s,\kappa_t)$ defined as

$$\mathrm{W}(\kappa_s,\kappa_t) = \frac{1}{(\kappa_s - 3/2)^{1/2}}\frac{\Gamma(\kappa_s)}{\Gamma(\kappa_s - 1/2)}\frac{(\kappa_t - 1/2)}{(\kappa_t - 3/2)}. \tag{79}$$

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Substituting equation (77) in equation (59), we can write the energy coefficient for Coulomb collision, as

$$\frac{\delta E_s}{\delta t} = \sum_t \frac{n_s m_s \nu_{st}^{\mathrm{Co}}}{m_s + m_t} \left[3k_B \, \Delta \mathrm{T} \; \mathrm{W}(\kappa_s, \kappa_t) \, \Psi_{\mathrm{Co}}(\varepsilon_{st}) + m_t \, \mathrm{D}(\kappa_s, \kappa_t) \, |\Delta \mathbf{u}|^2 \, \Phi_{\mathrm{Co}}(\varepsilon_{st})\right], \tag{80}$$

where Φ_{Co} and Ψ_{Co} are defined as in equation (153) and (155), respectively, and

$$\nu_{st}^{\mathrm{Co}} = \frac{4}{3} \frac{m_t}{m_s + m_t} \frac{n_t}{\pi^{1/2} \, b^3} \, \mathrm{Q}_{\mathrm{Co}}. \tag{81}$$

Hard-Sphere Interactions

Using equation (53), we can write the transport coefficients for hard-sphere interactions as

$$\frac{\delta n_s}{\delta t} = 0, \tag{82}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = -\sum_t m_{st} \, \mathrm{Q}_{\mathrm{HS}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_s f_t \, \mathrm{g}_{st} \, \mathbf{g}_{st} \, d\mathbf{c}_t \, d\mathbf{c}_s, \tag{83}$$

$$\frac{\delta E_s}{\delta t} = -\sum_t m_{st} \, \mathrm{Q}_{\mathrm{HS}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_s f_t \, \mathrm{g}_{st} \left(\hat{\mathbf{V}}_c \cdot \mathbf{g}_{st}\right) \, d\mathbf{c}_t \, d\mathbf{c}_s. \tag{84}$$

Calculating the transport coefficients for hard-sphere interactions follows the same steps that have been applied previously for the Coulomb collision, since the main difference between the integrals in equations (82–84) and those in equations (57–59) is that g_{st} has a power of 1 rather than -3. This results in a change only in the final integrals, where we use I_1^n, I_2^n, and I_3^n with $n = 1$, see Appendix B, to obtain the momentum coefficient for hard-sphere interactions.

$$\frac{\delta \mathbf{M}_s}{\delta t} = \sum_t n_s m_s \nu_{st}^{\mathrm{HS}} \, \mathrm{D}(\kappa) \, \Delta \mathbf{u} \; \Phi_{\mathrm{HS}}(\varepsilon_{st}), \tag{85}$$

and the energy coefficient for hard-sphere interactions

$$\frac{\delta E_s}{\delta t} = \sum_t \frac{n_s m_s \nu_{st}^{\mathrm{HS}}}{m_s + m_t} \left[3k_B \, \Delta \mathrm{T} \; \mathrm{W}(\kappa) \, \Psi_{\mathrm{HS}}(\varepsilon_{st}) + m_t \, \mathrm{D}(\kappa) \, |\Delta \mathbf{u}|^2 \, \Phi_{\mathrm{HS}}(\varepsilon_{st})\right], \tag{86}$$

where Φ_{HS} and Ψ_{HS} define as in equation (154) and (156), respectively, and

$$\nu_{st}^{\mathrm{HS}} = \frac{8}{3} \frac{m_t}{m_s + m_t} \frac{n_t}{\pi^{1/2}} \, b \, \mathrm{Q}_{\mathrm{HS}}. \tag{87}$$

Maxwell Molecule Collisions

Using equation (54), the transport coefficients for Maxwell molecule collisions become,

$$\frac{\delta n_s}{\delta t} = 0, \tag{88}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = -\sum_t m_{st} \, \mathrm{Q}_{\mathrm{MC}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_s^{\kappa_s} f_t^{\kappa_t} \, \mathbf{g}_{st} \, d\mathbf{c}_t \, d\mathbf{c}_s, \tag{89}$$

$$\frac{\delta E_s}{\delta t} = -\sum_t m_{st} \, \mathrm{Q}_{\mathrm{MC}} \int_{\mathbb{R}^3 \times \mathbb{R}^3} f_s^{\kappa_s} f_t^{\kappa_t} \left(\hat{\mathbf{V}}_c \cdot \mathbf{g}_{st}\right) \, d\mathbf{c}_t \, d\mathbf{c}_s. \tag{90}$$

There's no integration technique required to evaluate equations (88–90), since after rewriting the relative velocity in equation (4) and the dot product in equation (49) in terms of the random velocities c_s and c_t, as

$$\mathbf{g}_{st} = \mathbf{c}_s - \mathbf{c}_t - \Delta \mathbf{u}, \tag{91}$$

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and

$$\hat{\mathbf{V}}_c \cdot \mathbf{g}_{st} = \frac{m_s c_s^2 - m_t c_t^2 + (m_t - m_s)\mathbf{c}_t \cdot \mathbf{c}_s + (m_t - m_s)\,\mathbf{c}_s \cdot \Delta\mathbf{u} - 2m_t \mathbf{c}_t \cdot \Delta\mathbf{u} - m_t |\Delta\mathbf{u}|^2}{m_s + m_t}. \tag{92}$$

We can use the following expectation values, see Appendix A,

$$\left\langle c_\alpha^2 \right\rangle = \frac{3k_B T_\alpha}{m_\alpha}, \qquad \langle \mathbf{c}_\alpha \rangle = 0, \qquad \langle A \rangle = A\,,\ A \text{ is constant} \tag{93}$$

to obtain the momentum coefficient for Maxwell molecule collisions

$$\frac{\delta \mathbf{M}_s}{\delta t} = \sum_t n_s m_s \nu_{st}^{\mathrm{MC}}\, \Delta\mathbf{u}, \tag{94}$$

and the energy coefficient for Maxwell molecule collisions

$$\frac{\delta E_s}{\delta t} = \sum_t \frac{n_s m_s \nu_{st}^{\mathrm{MC}}}{m_s + m_t} \left[3k_B\,\Delta\mathrm{T} + m_t\,|\Delta\mathbf{u}|^2\right], \tag{95}$$

where

$$\nu_{st}^{\mathrm{MC}} = \frac{n_t\, m_t}{m_s + m_t}\, \mathrm{Q}_{\mathrm{MC}}. \tag{96}$$

Summary

The general expressions for the transport coefficients in the five-moment approximation, where the the distribution function is set to be drifting Kappa, are summarized as follows:

$$\frac{\delta n_s}{\delta t} = 0, \tag{97}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = \sum_t n_s m_s \nu_{st}\, \mathrm{D}(\kappa_s, \kappa_t)\, \Delta\mathbf{u}\, \Phi(\varepsilon_{st}), \tag{98}$$

$$\frac{\delta E_s}{\delta t} = \sum_t \frac{n_s m_s \nu_{st}}{m_s + m_t} \left[3k_B \Delta\mathrm{T}\ \mathrm{W}(\kappa_s, \kappa_t)\, \Psi(\varepsilon_{st}) + m_t\, \mathrm{D}(\kappa_s, \kappa_t)\, |\Delta\mathbf{u}|^2\, \Phi(\varepsilon_{st})\right]. \tag{99}$$

where

$$\varepsilon_{st} = \left(\frac{1}{2k_B} \frac{m_{st}}{T_{st}}\right)^{1/2} |\Delta\mathbf{u}|, \qquad \Delta\mathbf{u} = \mathbf{u}_t - \mathbf{u}_s, \qquad \Delta\mathrm{T} = T_t - T_s, \tag{100}$$

and

$$\mathrm{D}(\kappa_s, \kappa_t) = \frac{(\kappa_s - 1/2)}{(\kappa_s - 3/2)} \frac{(\kappa_t - 1/2)}{(\kappa_t - 3/2)}, \qquad \mathrm{W}(\kappa_s, \kappa_t) = \frac{1}{(\kappa_s - 3/2)^{1/2}} \frac{\Gamma(\kappa_s)}{\Gamma(\kappa_s - 1/2)} \frac{(\kappa_t - 1/2)}{(\kappa_t - 3/2)}. \tag{101}$$

The factors ν_{st}, Φ, and Ψ change depending on the collision type, such as Coulomb, hard-sphere, and Maxwell molecule collisions. These are summarized as follows:

Coulomb collisions:

$$\nu_{st} = \nu_{st}^{\mathrm{Co}} = \frac{4}{3} \frac{n_t}{\pi^{1/2}} \frac{m_t}{m_s + m_t} \left(\frac{1}{2k_B} \frac{m_{st}}{T_{st}}\right)^{3/2} \mathrm{Q}_{\mathrm{Co}}, \qquad \mathrm{Q}_{\mathrm{Co}} = 4\pi \left(\frac{e_s e_t}{4\pi\varepsilon_0 m_{st}}\right)^2 \ln\Lambda, \tag{102}$$

and

$$\Phi(\varepsilon_{st}) = \Phi_{\mathrm{Co}}(\varepsilon_{st}) = \frac{3\sqrt{\pi}}{4} \frac{\mathrm{erf}(\varepsilon_{st})}{\varepsilon_{st}^3} - \frac{3\, e^{-\varepsilon_{st}^2}}{2\,\varepsilon_{st}^2}, \tag{103}$$

$$\Psi(\varepsilon_{st}) = \Psi_{\mathrm{Co}}(\varepsilon_{st}) = e^{-\varepsilon_{st}^2}. \tag{104}$$

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Hard-sphere interaction:

$$\nu_{st} = \nu_{st}^{\mathrm{HS}} = \frac{8}{3}\frac{n_t}{\pi^{1/2}}\frac{m_t}{m_s+m_t}\left(2k_B\frac{T_{st}}{m_{st}}\right)^{1/2}\mathrm{Q_{HS}}, \qquad \mathrm{Q_{HS}} = \pi\,\sigma^2, \tag{105}$$

and

$$\Phi(\varepsilon_{st}) = \Phi_{\mathrm{HS}}(\varepsilon_{st}) = \frac{3\sqrt{\pi}}{8}\left(\varepsilon_{st}+\frac{1}{\varepsilon_{st}}-\frac{1}{4\varepsilon_{st}^3}\right)\mathrm{erf}(\varepsilon_{st}) + \frac{3}{8}\left(1+\frac{1}{2\varepsilon_{st}^2}\right)e^{-\varepsilon_{st}^2}, \tag{106}$$

$$\Psi(\varepsilon_{st}) = \Psi_{\mathrm{HS}}(\varepsilon_{st}) = \frac{\sqrt{\pi}}{2}\left(\varepsilon_{st}+\frac{1}{2\varepsilon_{st}}\right)\mathrm{erf}(\varepsilon_{st}) + \frac{1}{2}e^{-\varepsilon_{st}^2}. \tag{107}$$

Maxwell molecule collisions

$$\Phi(\varepsilon_{st}) = \Phi_{\mathrm{MC}}(\varepsilon_{st}) = 1/\mathrm{D}(\kappa_s,\kappa_t), \tag{108}$$

$$\Psi(\varepsilon_{st}) = \Psi_{\mathrm{MC}}(\varepsilon_{st}) = 1/\mathrm{W}(\kappa_s,\kappa_t), \tag{109}$$

and

$$\nu_{st} = \nu_{st}^{\mathrm{MC}} = \frac{n_t\,m_t}{m_s+m_t}\mathrm{Q_{MC}}, \qquad \mathrm{Q_{MC}} = \mathrm{g}_{st}Q_{st}^{(1)} = \text{constant}. \tag{110}$$

Hypergeometric Representation

Here, we rewrite the resulting transport coefficients in terms of the hypergeometric function, and this is done by expressing the Φ's and Ψ's in the hypergeometric representation, where they can be written as

Coulomb collisions:

$$\Phi_{\mathrm{Co}}(\varepsilon_{st}) = {}_1F_1\left(\frac{3}{2};\frac{5}{2};-\varepsilon_{st}^2\right), \tag{111}$$

$$\Psi_{\mathrm{Co}}(\varepsilon_{st}) = {}_1F_1\left(\frac{1}{2};\frac{1}{2};-\varepsilon_{st}^2\right). \tag{112}$$

Hard-sphere:

$$\Phi_{\mathrm{HS}}(\varepsilon_{st}) = {}_1F_1\left(-\frac{1}{2};\frac{5}{2};-\varepsilon_{st}^2\right), \tag{113}$$

$$\Psi_{\mathrm{HS}}(\varepsilon_{st}) = {}_1F_1\left(-\frac{1}{2};\frac{3}{2};-\varepsilon_{st}^2\right). \tag{114}$$

Maxwell molecule:

$$\Phi_{\mathrm{MC}}(\varepsilon_{st}) = 1/\mathrm{D}(\kappa_s,\kappa_t), \tag{115}$$

$$\Psi_{\mathrm{MC}}(\varepsilon_{st}) = 1/\mathrm{W}(\kappa_s,\kappa_t). \tag{116}$$

6 Special Cases

6.1 Limiting case

One of the special properties of the Kappa distribution is that as κ tends to infinity, the Kappa distribution turns to a Maxwellian distribution, where we have, Pierrard and Lazar [2010],

$$\lim_{\kappa\to\infty}\eta(\kappa) = 1 \qquad \text{and} \qquad \lim_{\kappa\to\infty}\left[1+\frac{a}{(\kappa-3/2)}\right]^{-\kappa-1} = e^{-a}, \tag{117}$$

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thus, the Kappa distribution becomes identically Maxwellian distribution

$$\lim_{\kappa\to\infty} f_\alpha^\kappa = \frac{n_\alpha}{\pi^{3/2}\, w_\alpha^3} \exp\left(-\frac{c_\alpha^2}{w_\alpha^2}\right) = f_\alpha^{\mathrm{M}}, \tag{118}$$

which gives us one more step to be sure that the derived formulas are correct by taking the limit of transport coefficients as $\kappa \to \infty$ and comparing the resulting limit with existing formulas for the Maxwellian. The transport coefficients in the three cases Coulomb collisions, hard sphere, and Maxwell molecule are given in equation (97–99), and in the derived formulas the kappa index exist only in the two function $\mathrm{D}(\kappa_s,\kappa_t)$ and $\mathrm{W}(\kappa_s,\kappa_t)$ which mean that the limit of the transport coefficient is simplified to the two limits

$$\lim_{\kappa\to\infty} \mathrm{D}(\kappa,\kappa) = \lim_{\kappa\to\infty} \mathrm{W}(\kappa,\kappa) = 1, \qquad \kappa = \kappa_s = \kappa_t \tag{119}$$

so the limit of the transport equations (97–99), as κ goes to ∞ is

$$\lim_{\kappa\to\infty}\left[\frac{\delta n_s}{\delta t}\right] = 0, \tag{120}$$

$$\lim_{\kappa\to\infty}\left[\frac{\delta \mathbf{M}_s}{\delta t}\right] = \sum_t n_s m_s \nu_{st}\, \Delta\mathbf{u}\, \Phi(\varepsilon_{st}), \tag{121}$$

$$\lim_{\kappa\to\infty}\left[\frac{\delta E_s}{\delta t}\right] = \sum_t \frac{n_s m_s \nu_{st}}{m_s + m_t}\left[3k_B\, \Delta\mathrm{T}\ \Psi(\varepsilon_{st}) + m_t\, |\Delta\mathbf{u}|^2\, \Phi(\varepsilon_{st})\right]. \tag{122}$$

The resulting limit in the equations gives exactly the same result as the Maxwellian, as seen in Schunk and Nagy [2009], with the same definitions of Φ, Ψ, and ν_{st}.

6.2 Non-Drifting Kappa Distribution

If we chose both species s and t to have non-drifting kappa distribution, then the drift velocities for both of the interacting particles is equal to zero, $\mathbf{u}_s = \mathbf{u}_t = 0$, and since we define, $\Delta\mathbf{u} = \mathbf{u}_t - \mathbf{u}_s$, we have

$$\Delta\mathbf{u} = 0, \qquad |\Delta\mathbf{u}| = 0 \tag{123}$$

then using the relation

$$\Psi_{\mathrm{Co}}(0) = \Psi_{\mathrm{HS}}(0) = 1, \qquad \Psi_{\mathrm{MC}}(0) = 1/\mathrm{W}(\kappa_s,\kappa_t), \tag{124}$$

the transport coefficient becomes

$$\frac{\delta n_s}{\delta t} = 0, \tag{125}$$

$$\frac{\delta \mathbf{M}_s}{\delta t} = 0, \tag{126}$$

$$\frac{\delta E_s}{\delta t} = \sum_t \frac{n_s m_s \nu_{st}}{m_s + m_t}\left[3k_B\, \Delta\mathrm{T}\ \mathrm{W}(\kappa_s,\kappa_t)\, \mathrm{F}(\kappa_s,\kappa_t)\right]. \tag{127}$$

where F is a function of κ_s and κ_t defined for *Coulomb and Hard-sphere*, as

$$\mathrm{F}(\kappa_s,\kappa_t) = \mathrm{F}_{\mathrm{Co}}(\kappa_s,\kappa_t) = \mathrm{F}_{\mathrm{HS}}(\kappa_s,\kappa_t) = 1, \tag{128}$$

and for *Maxwell molecule*, as

$$\mathrm{F}(\kappa) = \mathrm{F}_{\mathrm{MC}}(\kappa) = 1/\mathrm{W}(\kappa_s,\kappa_t). \tag{129}$$

When the drift velocities of the species s and t are equal, i.e. $\mathbf{u}_s = \mathbf{u}_t$, we obtain the same result as given in equations (125–127).

https://doi.org/10.5194/egusphere-2025-2874
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7 Variations Of Transport Coefficients As A Result Of Coulomb Collisions

The transport coefficients for the five-moment approximation, presented in equations (97−99), describe how density, momentum, and energy, for particles s change under the effect of collisions. These coefficients depend on three variables, number density n_s, drift velocity $\mathbf{u}_s$, and temperature T_s for particles s, as well as on the parameters of particles t, number density n_t, drift velocity $\mathbf{u}_t$, and temperature T_t. Additionally, two functions of κ_s and κ_t , $\mathrm{D}(\kappa_s, \kappa_t)$ and $\mathrm{W}(\kappa_s, \kappa_t)$. The masses of both interacting particles, s and t, (m_s, m_t) are constant and remain unchanged throughout the collision process for all types of collisions, this makes the density coefficient to be zero according to equation (97). In this section, we will examine how these coefficients, in the case of Coulomb collisions, vary with respect to the variables (n_s, $\mathbf{u}_s$, T_s), the parameters (n_t, $\mathbf{u}_t$, T_t), and the functions D and W.

Maxwellian Distribution

In the Maxwellian case, both functions $\mathrm{D}(\kappa_s, \kappa_t)$ and $\mathrm{W}(\kappa_s, \kappa_t)$ are set to one, Section 6.1. For better clarity, we will discuss the behaviour of the transport coefficients in three cases. In each case, we will consider a specific set of variables or parameters.

First case, the *number density* of the interacting particles n_s and n_t. From equations (97−99), we see that n_s and n_t appear as a product. This indicates that an increase in the number density of either particle s or t will increase the influence of collisions on both momentum and energy of the s particles. Such a result is reasonable because the number density is defined as the number of particles per unit volume. Increasing the number density requires increasing the number of particles within the same volume. This results in more collisions between the interacting particles s and t, leading to a greater change in both momentum and energy due to the collisions.

Second case, the *drift velocities* of the interacting particles $\mathbf{u}_s$, $\mathbf{u}_t$, and the *temperature* of the s particles T_s. Equations (97−99) show that the transport coefficients depend on the difference in drift velocity, $\Delta\mathbf{u} = \mathbf{u}_t - \mathbf{u}_s$, and T_s. In Figures (1a) and (1c), we plot the isolines of the transport coefficients for momentum and energy as functions of $\Delta\mathbf{u}$ and T_s, with all other constants set to 1.0 for simplicity. Additionally, we assume identical parameters for all t particles, so the summation over t, in equations (97−99), reduces to multiplication by the number of t particles, N_t, which is set to 1000 for easier comparison with other cases. Figure (1a) shows the magnitude of the momentum transport coefficient, assuming that the direction of $\Delta\mathbf{u}$ is along the z-axis. From the figure, we can see that the momentum of the particles s does not change when the difference in drift velocities, $\Delta\mathbf{u}$, is zero, regardless of the temperature T_s. In other words, the drift velocities of the two particles are either equal or both zero, meaning that the particles are not moving relative to each other, or the system is in a state where there is no net current flow (e.g., no applied electric field). Under these conditions, Coulomb collisions can be treated as an elastic collisions, so that the total kinetic energy and momentum of the system are conserved, resulting in no change in momentum caused by the collisions. To see how momentum transport coefficient changes relative to $\Delta\mathbf{u}$, we plot the cross-section of the momentum transport coefficient at T_s equals to zero as shown in Figure (1b). We can observe that when one particle's drift velocity is slightly larger than the other's, the change in momentum of particle s due to collisions will increase until it reaches its maximum. Beyond this maximum point, as absolute value of the difference between the drift velocities becomes very large, i.e., $\mathbf{u}_s \gg \mathbf{u}_t$ or $\mathbf{u}_t \gg \mathbf{u}_s$, collisions have less and less effect on the momentum, and it approaches zero when $\Delta\mathbf{u}$ becomes very large, because when the interacting particles are moving at significantly different speeds (i.e., have large differences in drift velocities), they are more likely to move past each other without interacting, leading to decreasing in the number of collisions. Referring back to Figure (1a), as the temperature of the s particles increases, the impact of collisions on their momentum decreases until it eventually vanishes. This occurs because the mean collision frequency, ν, between the particles decreases with increasing temperature. Specifically, for Coulomb collisions,

https://doi.org/10.5194/egusphere-2025-2874
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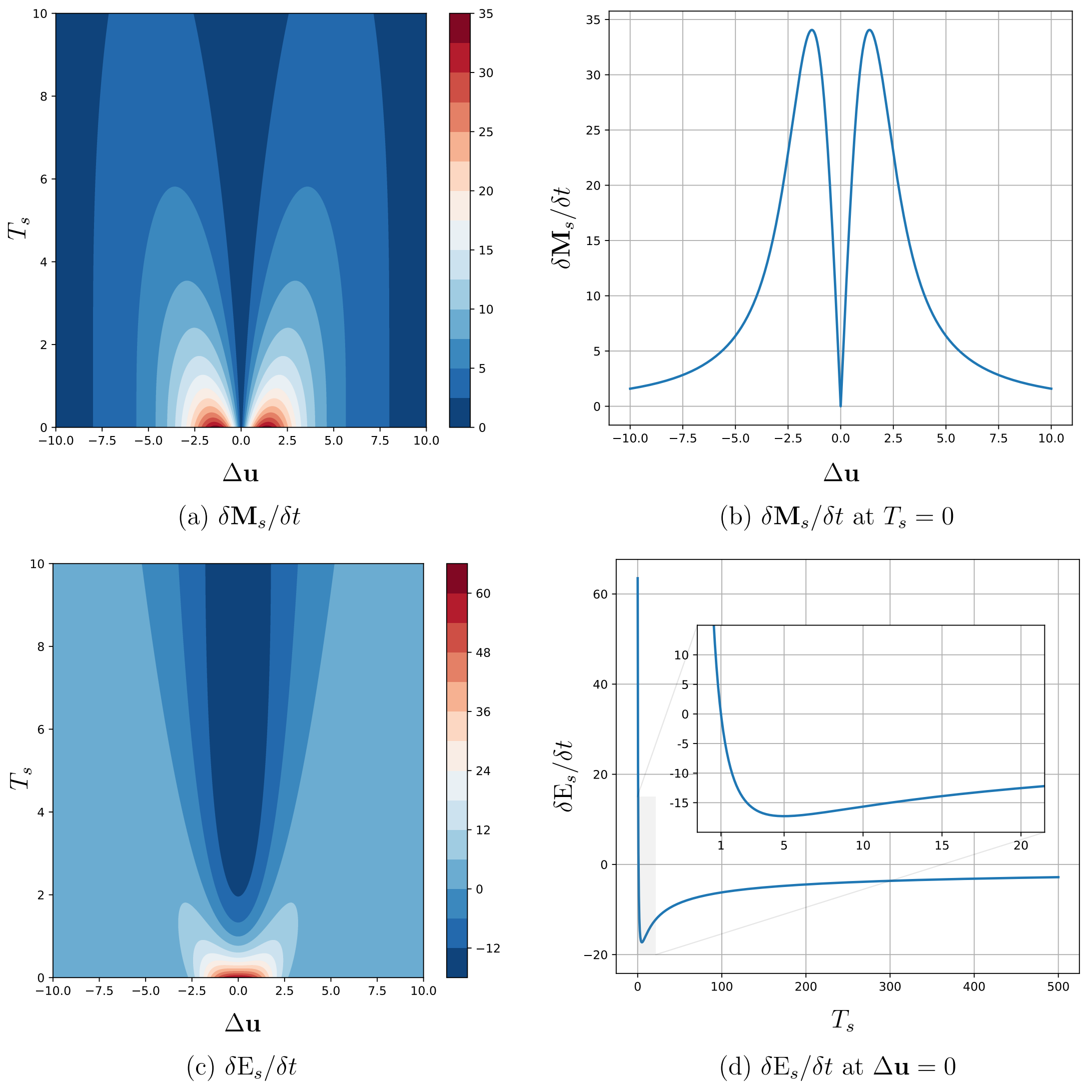
the relationship given by Freidberg [2008],

$$\nu \propto \frac{1}{T^{3/2}}, \tag{130}$$

showing that collisions become less probable at higher temperatures, leading to a reduction in their influence on momentum. Figure (1c) represents the energy transport coefficient, the graph shows that the maximum energy exchange occurs when the difference in the drift velocities is zero. As we mentioned earlier, in this case, both momentum and kinetic energy are conserved, meaning there is no change in these quantities. The change in the energy comes from the potential energy, specifically the difference between the temperatures of the interacting particles. Figure (1d) presents the cross-section of the energy transport coefficient when $\Delta\mathbf{u}$ equals to zero. At T_s equals to zero, collisions have their highest effect,since particles s gains temperature from particles t, the energy change of particles s decreases as T_s approaches T_t, reducing the temperature transfer. Eventually, when



(a) $\delta\mathbf{M}_s/\delta t$ (b) $\delta\mathbf{M}_s/\delta t$ at $T_s = 0$

(c) $\delta\mathrm{E}_s/\delta t$ (d) $\delta\mathrm{E}_s/\delta t$ at $\Delta\mathbf{u} = 0$

Figure 1: The transport coefficients for momentum and energy for the Maxwellian velocity distribution function in the case of Coulomb collisions.

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the temperatures of particles s and t are equal, $T_s = T_t$, the temperature difference becomes zero, $\Delta T = 0$, leading to no temperature transfer and the change in the energy of particle s is zero. This occurs in the Figure (1d) at $T_s = T_t = 1$. As the temperature of particles s increases above the temperature of particles t, $T_s > T_t$, particles s starts losing temperature to particles t, and this reduction increases as the temperature of particle s increases. Consequently, the energy transport coefficient becomes negative and continues decreasing until it reaches a minimum value, in our case, this happens at $T_s = 5$. After this point, the increase in the temperature of the particles s affects the collision frequency, meaning fewer collisions occur, as we see in equation (130). As a result, particles s keeps its temperature, without losing part of it to particles t. This explains the subsequent increase in the change of energy. Eventually, when the collision frequency reaches zero and the particles no longer collide, the change in the energy becomes zero, the limit of the transport coefficient as T_s approaches infinity is zero. Back to Figure (1c), as the absolute value of the difference in drift velocity becomes large, the distance between particles increase, which leads to a decrease in the energy change of particles s, because the particles become less likely to collide, the energy transport coefficient approaches zero as the drift velocity difference, $\Delta\mathbf{u}$, tends to $\pm\infty$, similar to the behaviour of the momentum transport coefficient when either $\mathbf{u}_s \gg \mathbf{u}_t$ or $\mathbf{u}_t \gg \mathbf{u}_s$.

Third case, the *temperature* of the t particles T_t. In Figure (2), we plot the isolines of the transport coefficients under the same conditions as in Figure (1), but for different values of T_t. As T_t increases the transport coefficients still exhibits the same behaviour, but the impact of collisions on the transport coefficients becomes smaller and smaller and the numerical values of the transport coefficients decreases. This can be seen by comparing the numbers on the color bars to those in Figure (1), (e.g. $\delta\mathbf{M}_s/\delta t$ equals 35 in Figure (1a) at $T_t = 1$, and 18 in Figure (2a) at $T_t = 2$; $\delta\mathrm{E}_s/\delta t$ equals 60 in Figure (1c) at $T_t = 1$, and 48 in Figure (2b) at $T_t = 2$), because as temperature increases, the number of collisions decreases as mentioned before. Another observation from Figure (2) is that the range of $\Delta\mathbf{u}$, which contributes to the transport coefficients (represented by the red area relative to the horizontal axis), expands as T_t increases. This occurs because a decrease in collision frequency results in fewer interruptions in particle motion. With fewer collisions, particles can accelerate more effectively under the influence of the external force, leading to an increase in their drift velocities.

Kappa Distribution

The Kappa distribution affects the transport coefficients through two functions $\mathrm{D}(\kappa_s, \kappa_t)$ and $\mathrm{W}(\kappa_s, \kappa_t)$, where we assume the kappa values for both species s and t to be equal, i.e., $\kappa_s = \kappa_t = \kappa$ so we can compare the results with the Maxwellian case. To understand how these two functions change the transport coefficients, we plot the isolines of the transport coefficients for momentum and energy as functions of $\Delta\mathbf{u}$ and T_s, as shown in Figures (3), with the same conditions as in Figure (1a), for various κ values. Additionally, in Figures (4), we show the cross-sections of these coefficients at $T_s = 0$. For the momentum coefficient, $\mathrm{D}(\kappa, \kappa)$ appears as a multiplication in equation (98), and this can be observed in Figures (3) and (4), as the behaviour of the momentum coefficient is similar to the Maxwellian case, with $\mathrm{D}(\kappa, \kappa)$ acting as a scaling parameter making the change of momentum increase at low kappa value. This occurs because at low κ, the suprathermal tails contain more particles with very high speeds. The extra particles increase the collision frequency, and the higher speeds lead to greater momentum transfer during collisions. For the energy coefficient, $\mathrm{W}(\kappa, \kappa)$ is multiplied in the first term of equation (99), which represents the change in the potential energy of the particles s, and $\mathrm{D}(\kappa, \kappa)$ is multiplied in the second term of the same equation, representing the change in the kinetic energy of the particles s. The general behaviour of the energy coefficient is approximately the same as in the Maxwellian case, particularly at high κ values. However, at low κ values, especially at $\kappa = 2$, the value of $\mathrm{D}(\kappa, \kappa)$ exceeds $\mathrm{w}(\kappa, \kappa)$, i.e., $\mathrm{D}(\kappa, \kappa) > \mathrm{W}(\kappa, \kappa)$, as outlined in Table (1). This causes the peaks near zero seen for $\kappa = 2$ in Figure (4b), due to the higher speeds of the extra particles, which cause greater kinetic energy exchange compared to the Maxwellian case. For both coefficients, as κ increases, the value of the transport coefficients decreases, and the result get closer to the result obtained by the Maxwellian distribution.

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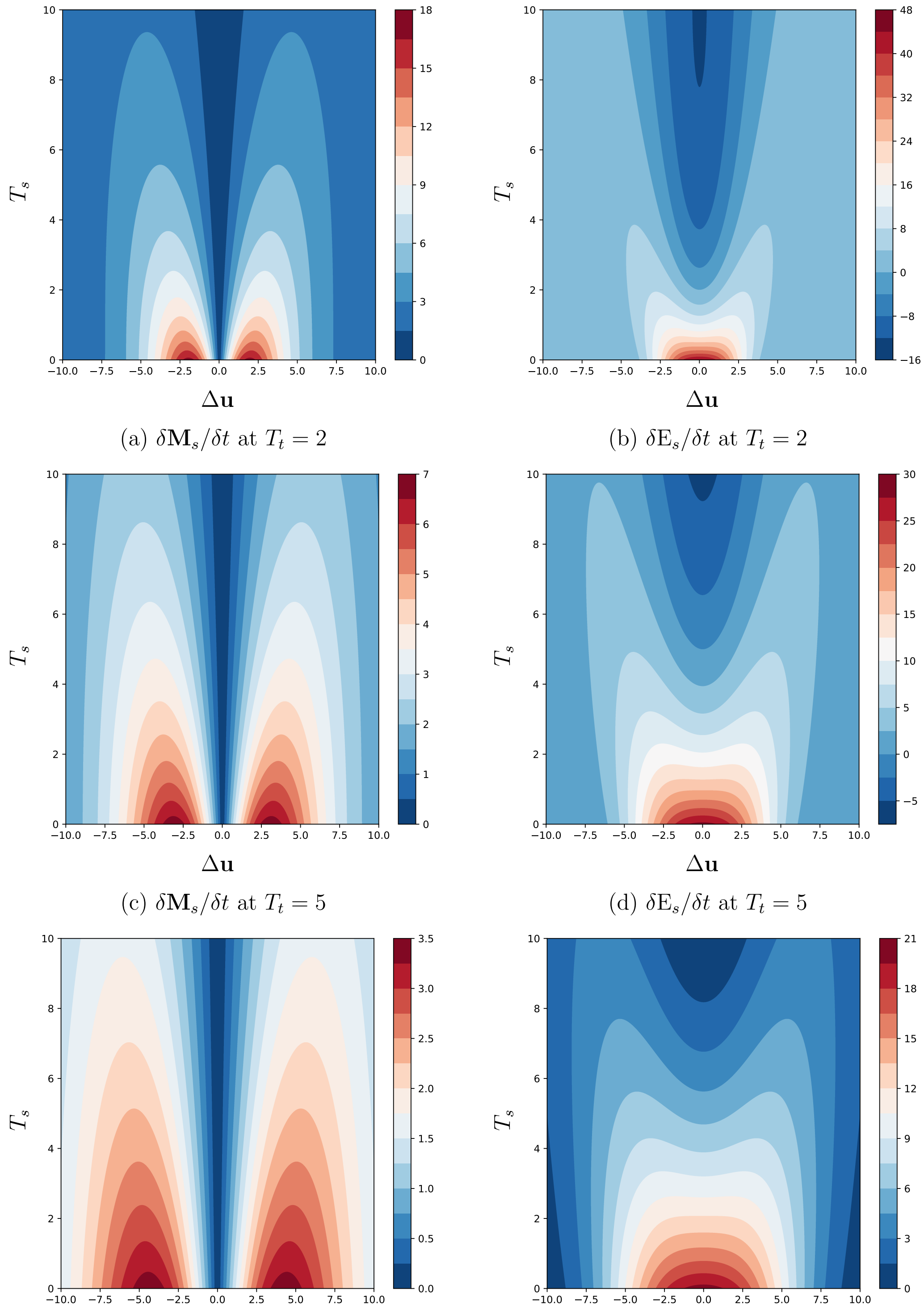

(a) $\delta\mathbf{M}_s/\delta t$ at $T_t = 2$

(b) $\delta\mathrm{E}_s/\delta t$ at $T_t = 2$

(c) $\delta\mathbf{M}_s/\delta t$ at $T_t = 5$

(d) $\delta\mathrm{E}_s/\delta t$ at $T_t = 5$

(e) $\delta\mathbf{M}_s/\delta t$ at $T_t = 10$

(f) $\delta\mathrm{E}_s/\delta t$ at $T_t = 10$

Figure 2: The transport coefficients for momentum and energy, for the Maxwellian velocity distribution function in the case of Coulomb collisions, at different values of T_t: 2, 5, and 10.

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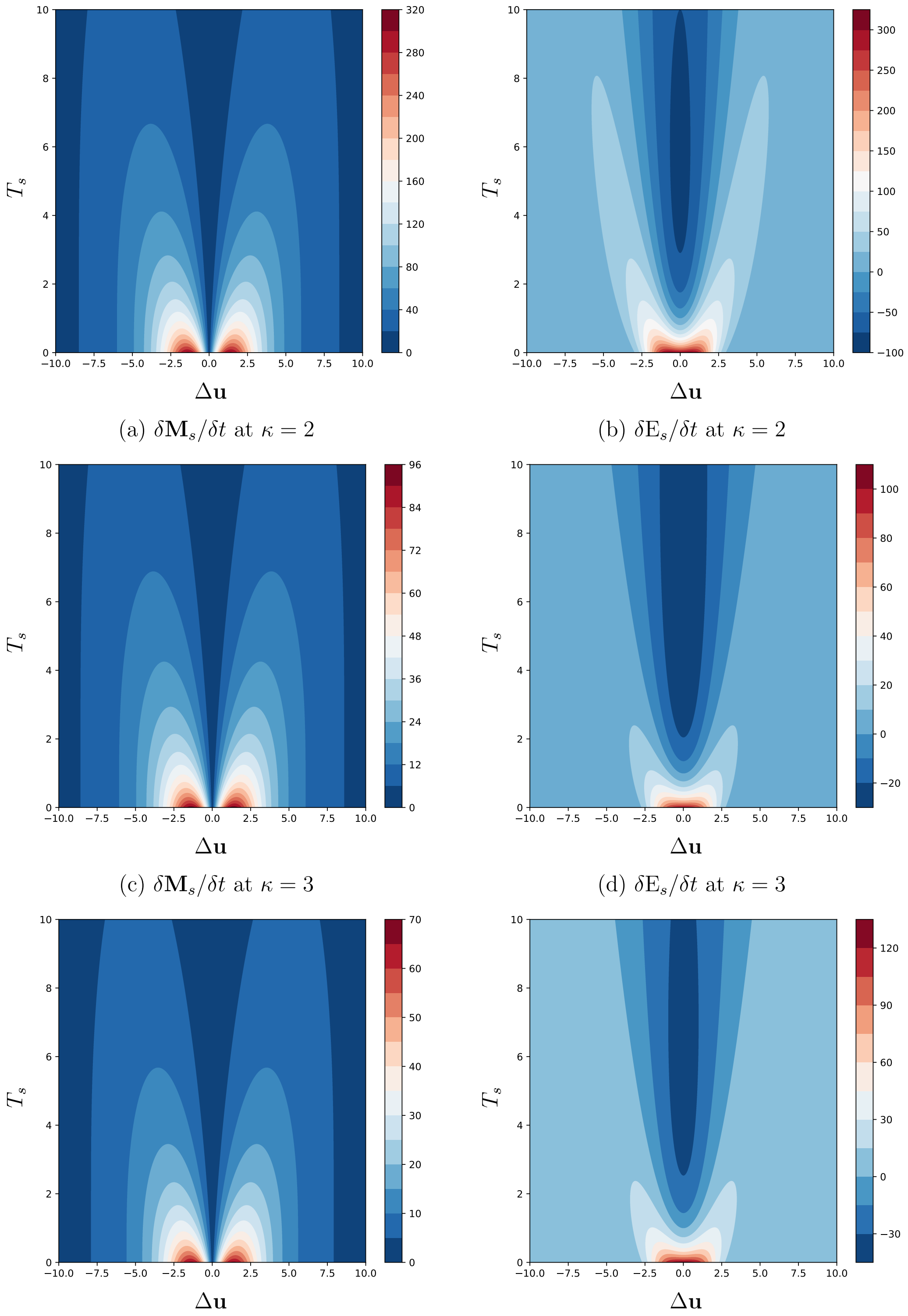

(a) $\delta \mathbf{M}_s/\delta t$ at $\kappa = 2$

(b) $\delta \mathrm{E}_s/\delta t$ at $\kappa = 2$

(c) $\delta \mathbf{M}_s/\delta t$ at $\kappa = 3$

(d) $\delta \mathrm{E}_s/\delta t$ at $\kappa = 3$

(e) $\delta \mathbf{M}_s/\delta t$ at $\kappa = 4$

(f) $\delta \mathrm{E}_s/\delta t$ at $\kappa = 4$

Figure 3: The transport coefficients, momentum and energy for the Kappa velocity distribution function in the case of Coulomb collisions at different values of κ: 2, 3, and 4.

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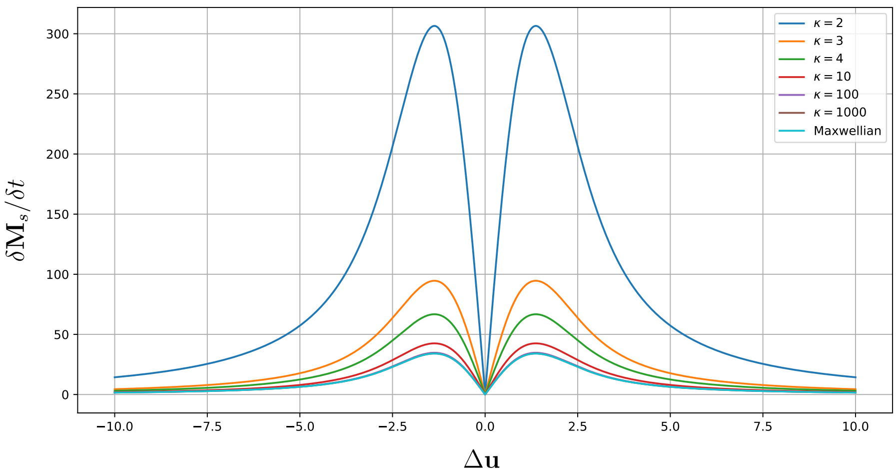
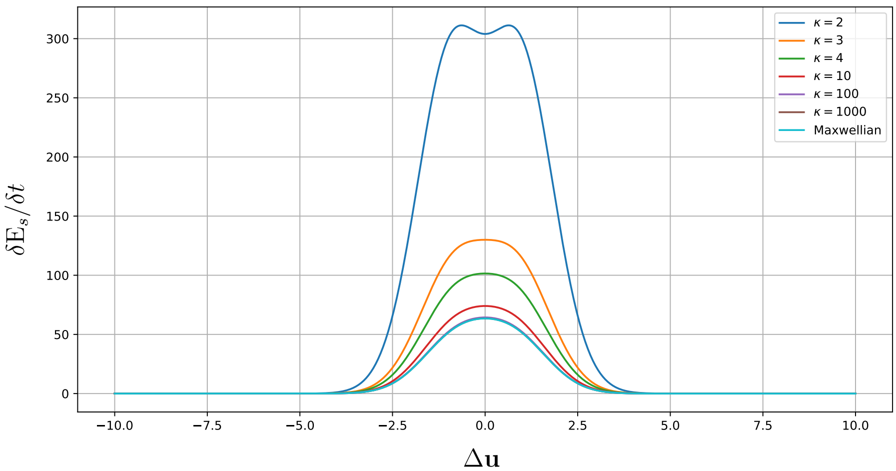


(a) $\delta\mathbf{M}_s/\delta t$ at $T_s = 0$



(b) $\delta\mathrm{E}_s/\delta t$ at $T_s = 0$

Figure 4: The cross-section of the transport coefficients, momentum and energy, for Kappa and Maxwell velocity distribution functions in the case of Coulomb collisions, at $T_s = 0$.

κ	$\mathrm{D}(\kappa,\kappa)$	$\mathrm{W}(\kappa,\kappa)$
2	9.0000	4.7873
3	2.7778	2.0474
4	1.9600	1.5986
10	1.2491	1.1661
100	1.0204	1.0140
1000	1.0020	1.0014

Table 1: $\mathrm{D}(\kappa,\kappa)$ and $\mathrm{W}(\kappa,\kappa)$ at different values of κ

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8 Conclusion

For isotropic plasmas, we have presented a closed system of transport equations. In these equations, the transport properties of a given species are defined with respect to the random velocity of that species, where the species velocity distribution function is expanded in an orthogonal polynomial series about a drifting Kappa weighting function. By taking the first term of the expansion, ignoring all higher moments of the velocity distribution, and showing that the drifting kappa has zero heat flux and zero stress pressure tensor, we obtain the five-moment approximation equation (25–27). We then obtained the transport coefficients for density, momentum and energy, as a result of collisions, based on a drifting Kappa velocity distribution function using the Boltzmann collision integral. We provide these coefficients for different types of collisions, including Coulomb collisions, hard-sphere interactions, and Maxwell molecules collisions in equation (97−99). Given the complexity of the terms involved in the transport coefficient, we have expressed the final results using hypergeometric functions in equations (111–116), to optimize and simplify the process for performing numerical calculations. Next, we investigate two special cases; The first case, when kappa index approaches infinity, which reproduces the result for the Maxwellian distribution case. In this limit, κ goes to infinity, the functions $\mathrm{D}(\kappa, \kappa)$ and $\mathrm{W}(\kappa, \kappa)$ approach 1. The second case involves taking the zeroth-order function to be a non-drifting Kappa distribution for the interacting particles by setting the drift velocity of the interacting particles to be zero. This case corresponds to an isolated system with no external force acting on the particles, and the collisions turns to be elastic. As a result, both total momentum and total kinetic energy are conserved, meaning the change in momentum is zero and the change in energy is affected by changes in the potential energy of the interacting particle. We concluded with a discussion on Coulomb collisions and how they affect the momentum and energy of the particles for both Maxwellian and Kappa velocity distributions, we found at low kappa index values, the number of collisions increased significantly, resulting in a greater changes in the momentum and energy of the particles, due to the increase in the number of particles with very high speeds. In general, as kappa index increases, the number of particles in the suprathermal tails decreases and the result become closer and closer to the Maxwellian distribution case.

Building on the foundation developed in this study, future efforts will focus on extending transport theory to more general plasma conditions. While the present work is limited to isotropic plasmas, a significant next step involves extending the model to account for anisotropic plasmas, where pressure and temperature vary with direction. Addressing anisotropy is particularly relevant for magnetized and space plasmas, where directional dependencies are essential for accurately describing transport processes. Another point of focus is providing a complete transport theory that covers the kappa velocity distribution case, as introduced by Schunk [1977], for the Maxwellian distribution function. This involves obtaining transport equations by expanding the species distribution function in a generalized orthogonal polynomial series with a kappa weighting function, resulting in various approximations, such as the eight-, ten-, thirteen-, and twenty-moment approximations. These advancements will allow for a more comprehensive understanding of collisional transport in a broader range of plasma environments.

A Appendix A

In this appendix, we want to calculate the following average values of the Kappa distribution function:

$$\langle A \rangle\,, \quad \langle \mathbf{c}_\alpha \rangle\,, \quad \langle c_\alpha^2 \rangle\,, \quad \langle \mathbf{c}_\alpha \mathbf{c}_\alpha \rangle\,, \quad \langle c_\alpha^2\, \mathbf{c}_\alpha \rangle\,,$$

where A is constant, and $\mathbf{c}_\alpha$ is the random speed. The Kappa distribution,

$$f_\alpha^\kappa = \frac{n_\alpha\, \eta(\kappa)}{\pi^{3/2}\; w_\alpha^3} \left(1 + \frac{c_\alpha^2}{\kappa_0\, w_\alpha^2}\right)^{-\kappa-1}, \tag{131}$$

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is normalized to n_α, so by the definition of the average value,

$$\langle \xi_\alpha \rangle = \frac{1}{n_\alpha} \int \xi_\alpha \, f_\alpha^\kappa \, d\mathbf{c}_\alpha, \tag{132}$$

we have

$$\langle A \rangle = A. \tag{133}$$

The Kappa distribution is an even function in $\mathbf{c}_\alpha$, and its components $c_{\alpha x}$, $c_{\alpha y}$, and $c_{\alpha z}$ imply that these average values are zero,

$$\langle \mathbf{c}_\alpha \rangle = \langle c_\alpha^2 \, \mathbf{c}_\alpha \rangle = 0, \tag{134}$$

and

$$\langle c_{\alpha i} \rangle = \langle c_{\alpha i} \, c_{\alpha j} \rangle = 0, \quad i \neq j \quad i, j = x, y, z. \tag{135}$$

To evolute the expectation value of c_α^2, we substitute f_α^κ into equation (132), to get

$$\langle c_\alpha^2 \rangle = \frac{1}{n_\alpha} \int_{\mathbb{R}^3} c_\alpha^2 \, f_\alpha^\kappa \, d\mathbf{c}_\alpha = \frac{\eta(\kappa)}{\pi^{3/2} \, w_\alpha^3} \int_{\mathbb{R}^3} c_\alpha^2 \left(1 + \frac{c_\alpha^2}{\kappa_0 \, w_\alpha^2}\right)^{-\kappa-1} d\mathbf{c}_\alpha, \tag{136}$$

by transforming the integral to the spherical coordinates, we have

$$\langle c_\alpha^2 \rangle = \frac{3}{2} w_\alpha^2 = \frac{3 k_B T_\alpha}{m_\alpha}. \tag{137}$$

In the same way, the expectation value of $\mathbf{c}_\alpha \mathbf{c}_\alpha$ is

$$\langle \mathbf{c}_\alpha \mathbf{c}_\alpha \rangle = \frac{1}{n_\alpha} \int_{\mathbb{R}^3} \mathbf{c}_\alpha \mathbf{c}_\alpha \, f_\alpha^\kappa \, d\mathbf{c}_\alpha, \tag{138}$$

$\mathbf{c}_\alpha \mathbf{c}_\alpha$ is second rank tenser, so

$$\langle \mathbf{c}_\alpha \mathbf{c}_\alpha \rangle = \begin{bmatrix} \langle c_{\alpha x}^2 \rangle & \langle c_{\alpha x} \, c_{\alpha y} \rangle & \langle c_{\alpha x} c_{\,\alpha z} \rangle \\ \langle c_{\alpha y} \, c_{\alpha x} \rangle & \langle c_{\alpha y}^2 \rangle & \langle c_{\alpha y} \, c_{\alpha z} \rangle \\ \langle c_{\alpha z} \, c_{\alpha x} \rangle & \langle c_{\alpha z} \, c_{\alpha y} \rangle & \langle c_{\alpha z}^2 \rangle \end{bmatrix},$$

all the off-diagonal expectation values are zero, according to equation (135), and the expectation values on the diagonal are equal

$$\langle \mathbf{c}_\alpha \mathbf{c}_\alpha \rangle = \langle c_{\alpha x}^2 \rangle \, \mathbf{I}, \tag{139}$$

smilier to $\langle c_\alpha^2 \rangle$, the average value of $\langle c_{\alpha x}^2 \rangle$ is

$$\langle c_{\alpha x}^2 \rangle = \frac{w_\alpha^2}{2} = \frac{k_B T_\alpha}{m_\alpha}, \tag{140}$$

which makes equation (139) become

$$\langle \mathbf{c}_\alpha \mathbf{c}_\alpha \rangle = \langle c_{\alpha x}^2 \rangle \, \mathbf{I} = \frac{k_B T_\alpha}{m_\alpha} \mathbf{I}. \tag{141}$$

B Appendix B

In the process of evaluating the transport coefficient, three kinds of integrals that takes the form

$$I_1^n = \int_{\mathbb{R}^3} \mathrm{g}_{st}^n \, e^{-|\mathbf{g}_{st} + \Delta \mathbf{u}|^2/b^2} \, \mathbf{g}_{st} \, d\mathbf{g}_{st}, \tag{142}$$

$$I_2^n = \int_{\mathbb{R}^3} (\mathrm{g}_{st}^{n+2} + \mathrm{g}_{st}^n \, \Delta \mathbf{u} \cdot \mathbf{g}_{st}) \, e^{-|\mathbf{g}_{st} + \Delta \mathbf{u}|^2/b^2} \, d\mathbf{g}_{st}, \tag{143}$$

$$I_3^n = \int_{\mathbb{R}^3} \mathrm{g}_{st}^n \, \Delta \mathbf{u} \cdot \mathbf{g}_{st} \, e^{-|\mathbf{g}_{st} + \Delta \mathbf{u}|^2/b^2} \, d\mathbf{g}_{st}, \tag{144}$$

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appears, with n equal -3 or 1. The evaluation of these integrals follows the same steps, where the first step is to use the vector property,

$$|\mathbf{g}_{st} + \Delta\mathbf{u}|^2 = \mathrm{g}_{st}^2 + |\Delta\mathbf{u}|^2 + 2\,\mathbf{g}_{st}\cdot\Delta\mathbf{u}, \tag{145}$$

By setting z-axis in the direction of vector $\Delta\mathbf{u}$, the dot product of the vectors $\mathbf{g}_{st}$ and $\Delta\mathbf{u}$ is

$$\mathbf{g}_{st}\cdot\Delta\mathbf{u} = \mathrm{g}_{st}\,|\Delta\mathbf{u}|\,\cos\theta, \tag{146}$$

where here θ is the angle between vectors $\mathbf{g}_{st}$ and $\Delta\mathbf{u}$ in the same time its the polar angle in the spherical coordinates, because vector $\Delta\mathbf{u}$ is in the direction of the z-axis. After using the expressions in equations (145) and (146), the second step is to transform the integrals to the spherical coordinates, and preform the integrals in the cases we are interested in, $n = -3$ and $n = 1$, to get

$$I_1^{-3} = -\frac{4}{3}\pi\,\Delta\mathbf{u}\;\Phi_{\mathrm{Co}}(\varepsilon_{st}), \tag{147}$$

$$I_1^{1} = -\frac{8}{3}\pi\,b^4\,\Delta\mathbf{u}\;\Phi_{\mathrm{HS}}(\varepsilon_{st}), \tag{148}$$

$$I_2^{-3} = 2\pi b^2\;\Psi_{\mathrm{Co}}(\varepsilon_{st}), \tag{149}$$

$$I_2^{1} = 4\pi b^6\;\Psi_{\mathrm{HS}}(\varepsilon_{st}), \tag{150}$$

$$I_2^{-3} = -\frac{4}{3}\pi\,|\Delta\mathbf{u}|^2\;\Phi_{\mathrm{Co}}(\varepsilon_{st}), \tag{151}$$

$$I_2^{1} = -\frac{8}{3}\pi\,b^4\,|\Delta\mathbf{u}|^2\;\Phi_{\mathrm{HS}}(\varepsilon_{st}). \tag{152}$$

where

$$\Phi_{\mathrm{Co}}(\varepsilon_{st}) = \frac{3\sqrt{\pi}}{4}\frac{\mathrm{erf}(\varepsilon_{st})}{\varepsilon_{st}^3} - \frac{3\,e^{-\varepsilon_{st}^2}}{2\,\varepsilon_{st}^2}, \tag{153}$$

$$\Phi_{\mathrm{HS}}(\varepsilon_{st}) = \frac{3\sqrt{\pi}}{8}\left(\varepsilon_{st} + \frac{1}{\varepsilon_{st}} - \frac{1}{4\varepsilon_{st}^3}\right)\mathrm{erf}(\varepsilon_{st}) + \frac{3}{8}\left(1 + \frac{1}{2\varepsilon_{st}^2}\right)e^{-\varepsilon_{st}^2}, \tag{154}$$

$$\Psi_{\mathrm{Co}}(\varepsilon_{st}) = e^{-\varepsilon_{st}^2}, \tag{155}$$

$$\Psi_{\mathrm{HS}}(\varepsilon_{st}) = \frac{\sqrt{\pi}}{2}\left(\varepsilon_{st} + \frac{1}{2\varepsilon_{st}}\right)\mathrm{erf}(\varepsilon_{st}) + \frac{1}{2}e^{-\varepsilon_{st}^2}. \tag{156}$$

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