Detailed mathematical calculation of the effect of merging on chord length distributions

Definitions

Let us consider a population of updrafts of density $\mathcal{D}_0$, with an exponential size distribution of characteristic size L_0. The effective updrafts have a characteristic size βL_0 and we assume that they have an exponential size distribution:

$$\mathbb{S}_0(x) = \frac{1}{\beta L_0} e^{-\frac{x}{\beta L_0}} \tag{S1}$$

We will denote $\mathbb{P}(x)$ the probability density function such that $\mathbb{P}(x)\mathrm{d}x\mathrm{d}y$ is the probability to find an effective updraft of size $x \pm \mathrm{d}x$ in a space interval of width $\mathrm{d}y$. If the updrafts are placed randomly in space, which we will assume in all the following, this probability is related to the effective updraft size distribution and the updraft density as follows:

$$\mathbb{P}_0(x) = \mathcal{D}_0 \mathbb{S}_0(x) \tag{S2}$$

Some of the effective updrafts will overlap, which means that some of the updrafts will merge. We want to compute two size distributions: the size distribution of the updrafts that will not merge, and the size distribution of the updrafts that are the product of this merging process.

S1 Control of coverage fraction by the cloud or updraft lifetime

Before proceeding, it is useful to compute the coverage fraction of the updrafts as a function of the updraft density $\mathcal{D}_0$ before merging and the updraft size L_0. To that end, we consider a population of updrafts that we separate into N sub-populations of equal number. We assume that $N \gg 1$ so that each part corresponds to a updraft coverage fraction of $\mathcal{D}_0 L_0 / N \ll 1$. Therefore, it is highly unlikely that two updrafts merge within a given sub-population.

Then, in a thought experiment, we populate the space successively with the N sub-populations of updrafts. After adding the first population the coverage fraction of effective updrafts is given by:

$$f_1^{eff} = \frac{\beta \mathcal{D}_0 L_0}{N} \tag{S3}$$

Now we want to compute the coverage fraction of effective updrafts after introducing n sub-populations by recurrence. Let us assume that the space is populated by n sub-populations, that yield a total coverage fraction of effective updrafts f_n^{eff}. We add one more sub-population of updrafts that has a coverage fraction $\beta \mathcal{D}_0 L_0 / N$. Because the new population is randomly placed, the overlap fraction between the already placed effective updrafts and the newly added effective updrafts is simply the product of the two fractions:

$$f_{n,n+1}^{overlap} = f_n^{eff} \times \frac{\beta \mathcal{D}_0 L_0}{N} \tag{S4}$$

so that the new total effective updraft fraction writes:

$$f_{n+1}^{eff} = f_n^{eff} + \frac{\beta \mathcal{D}_0 L_0}{N} - f_{n,n+1}^{overlap} \tag{S5}$$

This can be rewritten:

$$1 - f_{n+1}^{eff} = \left(1 - \frac{\beta \mathcal{D}_0 L_0}{N}\right)(1 - f_n^{eff}) \tag{S6}$$

The solution to this recurrence relation is straightforward:

$$f_n^{eff} = 1 - \left(1 - \frac{\beta \mathcal{D}_0 L_0}{N}\right)^n \tag{S7}$$

All the updrafts have been placed when $n = N$. Moreover, because our model is valid when $N \gg 1$ we can approximate the total coverage fraction of effective updrafts after merging by:

$$f_{th}^{eff} = \lim_{N \to \infty} f_N^{eff} \tag{S8}$$

Now, recalling that:

$$\left(1 - \frac{\beta \mathcal{D}_0 L_0}{N}\right)^N = \exp\left(N \ln\left(1 - \frac{\beta \mathcal{D}_0 L_0}{N}\right)\right) \tag{S9}$$

the limit can be computed and provides the effective updraft coverage fraction after merging as a function of the initial updraft size and density:

$$f_{th}^{eff} = 1 - e^{-\beta \mathcal{D}_0 L_0} \tag{S10}$$

which can be converted into the actual coverage fraction of updrafts:

$$f_{th} = \frac{1}{\beta}(1 - e^{-\beta \mathcal{D}_0 L_0}) \tag{S11}$$

For large updraft density, the coverage fraction converges towards the constant value:

$$f_{th}^{max} = \frac{1}{\beta} \tag{S12}$$

It is interesting to note that this maximal coverage fraction is not equal to 1, but depends on β, i.e. on the ratio between the lifetime and the transit time in the updraft.

S2 Merging effects on the size distribution

We now aim at computing the size distribution of the updrafts after merging. To that end, we will only focus on effective updrafts. We will compute separately the size distributions of the effective updrafts that merge, and the size distribution of the effective updrafts that do not merge.

S2.1 The size distribution of non merged updrafts

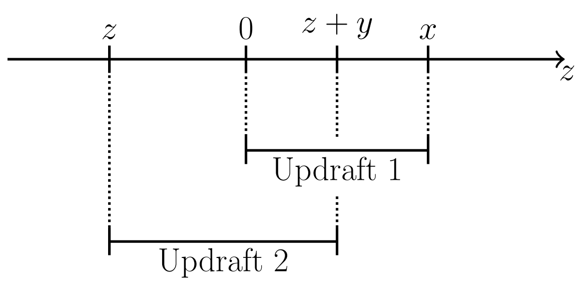

Figure 1: Schematic defining the mathematical variables discussed in the main text. For merging not to occur, z should not be between $-y$ and 0. See the text for details.

Let $\mathbb{P}_{merging}(x, y)\mathrm{d}y$ be the probability that a given effective updraft of size x merges with an effective updraft of size $y \pm \mathrm{d}y$. For convenience, we will denote z the horizontal coordinate, and assume that the initial effective updraft has its leftmost edge at $z = 0$, as shown on figure 1. Merging with an effective updraft of size y, then, occurs if and only if the left edge of the second effective updraft is located between $z = -y$ and $z = x$. As a result, the effective updraft will not merge if, and only if, there is no effective updraft of size $y \pm \mathrm{d}y$ with its left edge located in this interval. Using the definition of $\mathbb{P}_0$, this translates into:

$$1 - \mathbb{P}_{merging}(x,y)\mathrm{d}y = \prod_{z=-y}^{x} (1 - \mathbb{P}_0^{eff}(y)\mathrm{d}y\mathrm{d}z) \tag{S13}$$

Now, we compose by the logarithm function and use the fact that $\mathbb{P}_{merging}(x,y)\mathrm{d}y$ is infinitesimal to find that:

$$\mathbb{P}_{merging}(x,y)\mathrm{d}y = \int_{z=-y}^{x} \mathbb{P}_0^{eff}(y)\mathrm{d}y\mathrm{d}z = (x+y)\mathbb{P}_0^{eff}(y)\mathrm{d}y \tag{S14}$$

The probability that a given effective updraft of size x does not merge at all can then be computed by noting that this is equivalent with the effective updraft not merging with any effective updraft of size y for all y:

$$\mathbb{P}_{isolated}(x) = \prod_{y=0}^{\infty} (1 - \mathbb{P}_{merging}(x,y)\mathrm{d}y) \tag{S15}$$

Taking the logarithm of this probability, replacing $\mathbb{P}_{merging}(x,y)$ by its expression and using the fact that $\mathbb{P}_0^{eff}(y)\mathrm{d}y$ is infinitesimal we find:

$$\begin{aligned}
\ln(\mathbb{P}_{isolated}(x)) &= \int_0^{\infty} \ln(1 - \mathbb{P}_{merging}(x,y)\mathrm{d}y) && \text{(S16)}\\
&= -\int_0^{\infty} (x+y)\mathbb{P}_0^{eff}(y)\mathrm{d}y && \text{(S17)}\\
&= -\beta\mathcal{D}_0 L_0 \int_0^{\infty} e^{-\frac{y}{\beta L_0}} \frac{x+y}{\beta L_0} \frac{\mathrm{d}x}{\beta L_0} && \text{(S18)}\\
&= -\beta\mathcal{D}_0 L_0 e^{\frac{x}{\beta L_0}} \int_{\frac{x}{\beta L_0}}^{\infty} u e^{-u} \mathrm{d}u && \text{(S19)}\\
&= -\mathcal{D}_0(\beta L_0 + x) && \text{(S20)}
\end{aligned}$$

where the last equality is obtained through an integration by parts.

We can finally obtain the probability density of non-merged effective updrafts, by multiplying the probability density of all effective updrafts and the probability for an effective updraft to be isolated:

$$\begin{aligned}
\mathbb{P}_1^{eff}(x) &= \mathbb{P}_0^{eff}(x)\mathbb{P}_{isolated}(x) && \text{(S21)}\\
&= \frac{\beta\mathcal{D}_0}{L_0} e^{-\frac{x}{\beta L_0}} e^{-\mathcal{D}_0(\beta L_0 + x)} && \text{(S22)}
\end{aligned}$$

In other words the probability density of non merged effective updrafts $\mathbb{P}_1$, (defined as an equivalent of $\mathbb{P}_0$; see section) writes:

$$\mathbb{P}_1^{eff}(x) = \mathcal{D}_1 \mathbb{S}_1^{eff}(x) \tag{S23}$$

where the size distribution of the non-merged effective updrafts is given by:

$$\mathbb{S}_1^{eff}(x) = \frac{1}{\beta L_1} e^{-\frac{x}{\beta L_1}} \tag{S24}$$

This size distribution is, interestingly, also an exponential distribution. Moreover, it directly provides the size distribution of the real updrafts, by homothethy:

$$\mathbb{S}_1(x) = \frac{1}{L_1} e^{-\frac{x}{L_1}} \tag{S25}$$

The characteristic size of this exponential distribution, L_1, is given by:

$$\boxed{L_1 = \frac{L_0}{1 + \beta \mathcal{D}_0 L_0}} \tag{S26}$$

It decreases with the parameter $\beta \mathcal{D}_0 L_0$, which indicates that the efficiency of the merging process increases with the updraft lifetime, the updraft initial density, and the updraft size. We can also note that $L_1 \leq L_0$, which indicates that the smaller updrafts are more likely to remain unaffected by the merging process.

We also obtain the density of non merged updrafts:

$$\mathcal{D}_1 = \mathcal{D}_0 \cdot \frac{e^{-\beta \mathcal{D}_0 L_0}}{1 + \beta \mathcal{D}_0 L_0} \tag{S27}$$

The density of non merged updrafts increases with the initial density of updrafts and with their lifetime. It reaches a maximum beyond which merging is so efficient that only very few small updrafts do not merge, leading to a decrease in the total number of updrafts that do not merge.

Simple arithmetic computations show that there is an optimal initial updraft density that maximizes the total number of non merged updrafts at a critical initial effective updraft fraction $f = \beta \mathcal{D}_0 L_0$ such that:

$$f_{crit} = \frac{1}{\varphi} \approx 0.618 \tag{S28}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden number. Above that critical number, the actual density of non-merged updrafts is anticorrelated with the density of updrafts before merging.

S2.2 Size distribution of merged updrafts

Now, we focus on the distribution of merged updrafts. Again, for convenience, we will reason in terms of effective updrafts. The probability density of merged effective updrafts is the sum of all the different ways to form a merged effective updraft. To avoid double counting, we will number the effective updrafts that merge as follows. In a merged effective updraft, the initial effective updraft is considered to be the leftmost effective updraft in the merged effective updraft (effective updraft number 1). Then, the additional effective updrafts on the right that overlap with the initial effective updraft can be sorted by the position of their right edge, and numbered accordingly.

Updrafts merged only once

Let us start by computing the size distribution of effective updrafts that are the product of a single merging between two (and only two) initial effective updrafts. The situation is depicted on figure 2.

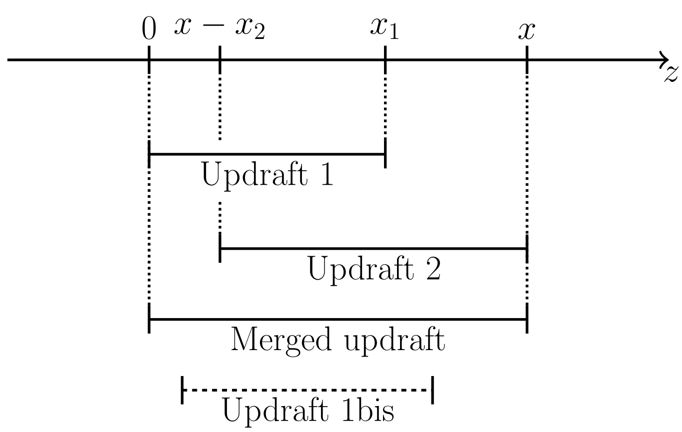

Figure 2: Schematic defining the mathematical variables discussed in the main text. See the text for details. The hypothetical updraft 1bis is discussed later in the text.

We will denote by z the horizontal axis. Let us consider one of those merged effective updrafts: it is composed of the effective updraft 1 on the left, of size x_1, and the effective updraft 2 on the right, of size x_2.

We want to compute the probability to have a merged effective updraft of size x from those two effective updrafts. Since the problem is invariant by horizontal translation, we can assume that the merged effective updraft has its left edge at position $z = 0$ and its right edge at position $z = x$. For the two effective updrafts of size x_1 and x_2 to merge into the effective updraft of size x, three conditions need to be satisfied:

- The left effective updraft needs to have a width $0 \leq x_1 \leq x$
- At fixed x_1, the right effective updraft needs to have a width $x - x_1 \leq x_2 \leq x$
- The right effective updraft needs to have its right edge located at position $z = x$

In probabilistic terms, we translate the two last conditions as follows: at fixed x_1, the probability to find the right edge of the second effective updraft of size $x_2 \pm \mathrm{d}x_2$ at position $x \pm \mathrm{d}x$ is $\mathbb{P}_0(x_2)\mathrm{d}x_2\mathrm{d}x$.

Moreover, the probability that the left effective updraft is of size $x_1 \pm \mathrm{d}x_1$ is $\mathbb{S}_0(x_1)\mathrm{d}x_1$.

Finally, the left effective updraft needs to have its leftmost edge free of any other effective updraft, which has a probability $1 - f_{th}$.

As a result the probability density to find a merged effective updraft of size $x \pm \mathrm{d}x$ that has its left edge free can be written as the sum of all the possible combinations leading to that effective updraft:

$$\mathbb{M}_1(x)\mathrm{d}x \approx \int_0^x \int_{x-x_2}^x (1 - f_{th})\mathbb{P}_{nooverlap}(x_1, x)\mathbb{P}_0^{eff}(x_1)\mathbb{P}_0^{eff}(x_2)\mathrm{d}x_2\mathrm{d}x_1 \tag{S29}$$

Note that we introduced in this formula the probability $\mathbb{P}_{no\,overlap}(x_1, x)$ in order to filter out some configurations that are actually the merging of more than two effective updrafts. Indeed, an effective updraft 1bis could have its right edge located between effective updrafts 1 and 2 (see figure 2). In such a case, the effective updraft 2 should not be considered, because it yields the probability to form a updraft that merges two times. Therefore, we need to account for the probability that this event does not happen, noted $\mathbb{P}_{no\,overlap}(x_1, x)$ which is the probability that not any effective updraft has its leftmost edge at $0 < z < x_1$ and its rightmost edge at $x_1 < z < x$.

This probability can be written as another continuous product:

$$\mathbb{P}_{no\,overlap}(x_1, x) = \prod_{z=0}^{x_1} \left(1 - \int_{x_1-z}^{x-z} \mathbb{P}_0^{eff}(y)\mathrm{d}y\mathrm{d}z\right) \tag{S30}$$

We can again compose by the logarithm to find that:

$$\begin{aligned} \ln(\mathbb{P}_{no\,overlap}(x_1, x)) &= -\int_0^{x_1} \int_{x_1-z}^{x-z} \mathbb{P}_0^{eff}(y)\mathrm{d}y\mathrm{d}z && \text{(S31)} \\ &= -\mathcal{D}_0 \int_0^{x_1} (e^{\frac{z-x_1}{\beta L_0}} - e^{\frac{z-x}{\beta L_0}})\mathrm{d}z && \text{(S32)} \\ &= -\beta\mathcal{D}_0 L_0(e^{\frac{x_1}{\beta L_0}} - 1)(e^{-\frac{x_1}{\beta L_0}} - e^{-\frac{x}{\beta L_0}}) && \text{(S33)} \\ &= -\beta\mathcal{D}_0 L_0(1 - e^{-\frac{x_1}{\beta L_0}})(1 - e^{-\frac{x-x_1}{\beta L_0}}) && \text{(S34)} \end{aligned}$$

So that:

$$\mathbb{P}_{nooverlap}(x_1, x) = \exp\left(-\beta\mathcal{D}_0 L_0(1 - e^{-\frac{x_1}{\beta L_0}})(1 - e^{-\frac{x-x_1}{\beta L_0}})\right) \tag{S35}$$

Reinjecting this expression in equation S29 leads to a complex integral that involves some superexponential functions. We therefore choose a simplified approach, and note that $\mathbb{P}_{nooverlap}(x_1, x)$ is bounded by two constant numbers:

$$e^{-\beta\mathcal{D}_0 L_0} \leq \mathbb{P}_{nooverlap}(x_1, x) \leq 1 \tag{S36}$$

In the following we will denote $\mathbb{P}_{nooverlap}(x_1, x) = \alpha$ and for simplicity, because α is bounded between two finite non-zero values, we will assume that α is more or less constant. We then have:

$$\mathbb{M}_1(x) = \alpha e^{-\beta \mathcal{D}_0 L_0} \int_0^x \int_{x-x_1}^x \mathbb{P}_0^{eff}(x_1)\mathbb{P}_0^{eff}(x_2)\mathrm{d}x_2\mathrm{d}x_1 \tag{S37}$$

$$= \alpha e^{-\beta \mathcal{D}_0 L_0} \frac{\mathcal{D}_0^2}{(\beta L_0)^2} \int_0^x \int_{x-x_1}^x e^{-\frac{x_1}{\beta L_0}} e^{-\frac{x_2}{\beta L_0}} \mathrm{d}x_2\mathrm{d}x_1 \tag{S38}$$

$$= \alpha e^{-\beta \mathcal{D}_0 L_0} \frac{\mathcal{D}_0^2}{\beta L_0} \int_0^x e^{-\frac{x}{\beta L_0}} (1 - e^{-\frac{x_1}{\beta L_0}})\mathrm{d}x_1 \tag{S39}$$

$$= \alpha e^{-\beta \mathcal{D}_0 L_0} \mathcal{D}_0^2 e^{-\frac{x}{\beta L_0}} \left(x + \beta L_0 (e^{-\frac{x}{\beta L_0}} - 1)\right) \tag{S40}$$

$$\approx \alpha \mathcal{D}_0 \left(\frac{x}{\beta L_0} - 1\right) e^{-\frac{x}{\beta L_0}} \tag{S41}$$

where in the last line we have considered that we only study large merged effective updrafts, and therefore we consider effective updrafts for which $x \gg \beta L_0$.

Updrafts merged multiple times

Now, it is possible to compute the distribution of merged effective updrafts made of three initial effective updrafts. Those updrafts are nothing else than one of the effective updrafts merged out of two effective updrafts, merged with one additional effective updraft. To avoid multiple counting, we continue to follow our counting convention, namely that the effective updraft number 3 is on the right. The size distribution of wide effective updrafts made of three initial effective updrafts can be written as:

$$\mathbb{M}_2(x) = \int_0^x \int_{x-x_1}^x \mathbb{P}_{nooverlap}(x_1, x)\mathbb{M}_1(x_1)\mathbb{P}_0^{eff}(x_2)\mathrm{d}x_2\mathrm{d}x_1 \tag{S42}$$

$$= \alpha^2 e^{-\beta \mathcal{D}_0 L_0} \mathcal{D}_0^3 \int_0^x e^{-\frac{x}{\beta L_0}} \left(\frac{x_1}{\beta L_0} - 1\right) (1 - e^{-\frac{x_1}{\beta L_0}})\mathrm{d}x_1 \tag{S43}$$

$$\approx \alpha^2 e^{-\beta \mathcal{D}_0 L_0} \mathcal{D}_0^3 \beta L_0 e^{-\frac{x}{\beta L_0}} \frac{1}{2} \left(\left(\frac{x}{\beta L_0}\right)^2 - 2\frac{x}{\beta L_0}\right) \tag{S44}$$

where the last equality is obtained through an integration by parts, and neglecting the high order terms in $e^{-\frac{x}{\beta L_0}}$. In a similar way, we can compute the probability densities of effective updrafts merged three and four times. The computations are not detailed here, but they yield:

$$\mathbb{M}_3(x) = \alpha^3 e^{-\beta\mathcal{D}_0 L_0}\mathcal{D}_0^4(\beta L_0)^2 e^{-\frac{x}{\beta L_0}} \times \frac{1}{6}\left(\left(\frac{x}{\beta L_0}\right)^3 - 3\left(\frac{x}{\beta L_0}\right)^2\right) \tag{S45}$$

$$\mathbb{M}_4(x) = \alpha^4 e^{-\beta\mathcal{D}_0 L_0}\mathcal{D}_0^5(\beta L_0)^3 e^{-\frac{x}{\beta L_0}} \times \frac{1}{24}\left(\left(\frac{x}{\beta L_0}\right)^4 - 4\left(\frac{x}{\beta L_0}\right)^3\right) \tag{S46}$$

This naturally lets us formulate the following hypothesis:

$$\mathbb{M}_n(x) = \frac{\mathcal{D}_0}{\beta L_0} e^{-\beta\mathcal{D}_0 L_0} e^{-\frac{x}{\beta L_0}} \frac{(\beta\mathcal{D}_0 L_0 \alpha)^n}{n!}\left(\left(\frac{x}{\beta L_0}\right)^n - n\left(\frac{x}{\beta L_0}\right)^{n-1}\right) \tag{S47}$$

This relationship is obviously true for $n = 1$ and $n = 2$ as shown by the previous results. It can be proved for all n through a recurrence relation:

$$\mathbb{M}_{n+1}(x) = \int_0^x \int_{x-x_1}^x \mathbb{P}_{nooverlap}(x_1, x)\mathbb{M}_n(x_1)\mathbb{P}_0^{eff}(x_2)\mathrm{d}x_2\mathrm{d}x_1 \tag{S48}$$

$$\begin{aligned} &= \alpha\mathcal{D}_0 \frac{\mathcal{D}_0}{\beta L_0} e^{-\beta\mathcal{D}_0 L_0} \frac{(\beta\mathcal{D}_0 L_0\alpha)^n}{n!} \\ &\quad \times \int_0^x e^{-\frac{x}{\beta L_0}}\left(\left(\frac{x_1}{\beta L_0}\right)^n - n\left(\frac{x_1}{\beta L_0}\right)^{n-1}\right)(1 - e^{-\frac{x_1}{\beta L_0}})\mathrm{d}x_1 \end{aligned} \tag{S49}$$

$$= \frac{\mathcal{D}_0}{\beta L_0} e^{-\beta\mathcal{D}_0 L_0} \frac{(\beta\mathcal{D}_0 L_0\alpha)^{n+1}}{n!} e^{-\frac{x}{\beta L_0}} \times J_n(x) \tag{S50}$$

with

$$J_n(x) = \int_0^{\frac{x}{\beta L_0}} (t^n - nt^{n-1})(1 - e^{-t})\mathrm{d}t \tag{S51}$$

$$= \left[\left(\frac{t^{n+1}}{n+1} - t^n\right)(1 - e^{-t})\right]_0^{\frac{x}{\beta L_0}} - \int_0^{\frac{x}{\beta L_0}} \left(\frac{t^{n+1}}{n+1} - t^n\right) e^{-t}\mathrm{d}t \tag{S52}$$

$$\approx \left[\left(\frac{t^{n+1}}{n+1} - t^n\right)(1 - e^{-t})\right]_0^{\frac{x}{\beta L_0}} - \int_0^{+\infty} \left(\frac{t^{n+1}}{n+1} - t^n\right) e^{-t}\mathrm{d}t \tag{S53}$$

$$\approx \frac{1}{n+1}\left(\left(\frac{x_1}{\beta L_0}\right)^{n+1} - (n+1)\left(\frac{x_1}{\beta L_0}\right)^n\right)(1 - e^{-\frac{x}{\beta L_0}}) - \left(\frac{(n+1)!}{n+1} - n!\right) \tag{S54}$$

$$\approx \frac{1}{n+1}\left(\left(\frac{x_1}{\beta L_0}\right)^{n+1} - (n+1)\left(\frac{x_1}{\beta L_0}\right)^n\right) \tag{S55}$$

where we have again kept only the dominant order in $e^{-x/\beta L_0}$ (we focus on the tail of the updraft size distribution). We can reinject this expression to find:

$$\mathbb{M}_{n+1}(x) \approx \frac{\mathcal{D}_0}{\beta L_0} e^{-\beta \mathcal{D}_0 L_0} e^{-\frac{x}{\beta L_0}} \frac{(\beta \mathcal{D}_0 L_0 \alpha)^{n+1}}{(n+1)!} \left(\left(\frac{x}{\beta L_0} \right)^{n+1} - (n+1) \left(\frac{x}{\beta L_0} \right)^{n} \right) \tag{S56}$$

The hypothesis of the mathematical form of $\mathbb{M}_n(x)$ is therefore proved by recurrence.

Putting all merged updrafts together

Now what is the size distribution of all the merged updrafts ? This can be computed by noting that the probability density of the merged effective updrafts is the sum of the probability densities of the effective updrafts resulting from the merging of one, two, three, etc... effective updrafts.

Not that we have to ensure that that each of those effective updrafts does not merge with another additional effective updraft on the right. The probability of this event is given by:

$$\mathbb{P}_{nooverlap}(x, +\infty) = \exp\left(-\beta \mathcal{D}_0 L_0 (1 - e^{-\frac{x}{\beta L_0}}) \right) \approx e^{-\beta \mathcal{D}_0 L_0} \tag{S57}$$

where the approximation stems from the fact that we only focus on the tail of the distribution.

Then, the distribution of merged effective updrafts asymptotes at large updrafts sizes the following distribution:

$$\begin{aligned}
\mathbb{P}_2^{eff}(x) &= \sum_{n=1}^{\infty} \mathbb{M}_n(x) \mathbb{P}_{nooverlap}(x, +\infty) && \text{(S58)} \\
&= \frac{\mathcal{D}_0}{\beta L_0} e^{-2\beta \mathcal{D}_0 L_0} e^{-\frac{x}{\beta L_0}} \sum_{n=1}^{\infty} \frac{(\beta \mathcal{D}_0 L_0 \alpha)^n}{n!} \left(\left(\frac{x}{\beta L_0} \right)^n - n \left(\frac{x}{\beta L_0} \right)^{n-1} \right) && \text{(S59)} \\
&= \frac{\mathcal{D}_0}{\beta L_0} e^{-2\beta \mathcal{D}_0 L_0} e^{-\frac{x}{L_0}} \left(e^{\alpha \mathcal{D}_0 x} - 1 - \alpha \mathcal{D}_0 \beta L_0 e^{\alpha \mathcal{D}_0 x} \right) && \text{(S60)} \\
&\approx \frac{\mathcal{D}_0}{\beta L_0} e^{-2\beta \mathcal{D}_0 L_0} (1 - \alpha \beta \mathcal{D}_0 L_0) e^{-\frac{x}{\beta L_0}} e^{\alpha \mathcal{D}_0 x} && \text{(S61)} \\
&\approx \frac{\mathcal{D}_0}{\beta L_2} e^{-2\beta \mathcal{D}_0 L_0} e^{-\frac{x}{\beta L_2}} && \text{(S62)}
\end{aligned}$$

where we have again used the fact that we focus on the tail of the distribution for which $\mathcal{D}_0 x \gg 1$, and L_2 will be defined a few lines later.

We can again rewrite this probability distribution in terms of a density of merged effective updrafts, and a size distribution:

$$\mathbb{P}_2^{eff}(x) = \mathcal{D}_2 \mathbb{S}_2^{eff}(x) \tag{S63}$$

where the size distribution of the non-merged effective updrafts asymptotes an exponential for large updraft sizes:

$$\forall x \gg \beta L_0, \quad \mathbb{S}_2^{eff}(x) \propto e^{-\frac{x}{\beta L_2}} \tag{S64}$$

Again, this can be converted into the size distribution of actual updrafts by homothethy and we are left with:

$$\forall x \gg L_0, \quad \mathbb{S}_2(x) \propto e^{-\frac{x}{L_2}} \tag{S65}$$

In other words, we have shown that if the initial size distribution of the updrafts without merging is exponential, then the distribution of the merged updrafts is asymptotically exponential for large updraft sizes, with a characteristic size:

$$L_2 = \frac{L_0}{1 - \alpha\beta\mathcal{D}_0 L_0} \tag{S66}$$

Note that L_2 is note the average size of the merged updrafts, because here we have only performed an asymptotic approximation for the largest updrafts.

Now, using again the bounds of α (equation S36), we obtain the following bounds for L_2:

$$\frac{L_0}{1 - \beta\mathcal{D}_0 L_0 e^{-\beta\mathcal{D}_0 L_0}} \leq L_2 \leq \frac{L_0}{1 - \beta\mathcal{D}_0 L_0} \tag{S67}$$

This equality is a strong constrain for small values of $\beta\mathcal{D}_0 L_0$ because performing a Taylor expansion of order 2 it yields:

$$L_0 \left(1 + \beta\mathcal{D}_0 L_0 + \mathcal{O}\left((\beta\mathcal{D}_0 L_0)^3\right)\right) \leq L_2 \leq L_0 \left(1 + \beta\mathcal{D}_0 L_0 + (\beta\mathcal{D}_0 L_0)^2 + \mathcal{O}\left((\beta\mathcal{D}_0 L_0)^3\right)\right) \tag{S68}$$

but at large $\beta\mathcal{D}_0 L_0$ both the left and right members of equation S67 are unphysical: the right member explodes as $\beta\mathcal{D}_0 L_0 \to 1$ while the left member decreases with $\beta\mathcal{D}_0 L_0$ for large $\beta\mathcal{D}_0 L_0$ which is also unphysical. We expect that L_2 not only satisfies equation S67, but also increases monotonically with $\beta\mathcal{D}_0 L_0$ and goes to infinity as $\beta\mathcal{D}_0 L_0 \to \infty$. These constrains are satisfied by the following expression:

$$\boxed{L_2 = L_0 e^{\beta\mathcal{D}_0 L_0}} \tag{S69}$$

Note that we have not proved this expression, and equation S69 is a conjecture. Indeed, the exponential function is only one of the many functions that satisfy all these constraints, but it is probably the simplest one.

Moreover, this expression has been validated against a numerical experiment by Brett McKim. He simulated the merging process and showed that equation S69 matches very well the simulations. Proving mathematically equation S69, by relaxing the hypothesis of a constant α, remains a perspective for future work.

Density of merged updrafts

Note that this new size distribution is only valid for large updraft sizes $L_0 x \gg 1$. This expression, therefore, is only valid for a small proportion of the merged updrafts, and it cannot be used to deduce the total number of merged updrafts. However, we can use the conservation of the total effective updraft coverage fraction during the process to deduce $\mathcal{D}_2$. This fraction, indeed, is conserved during the merging process. By definition, the two populations of merged and unmerged effective updrafts do not overlap, so this conservation writes:

$$f_{th}^{eff} \approx \mathcal{D}_1 \beta L_1 + \mathcal{D}_2 L_{merged}^{eff} \tag{S70}$$

where L_{merged}^{eff} is the average size of merged effective updrafts. Because $\mathbb{P}_2^{eff}$ is only an asymptotic approximation of the real density probability of merged updrafts $\mathbb{P}_{merged}^{eff}$, the average size of merged updrafts indeed differs from βL_2. We expect that the probability distribution of merged updrafts is equal to $\mathbb{P}_2^{eff}$ for updrafts much larger than βL_0, but is virtually zero for very small updrafts. We therefore can assume, under a very crude assumption:

$$\mathbb{P}_{merged}(x) \approx \mathbb{P}_2^{eff}(x) \Theta_H(x - \beta L_0) \tag{S71}$$

where Θ_H is the Heaviside function. This directly provides the average length of merged updrafts:

$$L_{merged}^{eff} = \frac{\displaystyle\int_0^\infty x \mathbb{P}_{merged}^{eff}(x) \mathrm{d}x}{\displaystyle\int_0^\infty \mathbb{P}_{merged}^{eff}(x) \mathrm{d}x} = \beta(L_0 + L_2) \tag{S72}$$

Reinjecting the equations for f_{th}^{eff} (S11), L_1 (S26), D_1 (S27) and L_2 (S69) one finds the following expression:

$$\mathcal{D}_2 = \frac{1 - e^{-\beta \mathcal{D}_0 L_0} \left(1 + \dfrac{\beta \mathcal{D}_0 L_0}{(1 + \beta \mathcal{D}_0 L_0)^2}\right)}{\beta L_0 (1 + e^{\beta \mathcal{D}_0 L_0})} \tag{S73}$$

The density of merged updrafts is usually low compared to the density of unmerged updrafts, and maximizes for $\beta \mathcal{D}_0 L_0 = 1.08$ at a value of $\beta \mathcal{D}_2 L_0 = 0.146$. Beyond that value, adding updrafts reduces the total number of merged updrafts because it makes merging very efficient and produces fewer but larger updrafts.

S3 The updraft distribution after merging

A double exponential distribution

After merging, the size distribution of updrafts can be written as:

$$\boxed{\mathbb{S}(x) = \frac{p_1}{L_1} e^{-x/L_1} + \frac{p_2}{L_2} e^{-x/L_2}} \tag{S74}$$

Indeed, for $x/L_0 \ll 1$ the unmerged updrafts dominate. Therefore, even though our expression for the size distribution of merged updrafts is not valid there (equation S65), its contribution is small enough to be neglected in the final size distribution. For $x/L_0 \gg 1$ equation S65 is valid. Finally, for $x/L_0 \approx 1$ the formula should be less accurate and probably slightly overestimates the actual size distribution of updrafts.

The relative magnitude of the two exponentials

In equation S74, p_1 and p_2 can be determined by the prefactors of $\mathbb{P}_1$ and $\mathbb{P}_2$. One finds:

$$p_1 = \frac{1}{1 + (1 + \beta\mathcal{D}_0 L_0) e^{-\beta\mathcal{D}_0 L_0}} \tag{S75}$$

and

$$p_2 = \frac{(1 + \beta\mathcal{D}_0 L_0) e^{-\beta\mathcal{D}_0 L_0}}{1 + (1 + \beta\mathcal{D}_0 L_0) e^{-\beta\mathcal{D}_0 L_0}} \tag{S76}$$

Note that p_1 and p_2 do not represent the total population of unmerged and merged updrafts, respectively, that are rather described by $\mathcal{D}_1$ and $\mathcal{D}_2$. For example, if $\beta\mathcal{D}_0 L_0 \to 0$, then $D_2 \to 0$, but $p_2 \to \frac{1}{2}$. This apparent contradiction is solved by noticing that only the tail of the size distribution of merged updrafts is exponential. Therefore, p_2 and is not necessarily related to the total density of merged updrafts $\mathcal{D}_2$.

It is interesting to note that as merging becomes more and more prominent, i.e. as $\beta\mathcal{D}_0 L_0$ increases, the characteristic size of the merged updrafts increases, but the relative weight of the second exponential decreases.

Densities of merged and non merged updrafts

The total density of updrafts is the sum of the density of non-merged and merged updrafts: $\mathcal{D}_{th} = \mathcal{D}_1 + \mathcal{D}_2$, which yields:

$$\boxed{\mathcal{D}_{th} \approx \mathcal{D}_0 \cdot \frac{e^{-\beta\mathcal{D}_0 L_0}}{1 + \beta\mathcal{D}_0 L_0} + \frac{1 - e^{-\beta\mathcal{D}_0 L_0}\left(1 + \frac{\beta\mathcal{D}_0 L_0}{(1 + \beta\mathcal{D}_0 L_0)^2}\right)}{\beta L_0 (1 + e^{\beta\mathcal{D}_0 L_0})}} \tag{S77}$$

For $\beta\mathcal{D}_0 L_0 = 0.83$, $\mathcal{D}_{th}$ reaches a theoretical maximal value:

$$\mathcal{D}_{crit} \approx \frac{0.34}{\beta L_0} \tag{S78}$$

It is interesting to note that the maximum density after merging decreases with β and therefore with the lifetime of the thermals or the clouds.