Von Neumann Analysis for Pure Damping Equation

For the pure damping equation neglecting advection term:

$$\frac{\partial u}{\partial t} = -\alpha u, \qquad \alpha > 0 \tag{1}$$

where $-\alpha u$ the damping term and α the damping coefficient. Using forward Euler in time, the discretization equation takes the form:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -\alpha_j u_j^n \tag{2}$$

Assume that the numerical solution at j^{th} point at n^{th} time step is a single Fourier component of the form:

$$u_j^n = V^n e^{ikj\Delta x} \tag{3}$$

where k is the wavenumber, and V^n the time-dependent complex amplitude at time step n. Substituting the Fourier mode from Eq. (3) into Eq. (2), the amplification factor is obtained:

$$G = \frac{V^{n+1}}{V^n} = 1 - \Delta t \alpha_j \tag{4}$$

Equation (4) shows that G is purely real so that the discretization scheme (2) is non-dispersive.

If the damping term is discretized in a staggered manner, similar to the Eq. (1) and (2) in our manuscript, Eq. (2) becomes:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = -\frac{1}{2}\left(\alpha_{j-\frac{1}{2}} \frac{u_{j-1}^n + u_j^n}{2} + \alpha_{j+\frac{1}{2}} \frac{u_j^n + u_{j+1}^n}{2}\right) \tag{5}$$

Substituting Eq. (3) into Eq. (5) yields the amplification factor:

$$G = 1 - \frac{\Delta t \bar{\alpha}}{2}\big(1 + \cos(k\Delta x)\big) - i\frac{\Delta t \Delta \alpha}{4}\sin(k\Delta x) \tag{6}$$

where $\bar{\alpha} = \frac{\alpha_{j-\frac{1}{2}} + \alpha_{j+\frac{1}{2}}}{2}$, $\Delta\alpha = \alpha_{j+\frac{1}{2}} - \alpha_{j-\frac{1}{2}}$.

For waves with wavelengths greater than $2\Delta x$, we have $0 < k\Delta x < \pi$ and therefore $\sin(k\Delta x) \neq 0$. If $\Delta\alpha \neq 0$, the imaginary part of Eq. (6) becomes non-zero. This indicates that the discrete scheme represented by Eq. (5) exhibits dispersive behavior, since the amplification factor now possesses an imaginary component, which introduces a phase shift between wave components. Variations in amplitude are therefore accompanied by changes in phase, leading to frequency-dependent wave propagation speeds.

This analysis leads to the same conclusion as presented in our manuscript.