A Nonlinear Generalized Boussinesq Equation ((2+1)-D) for Rossby-Khantadze Waves

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Abstract

In the following paper, we investigate nonlinear Rossby-Khantadze waves, by taking into account inhomogeneity in the geomagnetic field and angular velocity – due to Earth's differential rotation. Considering the system to be weakly nonlinear, we make use of perturbation theory to derive a new (2+1)–D generalized form of Boussineq equation. We evaluate the obtained equation by using the qualitative theory of ordinary differential equations (ODEs), and bifurcation theory of dynamical systems. The obtained numerical results show that the aforementioned solutions of the traveling waves correspond to Rossby-Khantadze solitons.

Keywords: Generalized Boussinesq model equation; nonlinear Rossby-Khantadze waves; nonlinearity; sheared zonal flow; traveling wave solutions; dynamical analysis

1. Introduction

Numerous investigations conducted by ground-based and satellite observations gives proof of the presence of zonal flows in various regions of the terrestrial atmosphere (Pedlosky, 1987). This is based on the fact of the non-uniform heating caused by the sun in the Earth's atmospheric regions. These ultra-low frequency (ULF) perturbations in ionosphere E and F regions occur due to the sheared flow with nonhomogeneous velocities along the meridians (Shukla et al., 2003; Onishchenko et al. 2004; Satoh, 2004; Kaladze et al., 2007; Kaladze et al., 2008). The effects of sheared flow on the properties of linear and non-linear waves in the ionosphere and under suitable conditions they give rise various nonlinear structures like zonal flows (ZFs), vortices, and solitons etc.

Sheared Rossby waves have gained much attention due to their prominent role in the global atmospheric circulation. Such slow long-period planetary waves have phase velocities ~ 1–100 m/s, which is around the velocity of the ionospheric (local) winds. Their frequency is in the order of 10^{-4}–10^{-6} s^{-1} at middle latitudes, whereas the period is at 2 h to 14 days. Besides the slow Rossby waves, fast perturbations also exist in the moderate-latitude ionosphere, which are created by the latitudinal inhomogeneity of the Earth'magnetic field and the Hall effect. The first theoretical evidence of such large-scale EM perturbations in the ionospheric E- and F-regions was made by Khantadze (Khantadze, 1986, 1999, and 2001), and in this work, he differentiated between fast and slow large-scale EM planetary waves. Consequently, fast EM planetary waves were named Khantadze modes, and these waves were observed by Soyuz and Proton rockets (Burmaka et al., 2006) at the middle latitude and by the world network of ionospheric and magnetic observations (Sharadze et al., 1988; Sharadze, 1991; Alperovich and Fedorov, 2007). Detailed analysis of such planetary EM waves was carried out by Kaladze et al., (2003, 2004) and Khantadze et al., (2010).

The spatial inhomogeneity along the meridians, of both the ambient magnetic field and the Coriolis force parameter generates coupled modes called the Rossby-Khantadze (RK) waves (see e.g., Kaladze et al., 2011). The existence of sheared RK electromagnetic vortices in the E region of Earth's ionosphere is studied thoroughly by Kaladze et al. (Kaladze et al., 2011; 2012; 2013a, 2013b; 2014). In those works, the authors have not only shown the self-organization of coupled RK waves into dipolar solitary vortices, but also predicted the generation of magnetic field in the system due to the aforementioned waves. More recently, different nonlinear processes having relevance to the generation of zonal flows (sheared) by Rossby waves are considered. The key factor for the generation of zonal flows in short-wavelength Rossby waves is Reynold's stress (Shukla et al., 2003 and Onishchenko et al. 2004). Rossby waves causes the generation of zonal flows in E layer of ionosphere (Kaladze et al., 2007). Such nonlinear Rossby wave structures splits into various parts, and this splitting is dependent on zonal flow's energy (Kaladze et al., 2008). Along with the analytical side, numerical work on RK waves with sheared zonal flows in the E layer of the ionosphere is worked out as well (Futatani et al., 2013; 2015). In these works, breaking of vortices is studied, where the energy is transferred from sheared flow into these multiple pieces (daughter waves). It is worth noting that equatorially propagating Rossby solitary waves by sheared flows have been predicted and discussed (Qiang et al., 2001) and their presence was confirmed through observations by *Freja and Viking satellites* (Bostrom, 1992; Lindqvist et al., 1994; Dovner et al. 1994; Qiang et al., 2001). In Jian et al., (2009)'s work, the authors studied the nonlinear propagation of sheared Rossby waves in stratified neutral fluids and obtained modified Korteweg-de Vries (MKdV) equation, which is characterised by a cubic nonlinearity. Kahlon et al., (2024), investigated the MKdV equation with cubic nonlinearity for Rossby-Khantadze nonlinear waves.

Zonal flow's generation in the ionosphere's E region by Rossby-Khantadze waves having magnetic field is also shown (Kaladze et al., 2012, Kahlon and Kaladze, 2015), where it has been predicted that there exists a possibility of the magnetic field generation, at the strength of $10^3\ nT$. Kaladze et al., (2019) studied the nonlinear interaction of magnetized Rossby waves with inclusion of zonal flows in the Earth's ionospheric E-layer, in which MKdV solitons were obtained. The possibility of planetary Rossby wave's existence in the dynamo E-area of weakly ionised ionosphere was predicted by Forbes, (1996). It was also shown that the theoretical work corresponds with the experimental interpretations. Much later, Vukcevic and Popovic, (2020) investigated the possibility of soliton formation at different latitudes in ionosphere. Direct observed data of satellites of such soliton structures from Earth's surface are discussed.

In the context of shallow water waves and in plasmas, several researchers have extended the KdV and MKdV equations to higher dimensions, in order to obtain realistically accurate results. Kadomstev-Petviashvilli (KP) equation and Zakharov-Kuznetsov (ZK) equation have gained much attention over the years (Vukcevic et al., 2020, Kadomstev et al., 1970, Groves et al., 2008, Infeld et al., 2000 and Zakharov et al., 1974). Both of these equations are (2+1) – dimensional in nature and are very useful in plasma models (as one can get almost complete information by taking parallel and perpendicular dimension into account). While modelling shallow water waves, Johnson (1996) investigated a (2+1) – dimensional Boussinesq equation for gravitational surface waves. Making use of the surface wave theory, Mitsotakis (2009) investigated the Boussinesq equation and simulated the propagation of such waves. In the context of geophysics, many authors (Gottwald, 2003, Yang et al., 2016, Yang et al., 2018, Zhang et al., 2017a, Zhang et al., 2017b) have investigated ZK equation by considering nonlinear Rossby waves from the quasi-geostropic potential vorticity equation. Although, the Boussinesq equation in the study of the nonlinear Rossby-Khantadze waves is not reported so far.

It is very useful to find exact solutions of nonlinear partial differential equations. Several techniques have recently been used to find such solutions, including but not limited to the method of trigonometric series (Ma and Fuchssteiner, 1996), the method of tan(ϕ(ξ)/2)-expansion (Manafian and Aghdaini, 2016), sine-cosine method (Wazwaz, 2005), Wronskian method (Ma and You, 2005), separation of variables approach (Lin and Zhang, 2007), Septic B-spline method (El-Danaf, 2008), the transformative functional rational method (Ma and Lee, 2009), the symmetry algebric method (Ma and Chen, 2009), the homotopy perturbation method (Ganji et al., 2009), the modified method of mapping and the extended mapping method (Zhang et al., 2010), qualitative theory of the bifurcation method and dynamical systems (Zhang et al., 2011), the multiple exp-function method (Ma and Zhu, 2012), the modified (G'/G)- method of expansion (Miao and Zhang, 2011), the modified trigonometric function series method (Zhang et al, 2011) infinite series method and Jacobi elliptic functional method (Zhang et al., 2012, Tasbozan et al., 2016), RBF approximation method (Uddin, 2014) (G′ /G−1/G)-expansion method (Zhang et al., 2014), Hirota bilinear method (Lü et al., 2016a, Ma et al., 1996, 2016, Lü and Ma, 2016b), lattice Boltzmann method (Wang and Yan, 2016) to name a few.

In the present work, for the partially ionized and conducting ionospheric E plasma we consider the stream-function and evolution of geomagnetic field for electromagnetic Rossby-Khantadze (RK) waves, which provides novelty to this work. In Sec. 2, we set the system of initial equations. In Se. 3, by using the reductive perturbation technique we obtain the linear dispersion equation from the lowest order of ε. In Sec. 4, we derive the Boussinesq equation for Rossby-Khantadze nonlinear waves from our considered set of equations. In Sec. 5, we study the dynamical analysis of the Boussinesq equation and get its exact traveling solitary solutions. In second last section, discussions are presented in Sec. 6. The summary and conclusion are made in Sec. 7.

2. Mathematical Preliminaries

We start by considering a weakly ionised system, as is characteristic to ionospheric plasmas. Here ions, electrons and neutral particles are embedded in a nonhomogeneous geomagnetic field, $\boldsymbol{B}_0(y) = \left(0, B_{0y}(y), B_{0z}(y)\right)$, and the angular velocity is taken into consideration as, $\boldsymbol{\Omega}(y) = \left(0, \Omega_{0y}(y), \Omega_{0z}(y)\right)$. We consider the 2D incompressible motion i.e., $\mathbf{v} = (u, \mathrm{v}, 0)$, which represents the velocity of the neutral gas where $u = -\frac{\partial\psi}{\partial y}$, $\mathrm{v} = \frac{\partial\psi}{\partial x}$ and $\psi(x, y, t)$ is the stream function.

We make use of a slab geometry with zonally x, latitudinally y, and locally vertical direction along z axis. Furthermore, the behavior of the nonlinear Rossby-Khantadze sheared electromagnetic waves is expressed by the 2D system of equations (e.g., Kaladze et al., 2011, Kaladze et al., 2014, Song et al. 2009; Liü et al. 2019) as given below:

$$\begin{cases} \frac{\partial\Delta\psi}{\partial t} + \beta\frac{\partial\psi}{\partial x} + \mathrm{J}(\psi, \Delta\psi) - \frac{1}{\mu_0\rho}\beta_B\frac{\partial h}{\partial x} = -\mu\,\Delta\psi + Q\ , & \text{(1a)} \\ \frac{\partial h}{\partial t} + \mathrm{J}(\psi, h) + \beta_B\frac{\partial\psi}{\partial x} + c_B\frac{\partial h}{\partial x} = 0\,, & \text{(1b)} \end{cases}$$

Here in the equation (1a) we consider vorticity as, $\zeta_z = \boldsymbol{e}_z \cdot \nabla \times \mathbf{v} = \Delta\psi = \nabla^2\psi = (\partial_x^2 + \partial_y^2)\,\psi$, from momentum equation of single fluid where $\beta = \frac{\partial f}{\partial y} = \frac{2\partial\Omega_{0z}}{\partial y}$ is the latitudinally inhomogenous angular velocity with $f = f_0 + \beta(y)y$. Here, $f_0 = 2\Omega_{0z} = 2\Omega_0\sin\phi_0$. While the parameter $c_B = \beta_B/en\mu_0$ with $\beta_B = \frac{\partial B_{0z}}{\partial y}$, being the nonhomogeneity in the geo-magnetic field, n is charged particles's number density, $J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$ is the Jacobian. Equation (1b) shows the z-component of perturbed magnetic field. Note that lesser contribution of charged particles (in comparison of neutrals) plays their role (Kaladze, et al. 2013a, 2013b) in the inductive current.

To solve the set of equation (1), we use the boundary condition

$$\left.\frac{\partial \psi}{\partial x}\right|_{y=y_1} = \left.\frac{\partial \psi}{\partial x}\right|_{y=y_2} = 0\,, \qquad (2)$$

representing the flow along the meridional directions, as explained by Pedlosky (1987) and Satoh (2004).

By introducing the following dimensionless parameters, we can express Eq. (1) in dimensionless form

$$(x, y) = L_\circ(x^*, y^*), \quad \psi = L_0 U_0\, \psi^*, \; t = \frac{L_\circ}{U_\circ} t^*\,, \beta = \frac{U_0}{L_0^2}\beta^*\,,\; \mu = \frac{U_0}{L_0}\mu^*\,, Q = \frac{U_0^2}{L_0^2}Q^* \qquad (3)$$

Here asterisk denotes the dimensional variables, which are further dropped in the equation below. Here L_0 is the zonally length; H is a vertically length and U_0 is the velocity. Finally, Eq. (1) takes the form

$$\begin{cases} \frac{\partial \Delta\psi}{\partial t} + \beta\frac{\partial \psi}{\partial x} + J(\psi, \Delta\psi) - \frac{1}{\mu_0\rho}\beta_B\frac{\partial h}{\partial x} = -\mu\,\Delta\psi + Q\,, \\ \frac{\partial h}{\partial t} + J(\psi, h) + \beta_B\frac{\partial \psi}{\partial x} + c_B\frac{\partial h}{\partial x} = 0\,, \end{cases} \qquad (4)$$

with the following boundary conditions

$$\left.\frac{\partial \psi}{\partial x}\right|_0 = \left.\frac{\partial \psi}{\partial x}\right|_1 = 0\;. \qquad (5)$$

3. Perturbation and weakly nonlinear approach

In this section, to investigate the non-linear Boussinesq equation describing the solitary Rossby-Khantadze waves. Here we make use of multiple scale and asymptotic expansion approach.

The expression

$$\psi = \bar{\psi}\,(y) + \psi'(x, y, t), \qquad (6)$$

describes the stream function with $\bar{\psi} = -\int_0^y [\bar{u}\;(s) - c_0]\, ds$ representing the background stream function where c_0 is a constant, $\bar{u}$(y) refers to background flow, and ψ' is the disturbance in stream function. While the perturbed magnetic field is:

$$h = \varepsilon h', \qquad (7)$$

Thus, the set of equation (4) can be expressed as

$$\begin{cases} \left(\frac{\partial}{\partial t} + (\bar{u} - c_0)\ \frac{\partial}{\partial\, t}\right) \Delta\ \ \psi' + p(\mathrm{y}) \frac{\partial \psi'}{\partial t} + \mathrm{J}(\psi', \Delta\psi') - \frac{\beta_B}{\mu_0\, \rho} \frac{\partial h'}{\partial x} \ \ = \ -\mu \Delta^2 \psi' \\ \frac{\partial h'}{\partial t} + \varepsilon \mathrm{J}(\psi', h') + (U(y) - c_0) \frac{\partial h'}{\partial x} + \beta_B \frac{\partial \psi'}{\partial x} + c_B \frac{\partial h'}{\partial x} = 0. \end{cases} \qquad (8)$$

where $p(y) = \ (\beta(y) y - \bar{u}')'$.

By applying the multiple scale approach we find the following stretched coordinates,

$$X = \varepsilon^{(1/2)} x, \quad Y = \varepsilon(y - c_1 t) \quad T = \varepsilon\, t\,, \qquad (9)$$

in the comoving frame of reference the differential operator can be expressed in the following manner

$$\frac{\partial}{\partial x} = \varepsilon^{(1/2)} \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial y} + \varepsilon \frac{\partial}{\partial Y}\ , \frac{\partial}{\partial t} = \varepsilon \frac{\partial}{\partial T} - c_1\, \varepsilon \frac{\partial}{\partial Y}\,. \qquad (10)$$

The perturbed stream function and perturbed magnetic fields are expanded as

$$\begin{cases} \psi' \ = \varepsilon \psi_1 + \varepsilon^{(3/2)} \psi_2 + \varepsilon^2 \psi_3 + \cdots, \\ h' \ = \varepsilon h_1 + \varepsilon^{(3/2)} h_2 + \varepsilon^2 h_3 + \cdots. \end{cases} \qquad (11)$$

Using (9), (10) and (11) into equation (7) we get from the lowest order i.e. $O(\varepsilon^{3/2})$:

$$\begin{cases} (\bar{u} - c_0) \frac{\partial}{\partial X} \left(\frac{\partial^2 \psi_1}{\partial y^2}\right) + p(y) \frac{\partial \psi_1}{\partial X} - \frac{\beta_B}{\mu_0 \rho} \frac{\partial}{\partial X}(h_1) = \ 0, \\ (\bar{u} - c_0 + c_B) \frac{\partial h_1}{\partial X} \ + \ \beta_B \frac{\partial}{\partial X}(\psi_1) = \ 0, \end{cases} \qquad (12)$$

Next order $O(\varepsilon^2)$ gives

$$\begin{cases} (\bar{u} - c_0) \frac{\partial}{\partial X} \left(\frac{\partial^2 \psi_2}{\partial y^2}\right) + p(y) \frac{\partial \psi_2}{\partial X} = -\frac{\beta_B}{\mu_0 \rho} \frac{\partial}{\partial X}(h_1) - \left(\frac{\partial}{\partial T} - c_1 \frac{\partial}{\partial Y}\right) \frac{\partial^2 \psi_1}{\partial y^2}, \\ \left(\frac{\partial}{\partial T} - c_1 \frac{\partial}{\partial Y}\right) h_1 + (\bar{u} - c_0 + c_B) \frac{\partial h_2}{\partial X} + \beta_B \frac{\partial \psi_2}{\partial X} \end{cases} \qquad (13)$$

From the second set of equation (13), we get

$$\frac{\partial h_2}{\partial X} = \frac{-\beta_B}{\bar{u} - c_0 + c_B} \frac{\partial \psi_2}{\partial X} - \frac{1}{(\bar{u} - c_0 + c_B)} \left(\frac{\partial}{\partial T} - c_1 \frac{\partial}{\partial Y}\right) h_1 \qquad (14)$$

Next order $O(\varepsilon^{5/2})$ gives

$$\begin{cases} (\bar{u} - c_0) \frac{\partial}{\partial X} \left(\frac{\partial^2 \psi_2}{\partial y^2}\right) + p(y) \frac{\partial \psi_3}{\partial X} - \frac{\beta_B}{\mu_0 \rho} \frac{\partial h_3}{\partial X} = -(\bar{u} - c_0) \frac{\partial^3 \psi_1}{\partial X^3} - 2(\bar{u} - c_0) \frac{\partial^3 \psi_1}{\partial X \partial Y \partial y} \\ \qquad - \left(\frac{\partial}{\partial T} - c_1 \frac{\partial}{\partial Y}\right) \frac{\partial^2 \psi_2}{\partial y^2} \ - \frac{\partial \psi_1}{\partial X} \frac{\partial^3 \psi_1}{\partial y^3} \ + \frac{\partial \psi_1}{\partial y} \frac{\square\square}{\partial X} \left(\frac{\partial^2 \psi_1}{\partial y^2}\right), \\ \left(\frac{\partial}{\partial T} - c_1 \frac{\partial}{\partial Y}\right) h_2 \ + \beta_B \frac{\partial \psi_3}{\partial X} = \ (\ \bar{u} - c_0 + c_B\ ) \frac{\partial h_3}{\partial X} + \frac{\partial \psi_1}{\partial X} \frac{\partial h_1}{\partial Y} - \frac{\partial \psi_1}{\partial Y} \frac{\partial h_1}{\partial X}\ . \end{cases}$$

(15)

Eq. (15b) gives

$$\frac{\partial h_3}{\partial X} = -\left(\frac{\partial}{\partial T} - c_1 \frac{\partial}{\partial Y}\right) h_2 \quad + \beta_B \frac{\partial \psi_3}{\partial X} + (\bar{u} - c_0 + c_B\ ) + \frac{\partial \psi_1}{\partial X}\frac{\partial h_1}{\partial Y} - \frac{\partial \psi_1}{\partial Y}\frac{\partial h_1}{\partial X} \qquad (16)$$

Assume that Eq. (12) has the solution

$$\psi_1 = A(X, Y, T)\ \varphi_1(y), \qquad (17)$$

Thus, from Eqs. (12) and (20) we get the following linear dispersion relation

$$\varphi_1'' \quad + \frac{p(y)}{(\bar{u}\ - c_0)} \varphi_1(\mathrm{y}) + \frac{\beta_B^2}{\mu_0\ \rho} \frac{1}{(\bar{u}\ - c_0)(\bar{u} - c_0 + c_B)} \varphi_1 = 0, \qquad (18)$$

and from the boundary condition given by Eq. (5) we get

$$\varphi_1(0)\ = \varphi_1\,(1) \quad = 0\ . \qquad (19)$$

The obtained Eq. (18) is the Rayleigh-Kuo equation describing the Rossby-Khantadze waves. By solving Eq. (12) simultaneously and the coefficients are locally constant and $U(y) = const.$, we get the following dispersion equation

$$\left(\left(\frac{\omega}{k_x} - U(y)\right) k_\perp^2 + p(y)\right)\left(\frac{\omega}{k_x} - U(y) - c_B\right) - \alpha\ = 0,\ , \qquad (20)$$

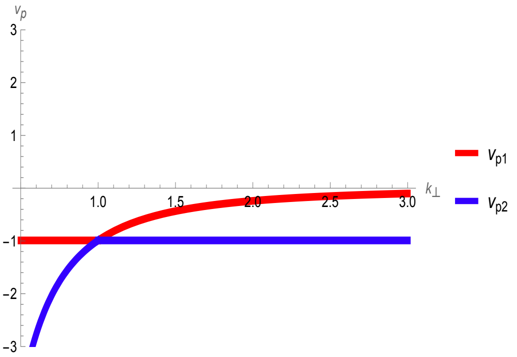
where $k_\perp^2 = k_x^2 + k_y^2$ and $\alpha\ = \frac{\beta_\beta^2}{\mu_0 \rho}$. Eq. (20) describes the dispersion equation of sheared Rossby-Khantadze waves. In the absence of α we get two solutions, one independent solution of Rossby waves and the second one for Khantadze waves.

By introducing the dimensionless variables $\frac{\omega}{k_x d} \Rightarrow \mathrm{v}_p$ and $\frac{k_\perp^2 d}{a} \Rightarrow k_\perp^2$ (with d = $\frac{b}{en\mu_0}$, a = $\frac{2\Omega_0}{R}$ and $b = \frac{2b_{eq}}{R}$) then we rewrite the dispersion relation (20)

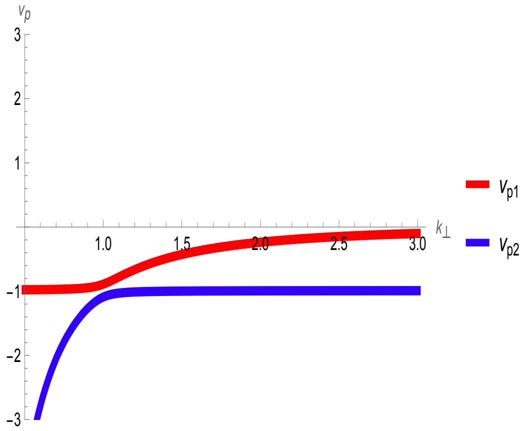
$$\mathrm{v}_p \ = -U + \frac{1}{2k_\perp^2} \cos\lambda_0 \left(-k_\perp^2 - 1 \pm \sqrt{(1 - k_\perp^2)^2 + k_\perp^4 \alpha_0}\right). \qquad (21)$$

Here $\alpha_0 = \frac{ben}{ap} = \frac{x}{|c_B|\beta}$. For the E-ionosphere layer, the parameters have the following $B_{eq} \cong 0.5\times 10^{-4}$T, $2\Omega_0 \cong 10^{-4}\ \frac{rad}{s}$, $\frac{n}{N} \sim 10^{-8}$- 10^{-6}, ρ = (10^{-7} -10^{-8}) kgm^{-3}, the parameter $\alpha_0 = (10^{-2} - 1)$ (Kaladze et al., 2011).

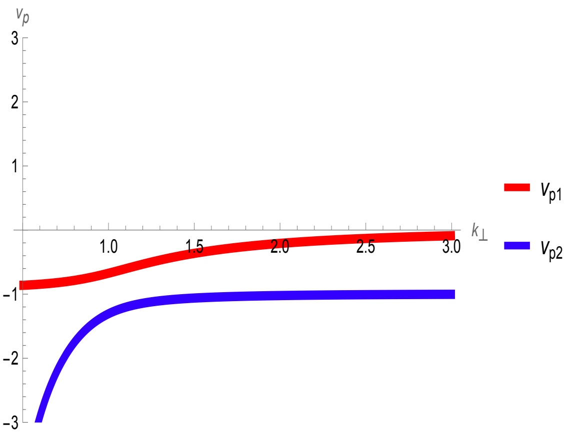
Fig. 1, represents the phase velocity v_p of the obtained coupled Rossby-Khantadze waves is plotted with wave number $k_\perp$ by varying α_0. Red curve v_{p1} is for “+” and blue v_{p2} is for “–” signs before the radicand in Eq. (21).



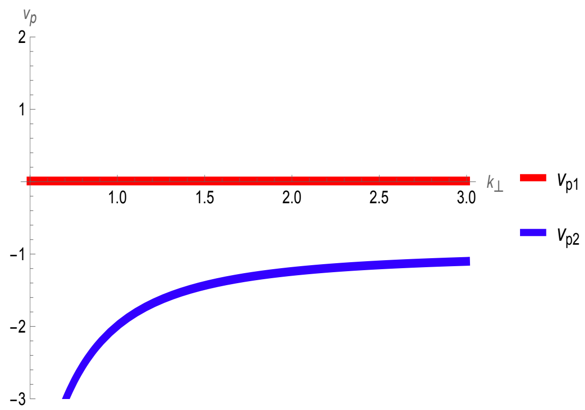
$a)\ \alpha_0 = 0$



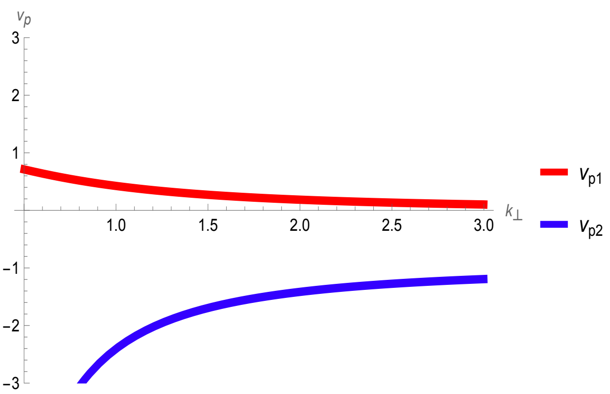
$b)\ \alpha_0 = 0.01$



$c)\ \alpha_0 = 0.1$



$d)\ \alpha_0 = 1$



e) $\alpha_0 = 2$

Fig.1 Normalized phase velocity vs normalized wave of coupled Rossby-Khantadze waves for $\lambda_0 = \pi/4$ is shown .

4 Derivation for the nonlinear Boussinesq Equation

In this section, by taking into account the separation of variables techniques we will derive the nonlinear Boussinesq Equation describing the solitary nonlinear structures.

Further, we assume that Eq. (13) has the solution

$$\psi_2 = \psi_{21} + \psi_{22}, \tag{22}$$

with

$$\psi_{21} = B_1(X,Y,T)\,\varphi_{21}(y), \quad \psi_{22} = B_2(X,Y,T)\,\varphi_{22}(y), \tag{23}$$

By using the separation of variables approach and using Eq. (22) and (23) in Eq. (13) we obtain

$$(\bar{u} - c_0)\frac{\partial B_2}{\partial X}\varphi_{22}'' + \left(p(y) + \frac{\beta_B^2}{\mu_0\,\rho(c_B+\bar{u}-c_0)}\right)\frac{\partial B_2}{\partial X} = c_1\frac{\partial A}{\partial y}\varphi_1'' - \frac{\alpha c_1}{(c_B+\bar{u}-c_0)^2}\frac{\partial A}{\partial Y}\varphi_1 \tag{24}$$

Put

$$\frac{\partial B_1}{\partial X} = \frac{\partial A}{\partial T}, \quad \text{and} \quad \frac{\partial B_2}{\partial X} = \frac{\partial A}{\partial Y}. \tag{25}$$

From Eq. (24) we get

$$\varphi_{21}'' + q(y)\varphi_{21} = -\frac{\varphi_1''}{\bar{u}-c_o} + \gamma\varphi_1 \tag{26}$$

$$\varphi_{21}(0) = \varphi_{21}(1) = 0. \tag{27}$$

with $q(y)$ and γ are given by $q(y) = \left. p(y) + \frac{\beta_B^2}{\mu_0\,\rho}\cdot\frac{1}{(\bar{u}-c_0)(u-c_0+c_B)} \middle/ (\bar{u}-c_o) \right.$; $\gamma = \frac{\beta_B^2}{\mu_0\,\rho}\frac{1}{(\bar{u}-c_0)(\bar{u}-c_0+c_B)^2}$.

And

$$\varphi_{22}'' + q(y)\varphi_{22} = \frac{c_1\varphi_1''}{\bar{u}-c_o} - c_1\gamma_1\varphi_1, \tag{28}$$

The boundary conditions are given by

$$\varphi_{22}(0) = \varphi_{22}(1) = 0. \tag{29}$$

From Eqs. (26) and (28) we have

$$\varphi_{22} = -c_1\,\varphi_{21} \tag{30}$$

In order to arrive at the evolution equation, we use Eqs. (20), (25) and (26) and substitute into Eq. (15)

$$(\bar{u}-c_0)\frac{\partial}{\partial X}\left(\frac{\partial^2\Psi_3}{\partial y^2}\right)+p(y)\frac{\partial\Psi_3}{\partial X}=F, \qquad (31)$$

where

$$F=-\varphi_{21}''\frac{\partial^2 B_1}{\partial T\partial X}-\varphi_{22}''\frac{\partial^2 B_2}{\partial T\partial X}+c_1\varphi_{21}''\frac{\partial^2 B_1}{\partial Y\partial X}+c_1\varphi_{21}''\frac{\partial^2 B_2}{\partial Y\partial X}(1+\frac{\alpha}{(c_B+\bar{u}-c_0)^2}-(\bar{u}-c_0)\varphi_1\frac{\partial^4 A}{\partial X^4}-$$

$$2(\bar{u}-c_0)\varphi_1\frac{\partial^3 A}{\partial X^2 Y}-(\varphi_1\varphi_1'''-\varphi_1'\varphi_1'')2\mathrm{A}\frac{\partial^2 A}{\partial X^2}+\frac{\alpha}{(c_B+\bar{u}-c_0)^3}(\frac{\partial^2}{\partial T^2}-2c_1\frac{\partial^2}{\partial T\partial Y}+c_1^2\frac{\partial^2}{\partial Y^2})\mathrm{A}\varphi_1$$

(32)

Eq. (31) is the evolution equation for Ψ_3 and we obtain its solution by multiplying by $\varphi_1(y)$ and then integrating over y to get

$$\int_0^1\frac{\varphi_1(y)}{\bar{u}-c_0}\text{-}[-\varphi_{21}''\frac{\partial^2 B_1}{\partial T\partial X}-\varphi_{22}''\frac{\partial^2 B_2}{\partial T\partial X}+c_1\varphi_{21}''\frac{\partial^2 B_1}{\partial Y\partial X}+c_1\varphi_{21}''\frac{\partial^2 B_2}{\partial Y\partial X}(1+\frac{\alpha}{(c_B+\bar{u}-c_0)^2}-(\bar{u}-c_0)\varphi_1\frac{\partial^4 A}{\partial X^4}-$$

$$2((\bar{u}-c_0)\varphi_1\frac{\partial^3 A}{\partial X^2 Y}-(\varphi_1\varphi_1'''-\varphi_1'\varphi_1'')2\mathrm{A}\frac{\partial^2 A}{\partial X^2}+\frac{\alpha}{(c_B+\bar{u}-c_0)^3}(\frac{\partial^2}{\partial T^2}-2c_1\frac{\partial^2}{\partial T\partial Y}+c_1^2\frac{\partial^2}{\partial Y^2})\mathrm{A}\varphi_1]\,\mathrm{dy}$$

(33)

$$I_1\frac{\partial^2 B_1}{\partial X\partial T}+I_2\frac{\partial^2 B_2}{\partial X\partial T}-c_1I_1\frac{\partial^2 B_1}{\partial X\partial Y}-c_1I_2\frac{\partial^2 B_2}{\partial X\partial Y}+I_3\frac{\partial^4 A}{\partial X^4}+I_4\frac{\partial^3 A}{\partial X^2\partial Y}+I_5\mathrm{A}\frac{\partial^2 A}{\partial X^2}+I_6(\frac{\partial^2 A}{\partial T^2}-2c_1\frac{\partial^2 A}{\partial T\partial Y}+c_1^2\frac{\partial^2 A}{\partial Y^2}=0$$

(34)

where the coefficients are:

$$
\begin{cases}
I_1 = \int_0^1 \frac{q(y)\varphi_1}{\bar{u}-c_0}\left(\varphi_{21} - \left(1 + \frac{\gamma(\bar{u}-c_0)}{q(y)}\right)\frac{\varphi_1}{\bar{u}-c_0}\right)\left(1 + \frac{\alpha}{(c_B + \bar{u} - c_0)^2}\right)dy\,; \\
I_2 = \int_0^1 \frac{q(y)\,\varphi_1}{\bar{u}-c_0}\left(\varphi_{22} + \left(1 + \frac{\gamma(\bar{u}-c_0)}{q}\right)\frac{\varphi_{1(y)}}{\bar{u}-c_0}\right)\left(1 + \frac{\alpha}{(c_B + \bar{u} - c_0)^2}\right)dy; \\
I_2 - c_1 I_1 = 2c_1 \int_0^1 \frac{q(y)\varphi_1^2}{(\bar{u}-c_0)^2}\,(1 + \frac{\gamma(\bar{u}-c_0)}{q}\left(1 + \frac{\alpha}{(\bar{u}-c_{0} + c_B)^2}\right)dy; \\
I_3 = -\int_0^1 \varphi_1^2\,dy\,; \\
I_4 = -2\int_0^1 \varphi_1\,\varphi_1' dy\,; \\
I_5 = \int_0^1 \frac{\varphi_1^3\,q'}{\bar{u}-c_0}\,dy\,; \\
I_6 = \int_0^1 \left(\frac{\partial^2 A}{\partial T^2} - 2c_1 \frac{\partial^2 A}{\partial T \partial Y} + c_1^2 \frac{\partial^2 A}{\partial Y^2}\right).
\end{cases}
$$

(35)

Noting that

$$\frac{\partial^2 B_1}{\partial X \partial T} = \frac{\partial^2 A}{\partial T^2}\,;\ \frac{\partial^2 B_2}{\partial X \partial T} = \frac{\partial^2 A}{\partial Y \partial T} \quad \text{as} \quad \frac{\partial B_1}{\partial X} = \frac{\partial A}{\partial T}\,; \quad \frac{\partial B_2}{\partial X} = \frac{\partial A}{\partial Y} \tag{36}$$

By using (36) in Eq. (34) we obtain

$$\frac{\partial^2 A}{\partial T^2} + \left(\frac{(I_2 - 2c_1 I_6 - c_1 I_1}{I_1 + I_6}\right)\frac{\partial^2 A}{\partial Y \partial T} - \left(\frac{I_6 c_1^2 - c_1 I_2}{I_1 + I_6}\right)\frac{\partial^2 A}{\partial Y^2} + \left(\frac{I_3}{I_1 + I_{6_3}}\right)\frac{\partial^4 A}{\partial X^4} + \left(\frac{I_5}{I_1 + I_6}\right) A \frac{\partial^2 A}{\partial X^2} = 0 \tag{37}$$

Rewriting Eq. (37) as

$$\frac{\partial^2 A}{\partial T^2} + a_1 \frac{\partial^2 A}{\partial T \partial Y} + a_2 \frac{\partial^2 A}{\partial Y^2} + a_3 \frac{\partial^4 A}{\partial X^4} + a_4 \frac{\partial^2 (A^2)}{\partial X^2} = 0. \tag{38}$$

where

$$a_1 = \frac{(I_2 - 2c_1 I_6 - c_1 I_1)}{I_1 + I_6} \qquad a_2 = -\frac{c_1 I_2}{I_1}, \qquad a_3 = \frac{I_3}{I_1}\ , \ a_4 = \frac{I_5}{2I_1}. \tag{39}$$

This equation describes the evolution of spatial-temporal amplitude $A(X, Y,\ T)$ of Rossby-Khantadze waves. When $I_2 = 2c_1 I_6 - c_1 I_1$ gives $a_1 = 0$, our equation (38) reduces to the standard Boussinesq equation ((2+1) – dimensional). Otherwise, equation (38) is the general form of Boussinesq equation (i.e. $a_1 = 0$).

5. Dynamical Analysis for the New Boussinesq equation

In order to solve the generalized Boussinesq equation, we follow the methodology developed by Kaladze et al., (2013b) and later make use of methods of dynamical analysis to get extended information about the solution of the equation, and to obtain its trajectories and fixed points in phase space.

We use the following co-moving frame A = $\emptyset(\xi)$ with $\xi = mX + nY + lT$ to turn Eq. (39) into an ordinary differential equation. Then after integrating it once over ξ gives us,

$$a_3 m^4 \emptyset'' + (l^2 + a_1 ln + a_2 n^2)\emptyset' + a_4 m^4 \emptyset^2 = g \quad (40)$$

with g as the constant of integration.

We can now express Eq. (40) as a set of two first order autonomous equations as

$$\begin{cases} \frac{d\emptyset}{d\xi} = y\,; \\ \frac{dy}{d\xi} = \frac{-a_4 m^2 \emptyset^2 - (l^2 + a_1 ln + a_2 n^2)\emptyset + g}{a_3 m^4}. \end{cases} \quad (41)$$

From (40) we express the Hamiltonian of the system as

$$H(\emptyset, y) = \frac{1}{2} y^2 - \frac{a_4 m^2 \emptyset^3}{3 a_3 m^4} - \frac{l^2 + a_1 ln + a_2 n^2}{2 a_3 m^4}\emptyset^2 + \frac{g}{a_3 m^4}\emptyset = h\,, \quad (42)$$

where h is a constant value.

In order to get the fixed points of our system, we suppose $\left(\frac{dy}{d\xi}\right)_{\emptyset_1} = 0$ where $\emptyset_1$ is the fixed point. Such that,

$$a_4 m^2 \emptyset_1^2 + (l^2 + a_1 ln + a_2 n^2)\emptyset_1 - g = 0. \quad (43)$$

Eq. (43) is a quadratic equation and has two roots, which are given below

$$\emptyset_1 = \frac{-(l^2 + a_1 ln + a_2 n^2)^2 - \sqrt{\Delta}}{2 a_4 m^2}, \quad (44)$$

and

$$\emptyset_2 = \frac{-(l^2 + a_1 ln + a_2 n^2)^2 + \sqrt{\Delta}}{2 a_4 m^2}. \quad (45)$$

where

$$\Delta = (l^2 + a_1 ln + a_2 n^2)^2 + 4 a_4 m^2 g. \quad (46)$$

Let $g_0 = |f(\phi_i) + g|$, then g_0 is the extremum values of $f(\phi) + g$.

Suppose $(\phi_i, 0)$ (where i= 1, 2) be one of the singular points of the system of equation, then from our system, the characteristic values

$$\lambda^2(\phi_i, 0) = \frac{f'(\phi_\iota)}{a_3 a^4}\,.$$

Based on the qualitative theory for the dynamical system we know that [44]

(i) If $\frac{f'(\phi_i)}{a^3} < 0$ then $(\phi_i$, o) is a center point

(i) If $\frac{f'(\phi_i)}{a^3} > 0$ then $(\phi_i$, o) is a saddle point

(ii) If $f'(\phi_i) = 0$ then $(\phi_i, 0)$ is degenerate saddle point

Thus, above analysis provides the bifurcations phase portraits of equation (42).

5. Solution for the Boussinesq equation

In this part, based on this dynamical theory, we will deduce the traveling wave solution to equation (42) by considering *g= 0.*

The equation (41) reduce to the system as follows

$$\begin{cases} \dfrac{d\emptyset}{d\xi} = y\,, \\ \dfrac{dy}{d\xi} = \dfrac{-a_4 m^2 \emptyset^2 - (\,l^2 + a_1 ln + a_2 n^2)\,\emptyset}{a_3 m^4}. \end{cases} \tag{47}$$

It is expected that equation (41) has a homoclinic orbits Γ_1 (which corresponds to a solitary wave profile).

In $\phi - y$ plane, Γ_1 is given as

$$y^2 = \frac{2a_4\, m^2}{3a_3 m^4}\phi^3 - \frac{(l^2 + a_1 l\, n + a_2\, n^2)}{a_3 m^4}\phi^2, \tag{48}$$

with $\phi_0 = 3(\,l^2 + a_1 ln + a_2 n^2)/2a_4 m^4$.

Equations (47) and (48) give

$$\pm\sqrt{\frac{1}{\frac{2a_4}{3a_3 m^2}\phi^3 - \frac{(l^2 + a_1 l\, n + a_2\, n^2)}{a_3 m^4}\phi^2}}\;\; \mathrm{d}\phi \quad = d\xi, \tag{49}$$

Here we suppose that $\phi(0) = \phi_o$ and integrate (49) along homoclinic orbits Γ_1, we get

$$\int_{\phi}^{\phi_o} \frac{ds}{\sqrt{\frac{2a_4}{3a_3\, m^2}s^3 - \frac{(l^2 + a_1 l\, n + a_2\, n^2)}{a_3 m^4}s^2}} = \int_{\xi}^{o} ds, \qquad \xi < 0 \tag{50}$$

and

$$\int_{\phi}^{\phi_o} \frac{ds}{\sqrt{\frac{2a_4}{3a_3\, m^2}s^3 - \frac{(l^2 + a_1 l\, n + a_2\, n^2)}{a_3 m^4}s^2}} = \int_{\xi}^{o} ds, \qquad \xi > 0 \tag{51}$$

Eqs. (50) and (51) give

$$\phi = \frac{-3\,(l^2+a_1 l\, n + a_2\, n^2)}{a_4 m^2[1 - cosh(\eta\xi)]}\,, \tag{52}$$

$$\phi = \frac{-3\,(l^2+a_1 l\, n + a_2\, n^2)}{a_4 m^2[1 + cosh(\eta\xi)]}\,, \tag{53}$$

where $\eta = \sqrt{\frac{(l^2+a_1 l\, n + a_2\, n^2)}{a_4 m^4}}$.

From (52) and (53) along with transformation A = $\phi\ (\xi)$, $\xi = m\,X\ +\ n\,Y\ +\ l\,T$ we get the solution of solitary wave,

$$u_1\,(X,\ Y,\ T) = \frac{-3\,(l^2+a_1 l\, n + a_2\, n^2)}{a_4 m^2\left[1 - cosh\sqrt{-\frac{(l^2+a_1 l\, n + a_2\, n^2)}{a_3 m^4}}\,\zeta\right]}. \tag{54}$$

and

$$u_2\,(X,\ Y,\ T) = \frac{-3(l^2+a_1 l\, n + a_2\, n^2)}{a_4 m^2\left[1 + cosh\sqrt{-\frac{(l^2+a_1 l\, n + a_2\, n^2)}{a_3 m^4}}\,\zeta\right]}. \tag{55}$$

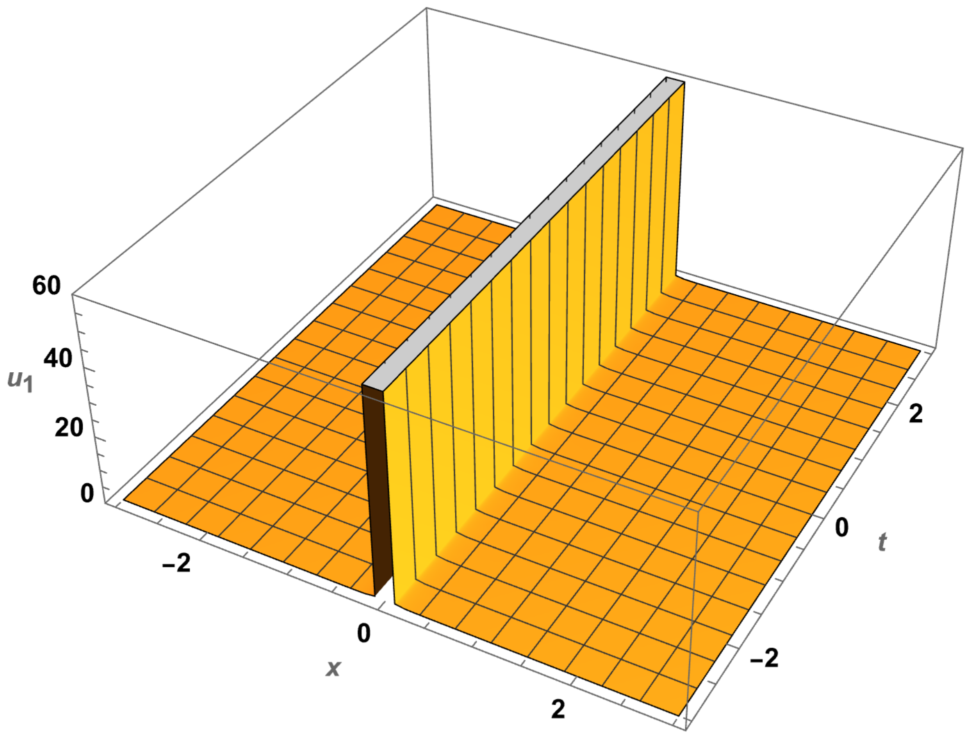


Fig. 2 the solutions (54) are plotted for the parameters $Y = 0$; $m = n = 1$; $a_1 = a_2 = 0.01$; $a_3 = -0.01$; $a_4 = 10$.

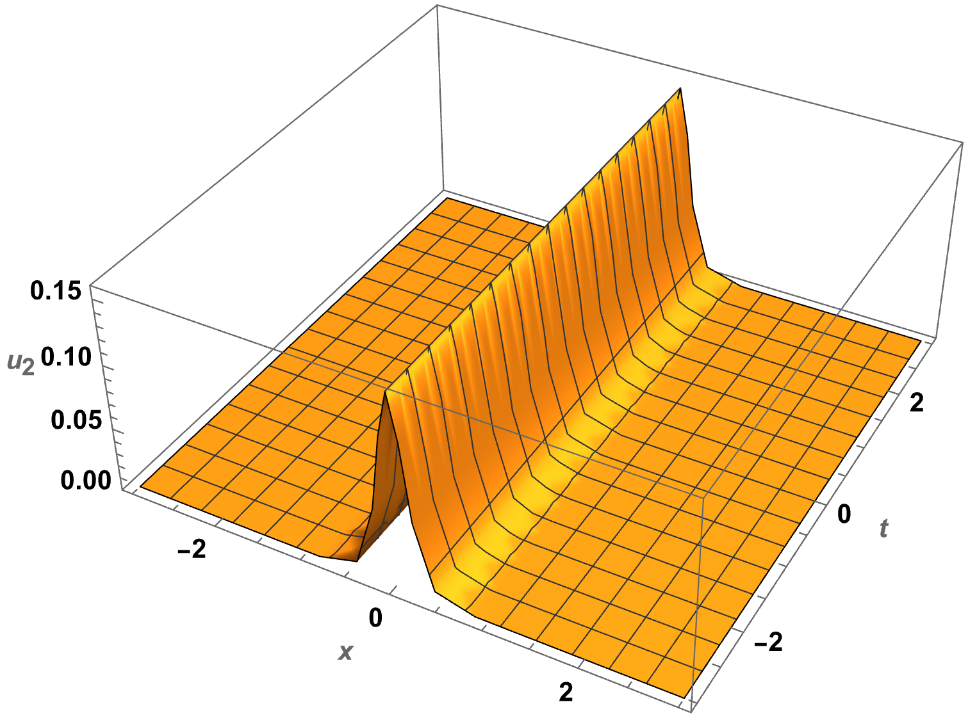


Fig. 3 the solutions (55) are plotted for the parameters $Y = 0$; $m = n = 1$; $a_1 = a_2 = 0.01$; $a_3 = -0.01$; $a_4 = 10$.

It is shown from the obtained solutions that the considered Rossby-Khantadze waves are solitary in nature.

6. Discussion

In this paper, investigation of large-scale Rossby-Khantadze nonlinear waves by incorporating sheared zonal flows in the ionospheric plasma found in the E-layer, is presented. The spatially nonhomogeneous Earth's angular velocity with the background magnetic field are taken. The spatial inhomogeneity in the magnetic field allows the coupling of Rossby and Khantadze waves named Rossby-Khantadze waves.

In this work, we considered a system of equations for Boussinesq model equation from the initial set of equations namely, momentum equation, continuity equation and Maxwell equation. This provides the nonlinear interaction of considered Rossby-Khantadze waves. By taking the curl of momentum equation, we obtain the vorticity equation which is the first system of equation. We obtain the equation of magnetic induction by using the Maxwell's equation, by taking the parameters of the E layer of ionosphere into account. The system of equations explains how Rossby-Khantadze nonlinear waves propagate in considered sheared zonal flow ionospheric E region. In earlier work, the authors take into account Rossby waves while here we take coupled Rossby and Khantadze waves. For the linear consideration, the linear dispersion relation of the fast (Khantadze) and slow (Rossby) electromagnetic (EM) wave in the ionospheric E - region is analyzed with two modes of frequency ω_1 and ω_2. The numerical work of obtained frequencies is shown. The phase velocities depending on wave number is shown in Figs. 1 - 5 (with red color describes ω_1 while blue ones to ω_2). For small wave vector, ω_1 approaches to the finite value, while for the ω_2 becomes $-\infty$. For small α_0, strong coupling is shown between two modes. With increasing α_0 the Rossby modes approaches to the positive values, ergo at α_0= 1, it approaches to zero and for the values $\alpha_0 > 1$, its phase velocity approaches to positive value, while the waves with ω_2 are always propagating along the latitudinally westward direction. For large wave vector, both modes lose their dispersing property.

In order to investigate the nonlinear behavior of coupled RKWs we use multiple scale analysis and asymptotic expansion, to derive nonlinear Boussinesq equation with spatially dependent coefficients. By using the method of multiple scale and hence considering finite amplitude perturbations, we obtain a new Boussinesq ((2+1) dimensional) equation. We have also presented the qualitative description of dynamical systems. Thus, based on the ideas of this work, we cannot only obtain the exact traveling wave solutions in the future research, but can also do the stability analysis, and determine the parameters at which the onset of chaos takes place. Furthermore, this can help us to understand not only the solitary profiles, but also the nonlinear periodic wave solutions associated to the Boussinesq equation.

By taking lowest order O ($\varepsilon^{3/2}$) of Eq. (7) we get an eigen-value Eq. (21). This order, however, does not bring information about the amplitude of the Rossby-Khantadze waves. Thenceforth we use the next order, O (ε^2) of Eq. (7) and obtain non-singular solutions. The obtained equation still doesn't provide information about the wave amplitude. Therefore, we need to go to the next order.

The next order of Eq. (7) provides a longitudinal dispersion effect, which competes with a weak nonlinear effect. This explains that if the perturbation problem has an effective solution, then the secular term F must be satisfied from Eq. (34), otherwise the wave's amplitude would be infinite and have no significance in practice. By doing some mathematical steps, from next order we get the nonlinear Boussinesq equation (41). By considering g=0, we also investigate the dynamical analysis and have done a fixed points analysis analytically. We also obtain the travelling solitary structures shown in Figs. 2-3. The obtained results might be helpful for understanding the data which is obtained by satellites orbiting the earth's ionosphere region.

The considered sheared RK waves give insights on large-scale processes and are observed mainly during magnetic storms as well as sub-storms, artificial explosions, earthquakes, etc. Hence, for the future experimental work, the theoretical findings of Rossby-Khantadze electromagnetic type oscillations will provide valuable information.

7. **Summary and Conclusion**

This study has explored the nonlinear dynamics of Rossby-Khantadze waves in weakly ionized ionospheric plasma, particularly emphasizing the presence of sheared zonal flows. By deriving the boussinesq equation, which incorporates nonlinearity, we have established a robust framework for analyzing the propagation characteristics of Rossby-Khantadze waves across the E-layer of the ionosphere.

The use of the multiple scale analysis and asymptotic expansion has led to the identification of solitary wave solutions that exhibit significant variations influenced by different parameter values. Overall, the findings of this research not only enhance our understanding of wave phenomena in the ionosphere but also have broader implications for various plasma environments, including those found in space and laboratory settings.

AUTHOR DECLARATIONS:

Conflict of Interest

The authors have no conflicts to disclose.

Data Availability

The data that support the findings of this study are available within the article.

Author contributions. LZK: conceptualization (equal); formal analysis (equal); investigation (equal); methodology (equal); writing; original draft (equal); supervision (equal); writing - review and editing (equal). TDK: conceptualization (equal); investigation (equal); methodology (equal); writing - review and editing (equal). HAS: methodology (equal); investigation (equal); supervision (equal); writing - review and editing (equal). TZ: formal analysis (equal); methodology (equal); writing; - original draft (equal). SAB: investigation (equal); writing– review and editing (equal).

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