Uncertainty on filtered transport

September 13, 2025

1 Context

This section aim to answer the following comment regarding the paper submitted to EGU-2025-2072 :

M1: Since the transports Q, QEu and Qr (and other quantitative estimates) are based on measurements which contain random error, it is important to have error bounds derived from the propagation of the instrumental noise. This would add confidence to the results, for example the comparison between experimental and theoretical Stokes drift fluxes in Section 4.1. Figures 1(c,e), 6, 9, and 10 would also benefit from this analysis.

2 Method

The first two sections list the general definition used to compute the propagation of uncertainty measurements.

2.1 General statistics

We recall the definition of the variance. We use it to express the uncertainty due to filtering in section 2.6.

2.1.1 Measured value vs. true value

We assume that the true physical quantity is denoted by X_{true}. The instrument provides a noisy measurement

$$X_{\text{mes}} = X_{\text{true}} + \delta_X \tag{1}$$

where δ_X is the measurement error (assumed to have zero mean if the instrument is unbiased).

2.1.2 Variance as the ensemble average of squared errors

The variance of the measurement error is defined as

$$\sigma^2 = \mathbb{E}\big[\,(X_{\text{mes}} - X_{\text{true}})^2\,\big] = \mathbb{E}\big[\delta_X^2\big], \tag{2}$$

that is, the ensemble average of the squared differences between the measured value and the true value.

2.2 The general propagation of uncertainty

We use the propagation of errors formula to express the error on a quantity F calculated from a combinaison of n variables x_i.

$$\sigma_F^2 = \left(\frac{\partial F}{\partial x_1}\right)^2 \sigma_{x_1}^2 + \left(\frac{\partial F}{\partial x_2}\right)^2 \sigma_{x_2}^2 + ... + \left(\frac{\partial F}{\partial x_N}\right)^2 \sigma_{x_N}^2 \tag{3}$$

The propagation error formula can be used in case of small error terms and uncorrelated, independant variables. It can be found using equation 1, more information on the formula in section 3.16.8 in the book [Emery and Thomson 2001 - ISBN 978-0-12-387782-6].

2.3 Total transport

The total transport express as :

$$Q(t,T) = < \int_{z(T_1,t)}^{z(T_2,t)} u(z,t)\ dz > \tag{4}$$

Where $z(T_i,t)$ is the height for which $T(z,t) = Ti$ and $u(z,t)$ is the horizontal cross-shelf velocity. The temperature and the velocity are independant and coloacted on a regular grid with vertical spacing δ_h. Notation $< . >$ represente low-pass filter.

2.3.1 The uncertainty on unfiltered transport

The unfiltered transport express as :

$$Q'(t,T) = \int_{z(T_1,t)}^{z(T_2,t)} u(z,t)\ dz \tag{5}$$

The discrete form is :

$$Q'(t;T_1,T_2) = \delta_h \sum_{k=k_1}^{k_2} u_k(t) \tag{6}$$

With $\quad M = k_2 - k_1 + 1.$ and k

Using equation 3 on equation 6 we find :

$$\sigma^2_{Q'}(t) = \delta^2_h \sum_{k=k_0}^{k_1} \sigma^2_u + u(z_{(T_1)},t)^2 \sigma^2_{z_{(T_1)}} + u(z_{(T_2)},t)^2 \sigma^2_{z_{(T_2)}} \tag{7}$$

$$\sigma^2_{Q'}(t) = M\delta^2_h\sigma^2_u + u(z_{(T_1)})^2\sigma^2_{z_{(T_1)}} + u(z_{(T_2)})^2\sigma^2_{z_{(T_2)}} \tag{8}$$

with σ_u is the standard deviation on one ping for a given ADCP configuration and σ_z is the uncertainty on the temperature height. The variance $\sigma_{z(T_i)}$ is defined in the next section.

2.4 Inteprolation and filter

2.4.1 The uncertainty on the temperature height

The uncertainty on the temperature height is denoted $\sigma_{z(T_i)}$. It depends on the temperature and pressure measurement noise and on the temperature interpolation. The measurements from sensors a and b are denoted $(T_{ma}\ ,\ z_{ma})$ and $(T_{mb}\ ,\ z_{mb})$. Between two sensor the temperature is linearly interpolated. The inteprolated temperature field is denoted Ti and express as :

$$T_i = (z - z_a)\frac{T_b - T_a}{z_b - z_a} + T_a \tag{9}$$

We define $\quad H = z_b - z_a,\ D = T_b - T_a,\ N = T_i - T_a,$ so that :

$$z(T_i) = z_a + H\frac{N}{D} \tag{10}$$

The variance is computed using equation 3 and we note $\sigma_{T_a} = \sigma_{T_b} = \sigma_{T_m}$ and $\sigma_{z_a} = \sigma_{z_b} = \sigma_{z_m}$.

$$\sigma^2_{z(T_i)} = \sigma^2_{T_m}\big((\tfrac{z_b - z_a}{(T_b - T_a)^2}(T_a - T_i))^2 + (\tfrac{z_b - z_a}{(T_b - T_a)^2}(Ti - Tb))^2\big) + \sigma^2_{z_m}\big((\frac{T_i - T_a}{T_b - T_a})^2 + (1 - \frac{T_i - T_b}{T_b - Ta})^2\big) \tag{11}$$

and using the expression defined above we can write :

$$\sigma^2_{z(T_i)} = \sigma^2_{T_m}\frac{H^2}{D^4}((N - D)^2 + N^2) + \sigma^2_{z_m}\big((1 - \tfrac{N}{D})^2 + (\tfrac{N}{D})^2\big) \tag{12}$$

The contribution of σ_{T_m} increases when the distance between the two measurementspoints increases, and when D decreases, i.e when the two measurements are similar. The contribution of σ_{z_m} is bounded by a multiplication factor 0.5 when $T_i = \frac{Tb+Ta}{2}$ and 1, when $T_i = T_a$ or $T_i = T_b$. Therefore, the main source of error in the temperature height is expected to be the temperature measurement. This error is reduced when the sensors are close to each other but distant enough to measure distinct temperature.

2.4.2 The uncertainty due to filtering

The filter used is a Butterworth filter of order 3 from scipy.signal python package. We applied it on a time series with a sampling frequency fs which depends on the data set (N-BoB or SE-BoB) and a cutting frequency f_c of 24 hours. The filter impulse response is denoted $h(t)$ (see Figure 1) and is applied forward and backward using scipy.signal.filtfilt().

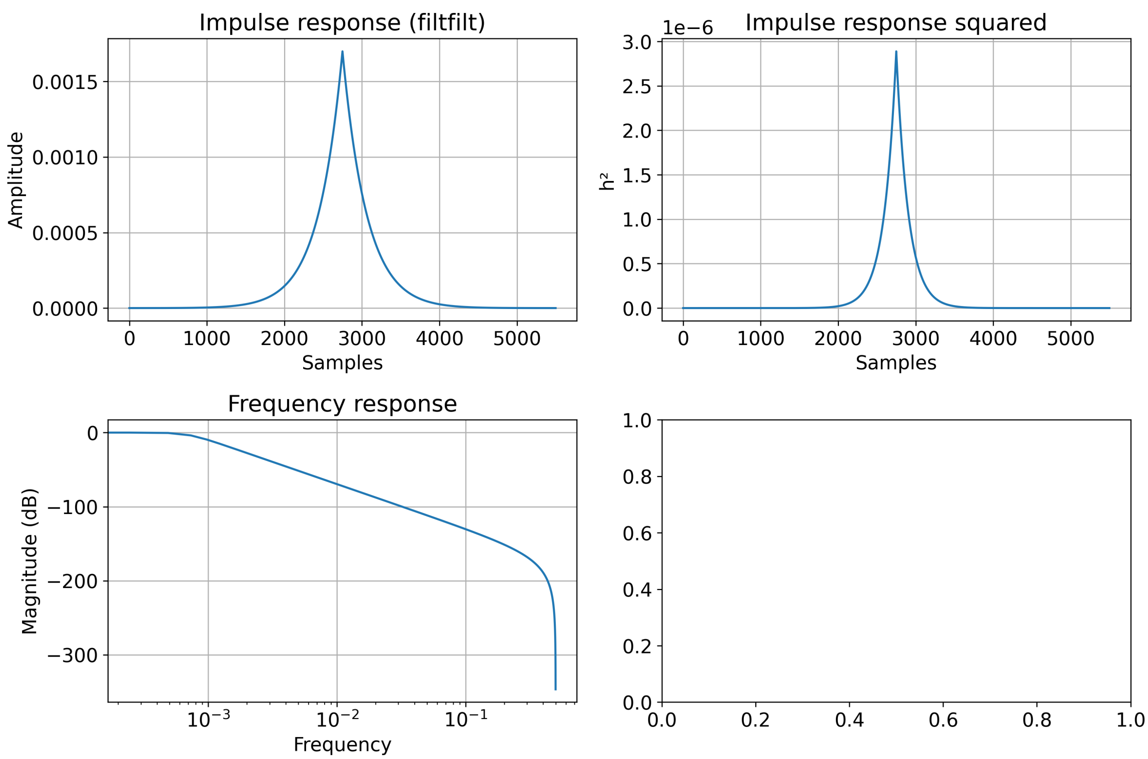

Figure 1: Caracteristics of the Butterworth low pass filter of 3rd order used on SE-BoB dataset : the cutting frequency is $f_c = 1/(24.60)$ and the frequency sampling is $fs = 1$.

We denote the unfiltered transport Q' and $\delta_{Q'}$ the measurement error. The error is non stationnary.

$$Q(t) = h * Q'(t) = \int_{-\infty}^{\infty} h(\tau)Q'(t-\tau)d\tau \tag{13}$$

Using the measured value vs. true value concept explicited in section 2.1.1 it can be shown that :

$$\delta_Q(t) = \int_{-\infty}^{\infty} h(\tau)\delta_{Q'}(t-\tau)d\tau$$
$$\mathbb{E}[\delta_Q^2(t)] = \iint_{-\infty}^{\infty} h(\tau)h(\tau')\mathbb{E}[\delta_{Q'}(t-\tau)\delta_{Q'}(t-\tau')]d\tau d\tau'$$

Where $\mathbb{E}$ is the ensemble average. We assume the errors to be δ-correlated so that :

$$\mathbb{E}[\delta Q(t-\tau)\delta Q(t-\tau')] = \sigma_{Q'}^2(t-\tau)\delta(\tau-\tau')$$

Where $\delta(t)$ is the Dirac function and therefore :

$$\sigma_Q^2 = \iint_{-\infty}^{\infty} h(\tau)h(\tau')\sigma_{Q'}^2(t-\tau)\delta(\tau-\tau')d\tau d\tau' \tag{14}$$

$$\sigma_Q^2 = \int_{-\infty}^{\infty} h^2(\tau)\sigma_{Q'}^2(t-\tau)d\tau \tag{15}$$

The variance σ_Q^2 is obtained by filtering the variance of the unfiltered transport $\sigma_{Q'}^2$ using the squared impulse response of the filter (see Figure 1).

2.5 Eulerian transport

Eulerian transport is computed over filtered velocity $< u(z,t) >$ and the filtered temperature field. The height of the filtered isotherm is denoted $z_{Eu}(T,t)$. We can write the Eulerian transport as :

$$Q_{Eu}(t,T) = \int_{z_{Eu}(T_1,t)}^{z_{Eu}(T_2,t)} < u(z,t) > dz \tag{16}$$

Using equation 3 we can write the uncertainty on Eulerian transport as :

$$\sigma_{Q_{Eu}}^2(t,T) = M\delta_h^2\sigma_{<u>}{}^2 + < u(z_{Eu(T_1)}) >^2 \sigma_{z_{Eu(T_1)}}^2 + < u(z_{Eu(T_2)}) >^2 \sigma_{z_{Eu(T_2)}}^2 \tag{17}$$

where M and δh definition are given in section 2.3. With the uncertainty due to filtering expressed in section 2.4.2 we can write :

$$\sigma_{<u>}^2 = \int_{-\infty}^{\infty} h^2(\tau)\sigma_u^2(t-\tau)d\tau \tag{18}$$

$$\sigma_{z_{Eu}}^2 = \int_{-\infty}^{\infty} h^2(\tau)\sigma_z^2(t-\tau)d\tau \tag{19}$$

2.6 Residual transport

Residual transport is the difference between Eulerian and total transport. We masked the residual transport for which the Eulerian and total transport were untrustable (error up to 50 % of the value).

3 Results

3.1 Total transport

We present the uncertainty on transport for the SE-BoB dataset. Equivalent work could be done on the N-BoB dataset. At the SE-BoB $\sigma_u = 0.08m/s$, $\sigma_{Tm} = 0.1C$ and $\sigma_{zm} = 0.1m$.

3.1.1 Error contribution before filtering

We identify which term contribute the most to the uncertainty in equation 8, between σ_u and $\sigma_{z(T_i)}$ (Figure 2). The velocity error dominates near the boundary, where the temperature is the measured temperature. In the interior, the Temperature/Pressure error dominates most of the time due to the distance between the sensors (H=10 m in SE-BoB dataset). At high frequency the velocity error can dominates (Figure 2). The velocity error dominates regularly at the same temperature range. This is likely due to an isotherm that regularly matched one of the temperature sensor and likely close to a stratified location. The values are set to NaN when no transport is measured or it is smaller than the error. To apply the filter we filled the variance time series with it's time average.

The maximum variance of the total transport after the 24h low pass filter is 0.035 m/s (Figure3). The maximum values are between 16 and 20 C before the 17/07, when the transport reach a maximum of -1.4 m/s (Figure 4).

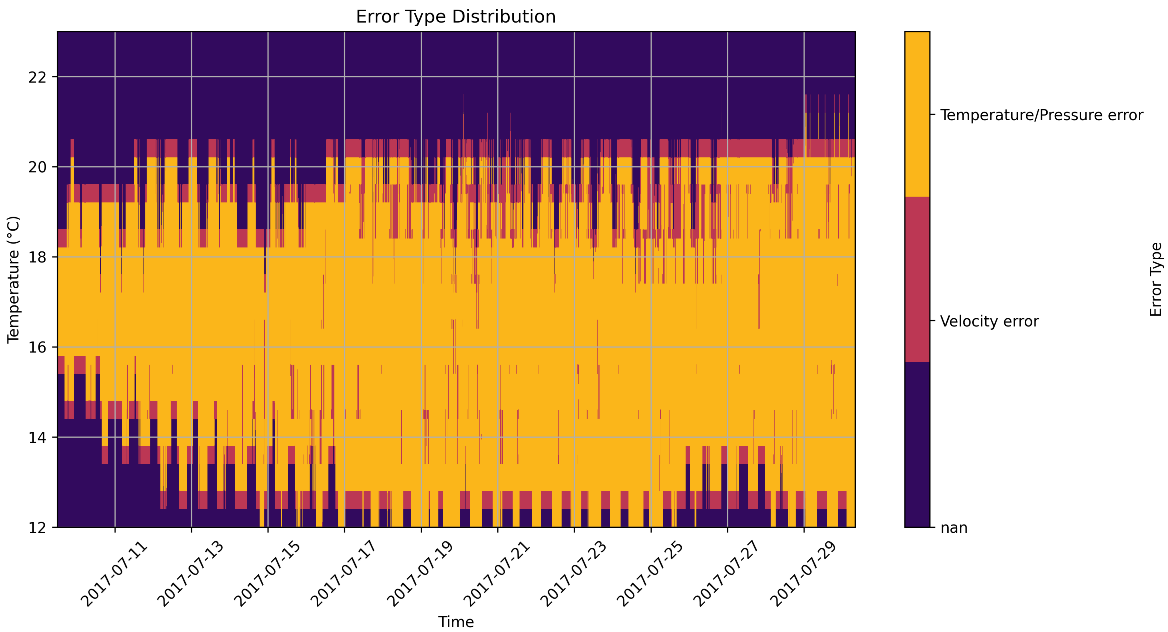

Figure 2: Error type at each time step and each temperature range from equation 8.

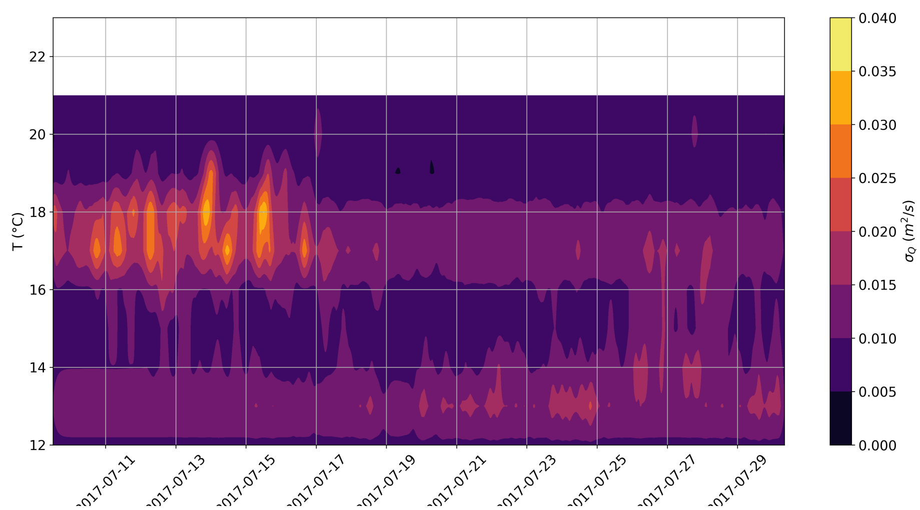

Figure 3: The variance on total transport σ_Q due to the propagation of the measurement uncertainties

3.1.2 Total transport time series and profile

Where the variance reached more than 50% of the transport we masked the transport (Figure4). The main structures were not masked.

3.2 Eulerian transport variance and mask

The variance was smaller for Eulerian transport than on total transport. The maximum variance on Eulerian transport was $0.008m/s$ (Figure 5). For ease of computation $\sigma_{z_{Eu}}$ was computed using a constant $\sigma_z = 0.7m$, obtained from equation 12 with $H = 10m$, $D = 1C$ and $N = 0.5C$. These values were chosen from the average temperature profile and sensor position shown in Figure 1 of the manuscript. Due to filter, uncertainty was reduced $\sigma_{z_{Eu}} = 0.02m$.

We masked the transport where the variance reached more than 50% of the transport (Figure6).

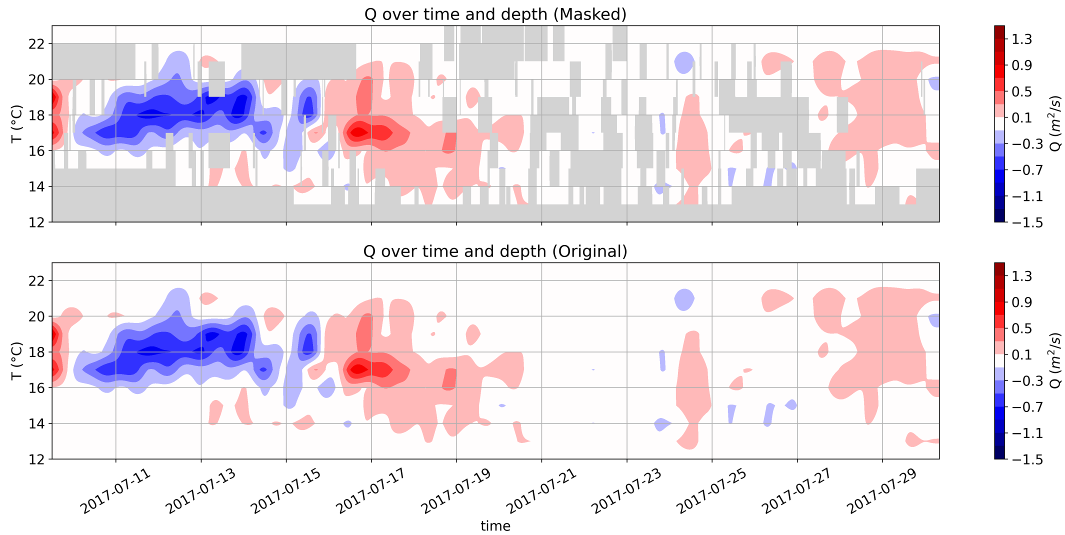

Figure 4: (Top) Total transport masked when the error is more than 50 % of the value (bottom) Original transport

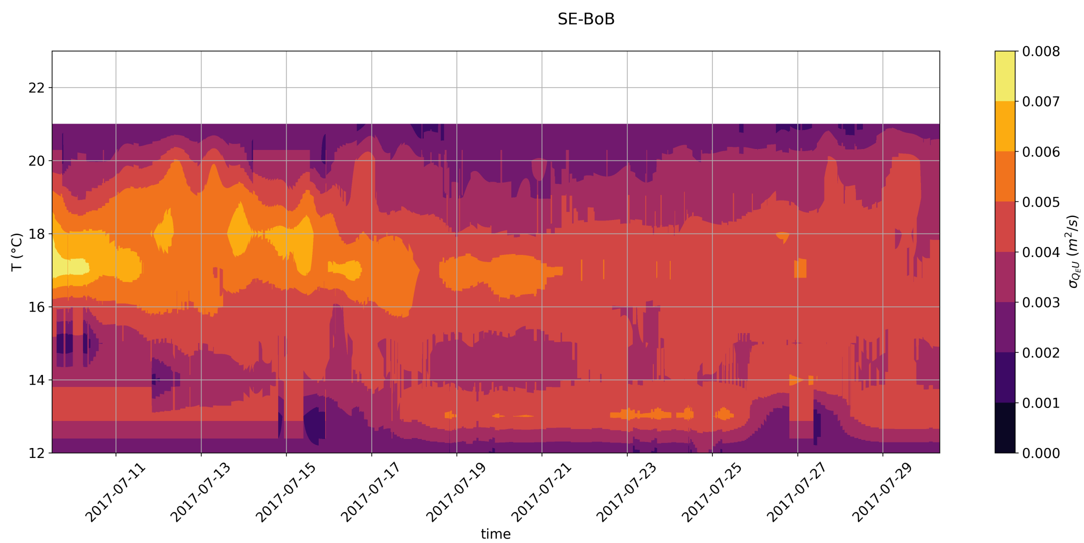

Figure 5: Variance on Eulerian transport as a function of time and temperature range. For the SE-BoB site.

Almost no data were masked on the Eulerian field.

3.2.1 Residual transport time series and vertical profile

The variance in residual transport was dominated by the variance on total transport. Based on Figure 3 the variance in residual transport was of the same order : $10^{-2}m/S$.

We applied the two masks from the total transport and Eulerian transport analyses to residual transport (Figure 7). Positive transport near the bottom (where the temperature is below 14°C) was partly masked.

The time average was mostly affected by the mask mostly for temperatures below 14°C (Figure 8 9). Where the transport was nan we considered the tranpsport to be nill. The positive transport for low temperature remained positive but was reduced between the original and masked transport.

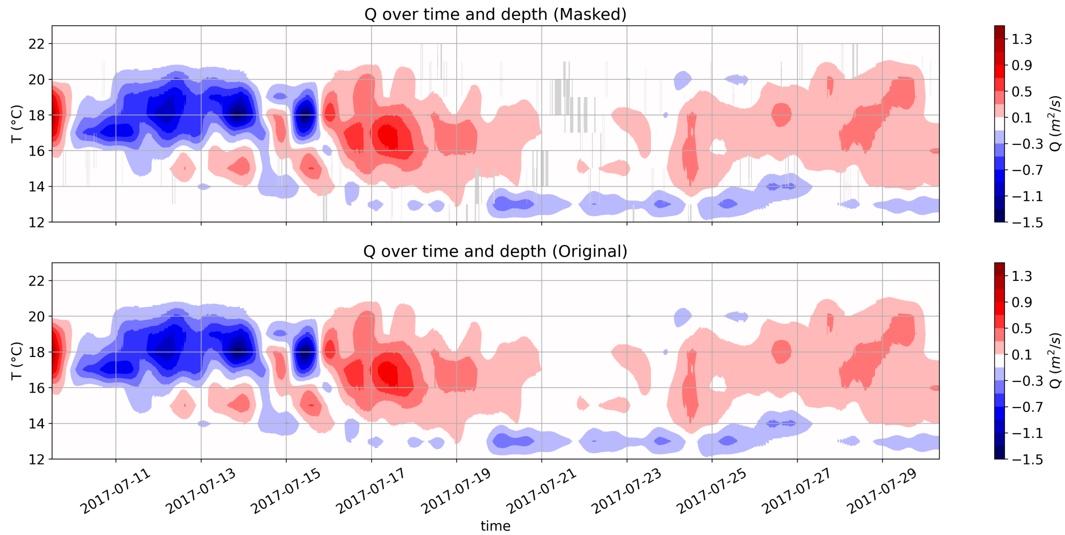

Figure 6: (Top) Eulerian transport masked when the error is more than 50 % of the value (bottom) Original eulerian transport

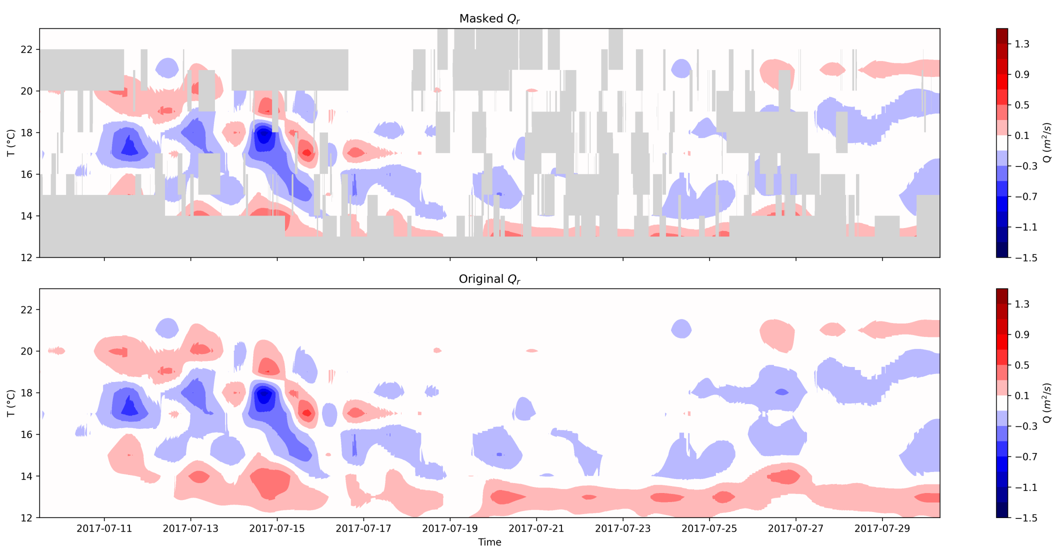

Figure 7: (Top) Residual transport masked with the combined mask from total and eulerian analysis. (bottom) Original residual transport

4 Conclusion and acknowledgement

Most of the variance in the measurements was due to temperature. The main source of uncertainty in residual transport was the uncertainty in total transport. Masking the value did not remove the main structures observed and described in the paper. The vertical profile of residual transport remained unchanged. The analysis confirmed the results observed in the manuscript.

We would like to express our sincere gratitude to Louis Marié (Ifremer LOPS) for his invaluable assistance in addressing the impact of filters on uncertainty measurements. We also acknowledge the use of the GPT-4o model in developing the code.

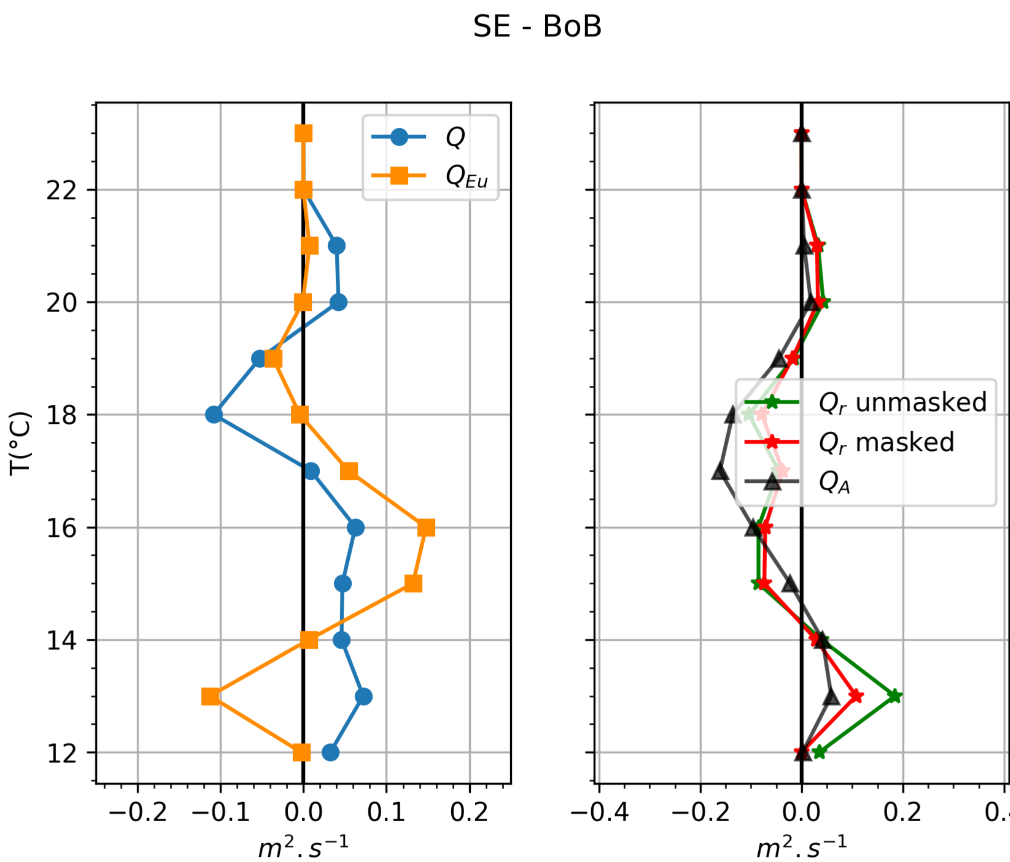

Figure 8: Vertical profile of original residual transport (black) and masked residual transport (red) averaged on the total time series

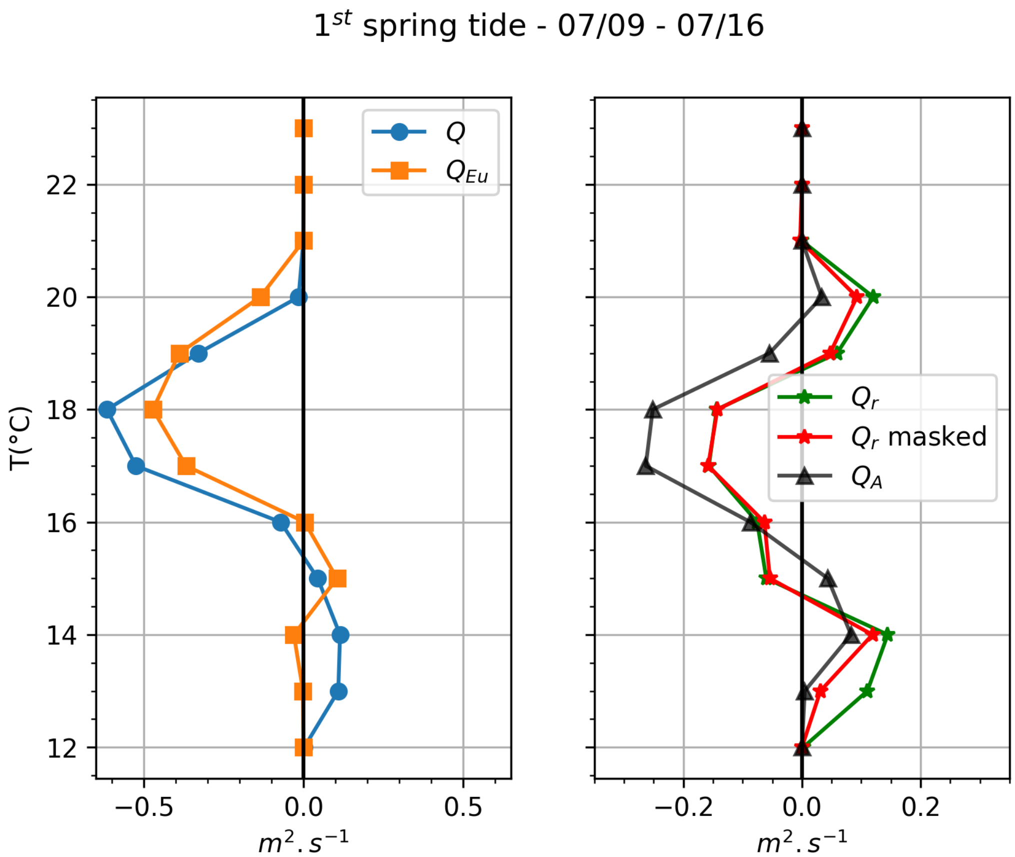

Figure 9: Vertical profile of original residual transport (black) and masked residual transport (red) averaged on the 1st spring tide

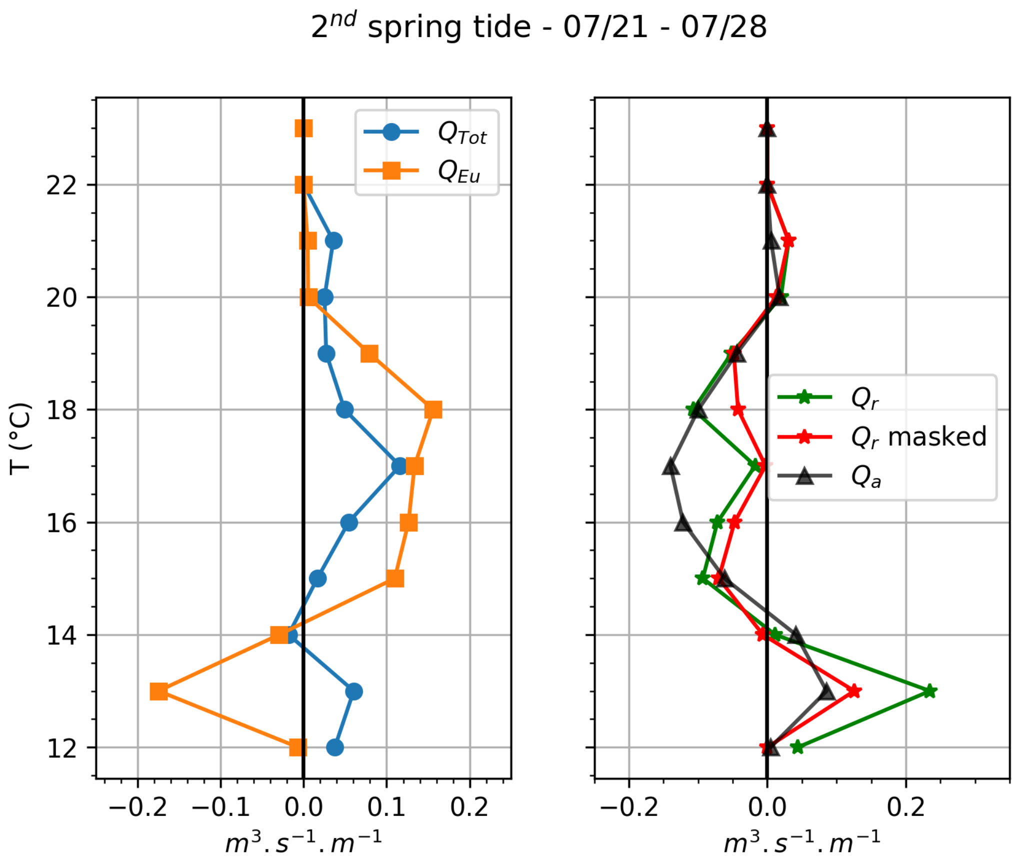

Figure 10: Vertical profile of original residual transport (black) and masked residual transport (red) averaged on the 1st spring tide