

**REFEREE REPORT FOR “CAN WE DEFINE CLIMATE BY MEANS OF AN
ENSEMBLE? A TALE OF TIME SCALES OF CONVERGENCE” BY G. DRÓTOS
AND T. BÓDAI**

1. EVALUATION

The manuscript addresses an interesting and important question: how to interpret ensemble-based descriptions of climate under time-dependent forcing. I found the discussion of the snapshot/pullback viewpoint, convergence time scales, and ensemble initialization particularly valuable. These aspects seem especially promising,

My main reservation concerns instead the way the manuscript presents its conceptual novelty. As I already tried to indicate in the first round, perhaps not clearly enough, I do not agree with the claim that a framework for defining climate in a non-autonomous setting is currently absent. In my view, such a framework already exists in the literature, and I do not see, at present, how the framework proposed here extends it at the conceptual level. For this reason, I remain unconvinced by the novelty claim as presently formulated. At the same time, this does not mean that I find the paper uninteresting. On the contrary, I think the discussion of time scales and the proposed initialization perspective are worthwhile. For this reason, my recommendation to the editor remains *revision*. The main reason for this recommendation is explained in detail in Section 2. The final section of this report lists minor comments and typos.

2. MAJOR COMMENT

A central concern raised in my first report remains, in my opinion, unresolved. Let me first apologise if my original report was not sufficiently explicit on this point. I deliberately tried to formulate the comment cautiously, because the relevant works include results obtained by myself and by some of my collaborators, and I did not want to sound self-referential or arrogant. However, since this issue is central to the authors’ stated claim of novelty, I now feel it is important to explain my concern more clearly.

My understanding is that the main aim of the manuscript is to provide a framework for defining climate in a non-autonomous setting. In the authors’ presentation, the difficulty is that climate is described through the distribution of ensemble members, but this identification is said to lack a careful mathematical framework. However, a rigorous mathematical framework for non-autonomous stochastic systems that leads essentially to this same notion of climate already exists; see [1, Sec. II.E], which was already cited in my first report on this paper; a much more extended framework, developed to try to give a mathematical definition of weather and climate, can be found in [2]. For this reason, I do not think it is accurate to state that no such framework or construction has been provided to date.

More specifically, the revised manuscript states:

“One assumption behind these interpretations is that climate, as a statistical description of weather, is represented by the distribution of the ensemble members; if one likes, this may be regarded as a definition of climate [...] However, no careful justification for this assumption has been provided to date. In fact, not even a framework or construction has been provided that would allow for such a justification. The aim of the present article is to provide one; i.e., not a justification, but a way of analysing the system that makes justification qualitatively well-posed.”

To my understanding, this statement is too strong, and indeed incorrect as written. A non-autonomous framework for defining climate is already available, and I had tried to point this out in my first report. Therefore, if the authors wish to maintain a novelty claim, they should explain clearly and explicitly in what sense their proposal goes beyond the already existing framework.

For the sake of clarity, let me briefly recall a non-autonomous framework for the definition of climate that is already available. Consider a non-autonomous stochastic system on $\mathbb{R}^d$ of the form

$$dX_t = b(q(t), X_t) dt + \sigma(t, X_t) dW_t, \quad t \in \mathbb{R},$$

where $q(t)$ is a slowly varying forcing, for instance representing the atmospheric CO₂ concentration. For every $s \leq t$, let $X_t^{s,x}$ denote the solution starting from x at time s . This induces a two-parameter Markov evolution operator on observables,

$$P_{s,t}\varphi(x) := \mathbb{E}[\varphi(X_t^{s,x})],$$

and, dually, an evolution on probability measures,

$$P_{s,t}^*\nu(A) := \int_{\mathbb{R}^d} \mathbb{P}(X_t^{s,x} \in A) \nu(dx).$$

The family $(P_{s,t}^*)_{s \leq t}$ satisfies the cocycle relations

$$P_{r,t}^* P_{s,r}^* = P_{s,t}^*, \quad P_{s,s}^* = \text{Id}.$$

This already provides a precise mathematical interpretation of the “distribution of the ensemble members” used by the authors. Indeed, if one starts at time s from an ensemble of initial conditions distributed according to some law ν , then the distribution of the ensemble members at time t is exactly the pushed-forward measure $P_{s,t}^*\nu$. In other words, the ensemble description fits naturally into the non-autonomous Markov framework.

Within this setting, the natural non-autonomous analogue of an invariant measure is a family of probability measures

$$(\mu_t)_{t \in \mathbb{R}}$$

such that

$$P_{s,t}^* \mu_s = \mu_t, \quad \forall s \leq t.$$

This is precisely the notion of a time-dependent invariant measure. In weak form, it reads

$$\int_{\mathbb{R}^d} \varphi d\mu_t = \int_{\mathbb{R}^d} P_{s,t}\varphi d\mu_s, \quad \forall \varphi \in C_b(\mathbb{R}^d), \quad \forall s \leq t.$$

Thus, in a non-autonomous setting, the mathematically natural object describing climate is not a single probability measure, but rather a *family* of time-dependent probability laws $(\mu_t)_{t \in \mathbb{R}}$ transported by the dynamics.

Assuming uniqueness of such a family, one obtains a mathematically precise candidate for “the climate” in the non-autonomous setting. Of course, whether such uniqueness holds is a delicate and highly nontrivial matter, and indeed this is one of the genuinely interesting mathematical issues in the subject. But this is precisely the point: the relevant framework already exists, and the main difficulty is not the absence of a framework, but rather the analysis of uniqueness of the relevant family and, in cases where uniqueness fails, the problem of providing a selection criterion to identify what “the climate” should be.

For this reason, my original concern remains. The present manuscript may well contain useful discussion, examples, or heuristics, but I do not think it should present as novel the mere existence of a framework for defining climate.

3. MINOR COMMENTS AND TYPOS

- Page 1, line 1. “It is hardly questioned today” seems a bit strange. Maybe reformulating as “It is now widely accepted” or “It is rarely questioned today” could be better.
- Page 1, lines 2–3. “It is also widely accepted that any evolution observed within a few decades after initialization is not relevant to climate” is a very strong sentence, and I would say that it is generally wrong in my opinion. Indeed, climate sometimes is seen as the long-time behavior of weather, and thus exactly the asymptotic regime is the one that it is looked for, and not what happens shortly after initialization.
- Page 2, line 29. “defintion” should be “definition”.
- Page 2, line 34. “analysi” should be “analysis”.
- Page 3, lines 57–58. “is given is Appendix C” should be “is given in Appendix C”.

- Page 3, line 82. “constucting” should be “constructing”.
- Page 7, line 190. “possibily” should be “possibility”.
- Page 9; the subsection title “What we can conclude about” seems a bit strange in English. I suggest either to remove “about” or to add an object to the sentence.
- Page 9, line 260. “coparison” should be “comparison”.
- Page 10, line 269. “In case the predictable context have” should be “In case the predictable context has”.
- Page 12, line 337. “assesement” should be “assessment”.

REFERENCES

- [1] G. Del Sarto and F. Flandoli. A non-autonomous framework for climate change and extreme weather events increase in a stochastic energy balance model. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 34(9):093122, 09 2024.
- [2] Franco Flandoli and Elisa Tonello. Towards a definition of climate and weather, 2019. Lecture notes. Available at: http://users.dma.unipi.it/flandoli/IHP_contribution_Flandoli_Tonello_v3.pdf.