A Bayesian statistical method to estimate the climatology of extreme temperature under multiple scenarios: the ANKIALE package

Yoann Robin[1], Mathieu Vrac[1], Aurélien Ribes[2], Occitane Barbaux[1,2,3], and Philipe Naveau[1]

[1]Laboratoire des Sciences du Climat et de l'Environnement, UMR 8212 CEA-CNRS-UVSQ, Université Paris-Saclay / IPSL, Sorbonne Université, Gif-sur-Yvette, 91191, France
[2]CNRM, Université de Toulouse, Météo France, CNRS, Toulouse, France
[3]Autorité de sûreté nucléaire et de radioprotection (ASNR), PSE-ENV/SCAN/BEHRIG, F-92260, Fontenay-aux-Roses, France

Correspondence: Yoann Robin (yoann.robin@lsce.ipsl.fr)

Abstract. We describe an improved method and the associated package for estimating the statistics of temperature extremes in a Bayesian framework. Building on previous work, this method uses a range of climate model simulations to provide a prior of the real-world changes, and then considers observations to derive a posterior estimate of past and future changes. The new version described in this study makes it possible to process several scenarios simultaneously, while keeping one single counterfactual world (i.e., the world without human influence). We offer a free licensed, easy-to-use command-line tool called ANKIALE (ANalysis of Klimate with bayesian Inference: AppLication to extreme Events), which can be used to reproduce the analyses presented here, as well as to process user-defined events. ANKIALE is based on a python code, but is designed to be used from the command line interface. ANKIALE is natively parallel, enabling it to be used on a personal computer as well as on a supercomputer. To derive the posterior, ANKIALE uses state of art MCMC-methods to sample the posterior distribution. The potential of this method and tool is illustrated via an application to maximum temperature from ERA5 (considered as observations) over Europe until 2100, at a 0.25~~°-~~° resolution, for a range of four emission scenarios, including a particular focus on the city of Paris (France).

1 introduction

Heatwaves are extreme phenomena whose frequency and intensity have increased with global warming ~~(e.g., Seneviratne et al., 2021; IPCC~~ ~~.~~ (see, e.g. Seneviratne et al., 2021; IPCC, 2022a). Humans (see, e.g. Campbell et al., 2018; Huang et al., 2022; Masselot et al., 2023) , plants (see, e.g. Hatfield and Prueger, 2015; Brás et al., 2021), ecosystems (see, e.g. Bastos et al., 2021) and infrastructure (see, e.g. Zuo e , can suffer significant damage beyond certain thresholds, so it has become necessary to be able to predict and anticipate these events. Over recent decades, these findings have led to the development of the so-called *attribution studies*, which ~~consists~~ consist in establishing the weight of human influence in the occurrence or intensity of an extreme event (Perkins-Kirkpatrick et al., 2024). A number of methods and protocols have been developed ~~(e.g., Ribes et al., 2020; Philip et al., 2020; Robin and Ribes, 2020a) ,~~ (see, e.g. Ribes et al., 2020; Philip et al., 2020; Robin and Ribes, 2020a) which have enabled the analysis and attribution of a number of extreme events. The World Weather Attribution (WWA, 2024) group has ~~made a specialty of~~ specialized in producing attribution studies within a short time (~~delay of the order of~~ typically within a week) following the occurrence of an event. Notable examples include the heatwave in Siberia in 2020 (Ciavarella et al., 2021), the heatwave in the USA and Canada in 2021 (Philip et al., 2022), and the wet heatwave in India in 2023 (Zachariah et al., 2023b). Other types of event can also be analyzed, such as extreme rainfall (Zachariah et al., 2023a, 2024; Clarke et al., 2024b), drought (Clarke et al., 2024a), or wildfire (Barnes et al., 2023), and others.

The attribution methods listed above typically infer the climatology (i.e. the *statistical distribution*) of the extremes of interest by assuming that the maxima of a variable (such as annual temperature) follow a Generalized Extreme Value distribution ~~(GEV, see, e.g., Coles, 2001)~~(see, e.g. Coles, 2001). This distribution is characterized by three parameters, which vary with external forcings (such as global or regional mean temperature). This statistical model is inferred either independently from observations, from climate models (Philip et al., 2020), or from both.

Several recent studies have proposed to implement the later option, i.e., ~~compbining~~ combining models and observations, within a Bayesian ~~(Auld et al., 2023; Zeder and Fischer, 2023)~~framework. In this context, a synthesis of climate models is used as *a prior* of the reality, and then observations are used to derive *the posterior* distribution of past and future changes ~~(Robin and Ribes, 2020a; Ribes et al., 2022). The Bayesian approach~~ . For example, Harris et al. (2013) proposes a Bayesian approach to predicting regional climate change. Several methods for synthesizing climate models have also been proposed by Sanderson et al. (2015); Knutti et al. (2017); Brunner et al. (2019). Brunner et al. (2020) also provides a summary of different observational constraint methods. More recently, Smith et al. (2021) studied an observational constraint approach on an Energy Balance Model trained on multiple CMIP6 models to derive estimates of historical aerosol forcing. Zeder et al. (2023) studied the effect of short observations on the statistics of extreme events considering a Bayesian approach. Finally, Auld et al. (2023) also works on the changes in the distribution of the annual maximum daily maximum temperature (TXx) over Europe with CO2 as covariate.

We describe several improvements to the Robin and Ribes (2020a); Ribes et al. (2022) method, where the Bayesian approach enables the estimation of the statistics of extremes at the end of the century according to a climate scenario and conditioned on observed data over the historical period. ~~However, inferences made without simultaneously considering multiple scenarios may lead to two potential inconsistencies.~~

~~Here, we describe several improvements to the Robin and Ribes (noted RR20, 2020a) method. Firstly, in RR20~~Firstly, the statistical method has to be re-run separately for each emission scenario considered, with no guarantee for consistency across scenarios, especially for the confidence intervals. In particular, the inferred counter-factual world (i.e., the world without human influence), is different according to the scenario, leading to communication issues for key attribution diagnoses such as the probability ratio. Our improved implementation enables us to infer all scenarios simultaneously, which ensures that only one counterfactual world is calculated. Second, we revise the sampling procedure – based on a Metropolis Hasting Monte-Carlo ~~(MCMC, Metropolis et al., 1953; Hastings, 1970)~~ (MCMC Metropolis et al., 1953; Hastings, 1970) method–, to make it consistent with recent progresses in the Bayesian community. This revised implementation runs much faster than the previous one, and offers many guarantees in terms of properties and convergence of the MCMC chain.

The improved method comes with a deeply revised python package and command-line tool. The original method of~~RR20~~ Robin and Ribes (2020a) used a python code ~~(Robin and Ribes, 2020b)~~ (Non-Stationary Statistics for Extreme Attribution, NSSEA Robin , developed for the attribution of a univariate extreme event. This code is not parallelized and required advanced knowledge of python in order to be used. Running this package over a high-resolution grid could require as long as around 20 years of CPU time. We therefore propose a new massively parallel code, developed in Python but with a command-line interface, which can process the entire domain in ~~less than a week (in wall time~~10000 hours, with approximately 2.5 hours per grid point (in CPU time, so about a week with 60 cores), and which is designed to be more accessible.

An ~~an~~ illustration of the potential of our revised method and packages, we analyse extreme temperature over Europe, extended to the Mediterranean basin, giving us a box from 22°W to 45.5°E, and from 26.5°N to 72.5°N, as shown in Fig. **??**a. This domain contains 54 countries, 11 of which are only partially included. The exact ratio and list are given in Tab. ~~??~~**??**. Our attribution study focuses on the analysis of observed events as represented in ERA5, but the statistical methods and models used can describe their future evolution. Here, we focus on estimating the climatology of the strongest temperature events already observed for each grid point in Europe (see Fig. **??**b).

The paper is organized as follows. In ~~See~~Sect. 2, we present the data used: observations, climate models, and the variables we derive from them. In ~~See~~Sect. 3, the methodology is presented, using extreme temperatures at the Paris location (France) as an example. ~~Sec. ?? analyses~~ We also analyse the improvements of the new method compared to the case where the scenarios are estimated independently of each other. Section 4 describes the new code, how it is used and what is implemented. Section 5 then looks at current and future ~~worst possible events~~ maxima over Europe, ~~on the one hand~~ using a method derived from ~~classical attribution, and on the other based on a specific definition~~attribution. Finally, conclusions and perspectives are provided in ~~See~~Sect. 6.

2 Data used

2.1 Observations

We use the "European Reanalysis of the Atmosphere, version 5" data~~(ERA5, Hersbach et al., 2020)~~ (ERA5 Hersbach et al., 2020) to characterize the historical observation-based extremes, and in the following, ERA5 will be ~~refer~~ referred to as "observations". This atmospheric reanalysis combines data from weather forecasting models with observations using assimilation to produce a large number of atmospheric variables. The ERA5 reanalysis provides variables by pressure level at hourly time steps, with surface values obtained by interpolation. Note that ERA5 is a dataset that may be biased compared to actual observations, particularly for extreme events. Figure **??** shows the difference compared to E-OBS (Cornes et al., 2018) – which is constructed by spatializing surface observations – on average (Fig. **??**a) and at maximum (Fig. **??**b). The mean bias varies from 1K to 2K, but locally can grow up to more than 10K, particularly over North Africa. Despite such well-known limitations, ERA5 has decisive advantages in our context: a global coverage (E-OBS is only available for Europe) for the entire time period, with some spatial consistency.

From this dataset, we retain temperatures over our European zone, aggregated on a daily time step, taking daily maxima between ~~0h and 23h~~ 0:00 and 23:00 (UTC), from 1940 to 2024, at the spatial resolution ~~0.25°~~0.25°. We only retain the land grid points (~~$\sim$ 52% of 185 × 271 = 50 135~~ $\sim$ 52% of 185 × 271 = 50135 grid points), see Fig. **??**~~.~~ a (the Fig. **??**b is used in the Sect. 5).

Let us now take a look at how the variable representing a heatwave is constructed for each ERA5 grid point. Starting with daily maximum temperatures (~~TX~~TX), to account for a heatwave extending over several days, we work with the *annual maximum of the ~~3-day~~ 3-day moving average*, noted ~~TX3x.~~ TX3x. In general, mortality increases sharply with the duration of heatwaves (D'Ippoliti et al., 2010), and a duration of three days allows us to capture this effect. To illustrate our methodology we zoom over the location of Paris (France)~~, and we use observations from a weather station located at the Paris-Montsouris site, provided by Météo-France (2023). This station has a much longer chronology (1873 / now), which will allow us to better verify the contribution of our approach by reducing the uncertainty in the estimation of the statistical model.~~. The bias of this time series compared to E-OBS has been represented on the Fig. **??**c-d.

~~For external forcing, the annual mean~~ In the statistical method used, changes in extremes are assumed to scale with global or regional average temperature – which is used as a covariate. Change is this average temperature are assumed to capture the response to external forcings. The temperature over Europe ~~(regional mean) or the world (global mean) is inferred~~ will be taken from HadCRUT5 (Morice et al., 2021; Osborn et al., 2021)~~and GISTEMP (Lenssen et al., 2019), respectively,~~, available from 1850 to the present day. We have chosen to use GISTEMP (Lenssen et al., 2019) for the global temperature.

2.2 Climate models used in this study

Global Climate Models (GCMs) from the Climate Model Intercomparison Project phase 6 ~~(CMIP6, Eyring et al., 2016)~~ (CMIP6 Eyring et al., 2016) simulate climate evolution on a global scale, with a spatial resolution of the order of 100 to 200 km. The simulations feed into numerous scientific projects to understand physical mechanisms, evaluate models, lead multidisciplinary impact studies and serve as a reference for IPCC reports ~~(see, e.g., AR6 reports, IPCC, 2021, 2022a, b)~~(see, e.g., AR6 reports IPCC .

~~Several emission scenarios covering the historical period (~~These simulations consist of a historical part, covering the period from 1850 ~~/~~ to 2014~~) to the end of the century (~~, and several future emission scenarios ranging from 2015 ~~/ 2100) are called Shared Socio-economic Pathways (SSP, O'Neill et al., 2014; van Vuuren et al., 2014; O'Neill et al., 2016)~~to 2100. These scenarios are called *Shared Socio-economic Pathways* (SSP O'Neill et al., 2014; van Vuuren et al., 2014; O'Neill et al., 2016) , and describe climate evolution under assumptions of socio-economic evolution of human societies. Four scenarios will be used in this study, describing four levels of warming: the SSP1-2.6 (~~+1.8~~+1.8K by the end of the century with respect to 1850/1900 period), the SSP2-4.5 (~~+2.8~~+2.8K), the SSP3-7.0 (~~+4.1~~+4.1K) and the SSP5-8.5 (~~+5.2K~~+5K) see, e.g. Ribes et al. (2021). ~~The current trajectory takes us towards a warming of the order of magnitude of the SSP2-4.5 scenario, estimated at +2.8K (C.I. from +2.1K to +3.4K) by the IPCC (2023).~~

For each model we take the same variables as for the observations: TX3x TX3x on each European grid point, mean annual temperature over Europe, and over the world. These variables cover the period from 1850 to 2100, thus including the historical part as well as the future projections of the four SSPs scenarios described above.

3 Methodology

The method described here uses the same steps as the RR20 method. We therefore only describe the differences between the two approaches, and refer to (Robin and Ribes, 2020a) for the other elements.

3.1 The statistical model

The aim of this section is to calculate the statistical parameters (of the law of extremes and covariates, but this could be more general) describing these variables, based on global and regional average temperatures and local extremes (for several simulations from climate models and observations). The method we will present here uses a Bayesian approach, where we first seek to construct a prior distribution of reality (using climate models), which we then constrain using observations, defining the posterior distribution. A key point of this approach is that the prior describes a much longer period than the one observed –typically 1850–2100 for climate models versus 1940–2024 for observations– which allows the construction of a posterior over a period where observations are absent.

In order to analyze how the occurrence of extreme temperature is modified in the future, we will first model their probability distribution. The variable studied will be noted T_t (it varies along time, and will represent either the TX3x or any maximum based temperatures), and its observation will be noted T_t^o. As T_t models maxima, the statistical model inferred will be a GEV, whose parameters μ_t and σ_t depend on a regional covariate X_t^{R}:

$$\begin{cases} T_t \sim \mathrm{GEV}(\mu_t, \sigma_t, \xi_t) \\ \mu_t := \mu_0 + \mu_1 X_t^{\mathrm{R,F}} \\ \log(\sigma_t) := \sigma_0 + \sigma_1 X_t^{\mathrm{R,F}} \\ \xi_t \equiv \xi_0 \\ X_t^{\mathrm{R,F}} := X^{\mathrm{R,0}} + X_t^{\mathrm{R,N}} + X_t^{\mathrm{R,A}} \end{cases}$$

The covariate $X_t^{\mathrm{R,F}}$ is a proxy for the temperature response to external forcings that apply to the climate system. The index F means the Factual world, including natural and anthropogenice forcings. The covariate $X_t^{\mathrm{R,F}}$ is splitted as the sum of a constant $X^{\mathrm{R,0}}$, a response to natural forcings $X_t^{\mathrm{R,N}}$ and a response to anthropogenic forcings $X_t^{\mathrm{R,A}}$. We also add to the model defined by Eq. **??** the global mean temperature $X_t^{\mathrm{G,F}}$, constructed in a similar way to the regional temperature. This allows the effect of the global temperature on the regional temperature and local extremes to be taken into account indirectly. The observations of $X_t^{*,\mathrm{F}}$ are denoted $X_t^{*,\mathrm{o}}$. Note that this model can be seen as a linear model, where the noise of T_t, instead of being Gaussian, follows the GEV distribution.

When $X_t^{*,\mathrm{F}}$ is constructed from historical forcings or scenarios, so we are in the so-called *Factual* world. Once $X_t^{*,\mathrm{F}}$ is known, we can construct its *Counterfactual* equivalent $X_t^{*,\mathrm{C}}$ (i.e. without human influence) by setting $X_t^{*,\mathrm{A}} \equiv 0$.

An important point of Eq. (??) is that the choice of factual world, counterfactual world, or the scenario in projection, are entirely defined by the term $X_t^{*,\mathrm{A}}$. The other parameters μ_0, μ_1, σ_0, σ_1, ξ and $X_t^{*,\mathrm{N}}$ are supposed to be the same whatever the world or scenario chosen.In Robin and Ribes (2020a, for the GEV part), and The Sect. 3.1 will present the statistical model. The inference method is described in Sect. Ribes et al. (2022, for the multi-covariate part), this leads to work on the following vector of parameters:

$$\theta := (\underbrace{X^{\mathrm{R},0}, X_t^{\mathrm{R},\mathrm{N}}, X_t^{\mathrm{R},\mathrm{A}}}_{\text{Regional covariate}}, \underbrace{X^{\mathrm{G},0}, X_t^{\mathrm{G},\mathrm{N}}, X_t^{\mathrm{G},\mathrm{A}}}_{\text{Global covariate}}, \underbrace{\mu_0, \mu_1, \sigma_0, \sigma_1, \xi}_{\text{GEV}}).$$

3.2 and illustrated with a concrete example in Sect. 3.3. The Sect. 3.4 will discuss the benefits of using or not using several climate scenarios simultaneously.

When θ is estimated on the climate scenarios independently from each other, even with the common historical part, different values can be found. In particular, the parameters may not coincide on the historical part, and the counterfactual becomes scenario dependent. We then propose the following θ vector for the scenarios SSP1-2.6, SSP2-4.5, SSP3-7.0 and SSP5-8.5 simultaneously (a more general definition is given in App. ??), which imposes the same natural forcings for all the scenarios – thus imposing a common counterfactual – and an anthropic term specific to each scenario:

$$\begin{aligned}\theta := &(X^{\mathrm{R},0}, X_t^{\mathrm{R},\mathrm{N}}, X_t^{\mathrm{R,A,SSP1\text{-}2.6}}, X_t^{\mathrm{R,A,SSP2\text{-}4.5}}, X_t^{\mathrm{R,A,SSP3\text{-}7.0}}, X_t^{\mathrm{R,A,SSP5\text{-}8.5}},\\ &X^{\mathrm{G},0}, X_t^{\mathrm{G},\mathrm{N}}, X_t^{\mathrm{G,A,SSP1\text{-}2.6}}, X_t^{\mathrm{G,A,SSP2\text{-}4.5}}, X_t^{\mathrm{G,A,SSP3\text{-}7.0}}, X_t^{\mathrm{G,A,SSP5\text{-}8.5}},\\ &\mu_0, \mu_1, \sigma_0, \sigma_1, \xi)\\ =&(\theta^{\mathrm{R}}, \theta^{\mathrm{G}}, \theta^{\mathrm{GEV}})\end{aligned}$$

3.1 Definition of the statistical model

To estimate θ, we adopt the strategyinitially described The aim of ANKIALE is to enable the inference of a statistical model describing a climate variable (such as the annual maximum temperature) from either a climate model or observations (or equivalent product, such as reanalyses). The inference strategy, developed by Ribes et al. (2020) in the case of the Normal distribution, and extended to laws of extremes a normal distribution, then by Robin and Ribes (2020a) . This procedure can be summarized as follows: Fit of an estimation θ^m in the case of extremes, is based on a *frequency analysis* for climate models and a *Bayesian analysis* for observations. This difference in treatment stems from the idea, already exploited by Ribes et al. (2017) , that a set of climate models can be used to construct an approximation of θ for each climate model m (see Tab. ??) in a frequentist way, with an uncertainty covariance matrix Σ_{θ^m}, see Sec. ??. Construct the random variable κ of a multi-model synthesis, incorporating model uncertainty, to be used as a *a prior* of reality, see Sec. ??, Derive a called a *posterior distributionprior* given observations , i.e. $(\kappa \mid X_t^{*,\mathrm{o}}, T_t^o)$, using Bayesian methods, see Sec. ??, ., and that observations can be used to *constrain* this prior to what has been observed, allowing the construction of what is called the *posterior*. This posterior therefore incorporates information from climate models, constrained in such a way as to be made compatible with observations.

In what follows, we will review the main elements of this method, detailing only the improvements and referring to Robin and Ribes (2020 work for the parts of the methodology that have not changed.

3.2 Estimation in climate models

Our aim is to estimate the parameter vector θ^m defined by Eq. (**??**) for each climate model mFormally, we have a climate variable T_t that follows a parametric probability distribution, whose parameters can be summarised in a vector θ. This vector θ can incorporate parameters that control a large number of elements, such as those parameterising the intensity of external forcings that apply at a time t, as well as a covariance matrix Σ_{θ^m} describing the uncertainty of this estimate. Recall that for each climate model m, and for each scenario, we have three time series: European annual mean temperatures $X_t^{\mathrm{R,SSP}}$, Global annual mean temperatures $X_t^{\mathrm{G,SSP}}$ and annual maxima of the 3-days moving average at the Paris-Montsouris station (France) T_t^{SSP}; SSP can be scenarios SSP1-2.6, SSP2-4.5, SSP3-7.0 or SSP5-8.5. Let us start with the covariate $X_t^{\mathrm{R,F}}$ and $X_t^{\mathrm{G,F}}$ (defining the parameters θ_t^{R} and θ_t^{G}), which, following Robin and Ribes (2020a) are Generalized Additive Models (GAM, see e.g., Hastie, 2017): the parameters of the underlying distribution. In this paper, we take as an example the annual maximum temperatures (over 3 days), which are assumed to follow a GEV (Generalised Extreme Value) distribution (see App. A). This distribution has three parameters: μ (location parameter, similar to the mean), σ (scale parameter, similar to the standard deviation) and ξ (shape parameter, controlling whether or not the extremes are bounded). The first two of these parameters are assumed to evolve over time scaling with a covariable X_t, which is representative of a mean climate change.

A statistical model typically used in attribution, as for example by Philip et al. (2020) or Otto et al. (2024) (assuming X_t is known), is given by:

$$X_t^{*,\mathrm{SSP}} = X_t^{*,\mathrm{F,SSP}} + \varepsilon^* = X^{*,0} + X_t^{*,\mathrm{N}} + X_t^{*,\mathrm{A,SSP}} + \varepsilon^* \begin{cases} T_t \sim \mathrm{GEV}(\mu_t, \sigma_t, \xi_t) \\ \mu_t = \mu_0 + \mu_1 X_t \\ \log \sigma_t \equiv \sigma_0 \\ \xi_t \equiv \xi_0 \end{cases} \tag{1}$$

The model is composed of the following elements: The idea is that climate change modifies the location parameter μ_t over time (which is similar to the increase of the mean), but that the variability and shape of the extremes remain unchanged. The indicator describing climate change X_t is generally given by a smoothing of the global temperature (e.g. a 15-year moving average). Then, the vector θ of the parameters of our statistical model can be written:

$$\theta := (\mu_0, \mu_1, \sigma_0, \xi_0). \tag{2}$$

θ can be estimated directly from observations or from climate simulations using maximum likelihood. Confidence intervals on θ are constructed using bootstrap. In this model, uncertainty on the climate change indicator X_t is not taken into account.

– $X^{*,0}$: a constant;

- $X_t^{*,\mathrm{N}}$: either the response of an Energy Balance Model (Energy Balance Model, Held et al., 2010; Geoffroy et al., 2013) for CMIP5, or radiative forcings for CMIP6 (?), modeling natural forcings;

- $X_t^{*,\mathrm{A,SSP}}$: a smoothing spline of residuals with 6 degrees of freedom, modelling the anthropogenic forcing of the scenario SSP;

- ε^*: a white noise Gaussian error term describing natural variability.

In general, additive models can be inferred using *backfitting algorithms* (see, e.g., Breiman and Friedman, 1985). These methods iteratively infer the coefficients of each component on the residuals of the previous component until convergence, (see Alg. **??**). Here, we have a modelwhere the term $X^{*,\mathrm{A,SSP}}$ is specific to each scenario, but $X^{*,\mathrm{N}}$ is common, which does not exactly fit within the scope of this algorithm. One further difficulty for practical application to climate data, is that the covariate X_t representing the response to climate change, is not fully well-known in general, and brings it's own uncertainty. This also applies to the breakdown between the response to natural forcings X_t^N and the response to anthropogenic forcings X_t^A. One possibility to include this additional uncertainty in our statistical model is to include X_t on the model parameters, thus extending the vector θ. This type of statistical model has been studied by Ribes et al. (2020) for the normal distribution and Robin and Ribes (2020a) for the GEV distribution, including a dependence on X_t for the scale parameter σ_t. It is written as follows: To estimate the coefficients θ^{R} and θ^{R}, we propose a modification of the backfitting algorithm, detailed in Alg. **??**. We retain the iterative approach: the anthropogenic terms $X^{*,\mathrm{A,SSP}}$ are inferred for each scenario, but the natural term $X^{*,\mathrm{N}}$ is calculated on the *average (with respect to the scenarios) of the residuals*

$$\begin{cases} T_t \sim \mathrm{GEV}(\mu_t, \sigma_t, \xi_t) \\ \mu_t = \mu_0 + \mu_1 X_t \\ \log \sigma_t = \sigma_0 + \sigma_1 X_t \\ \xi_t \equiv \xi_0 \\ X_t = X^0 + X_t^N + X_t^A \end{cases} \tag{3}$$

For this statistical model, θ therefore takes the following form:

$$\mathrm{residual}_t := \sum_{\mathrm{SSP}} X_t^{*,\mathrm{SSP}} - X_t^{*,\mathrm{A,SSP}}.$$

$$\theta := (X^0, X_t^N, X_t^A, \mu_0, \mu_1, \sigma_0, \xi_0). \tag{4}$$

This approach enables us to find an estimate $\hat{\theta}^{\mathrm{R}}$ of θ^{R}, and an estimate $\hat{\theta}^{\mathrm{G}}$ of θ^{G}.

Figure **??** shows the result ofthis decomposition for the global mean temperature of the IPSL model (IPSL-CM6A-LR), for four scenarios. We can see that $X_t^{\mathrm{G,F}}$ represents the mean value of the model realizations, with a 95% confidence interval encompassing these values. The peaks are due to the volcanoes, while the small oscillations are due to the sun cycles. The inferred counterfactual forcings do have a constant signal, with the exception of volcanoes and the sun.

It remains to estimate $\hat{\theta}^{\mathrm{GEV}}$, the coefficients of the GEV, as well as the covariance matrix $\hat{\Sigma}_{\theta^m}$ describing the uncertainty. We propose the following method to draw the vector $(\hat{\theta}^{\mathrm{R}}, \hat{\theta}^{\mathrm{G}}, \hat{\theta}^{\mathrm{GEV}})$ taking into account the four scenarios: statistical model has several limitations:

- The following n-uplet is resampled:

  $(X_t^{\mathrm{R,SSP1\text{-}2.6}}, \dots, X_t^{\mathrm{R,SSP5\text{-}8.5}}, X_t^{\mathrm{G,SSP1\text{-}2.6}}, \dots, X_t^{\mathrm{G,SSP5\text{-}8.5}}, T_t^{\mathrm{SSP1\text{-}2.6}}, \dots, T_t^{\mathrm{SSP5\text{-}8.5}})$;

  It does not incorporate the work of Qasmi and Ribes (2022), who worked on how to simultaneously account for global and regional covariates (which is important, for example, when the regional response differs significantly from the global response, due to aerosols).

- Using the method described above, the vector $(\hat{\theta}^{\mathrm{R}}, \hat{\theta}^{\mathrm{G}})$ is found; Only one scenario for the future period can be used at a time, which may lead to different estimates for the historical period;

- The parameters X^0 and X^N, which model the response to natural forcings, should not depend on the scenario.

In ANKIALE, we propose using a statistical model that meets the following constraints:

- The vector $\hat{\theta}^{\mathrm{R}}$ is used to generate the four smoothed covariates $X_t^{\mathrm{R,F,SSP}}$ covariate can be global (denoted X_t^G) and / or regional (denoted X_t^R);

- Allow for the simultaneous consideration of multiple future SSP scenarios;

- These covariates are used to calculate the GEV coefficients by maximum likelihood estimation (see Robin and Ribes, 2020a, App. A) with the T_t^{SSP} series, generating four estimates of the $\hat{\theta}^{\mathrm{GEV}}$ vector;

- Four vectors $(\hat{\theta}^{\mathrm{R}}, \hat{\theta}^{\mathrm{G}}, \hat{\theta}^{\mathrm{GEV}})$ are thus constructed, differing only in the GEV part. These four vectors are draws of θ^m The response to natural forcings do not depend on the SSP scenario.

This method is applied 1000 times, generating $4 \times 1000 = 4000$ estimates of θ^m. We then define $\hat{\theta}^m$ as the mean of these draws, Starting from several climate variables T_t^{SSP}, $\mathrm{SSP} \in \{\mathrm{SSP}_1, \dots, \mathrm{SSP}_{N_{\mathrm{SSP}}}\}$ (the term SSP here referring to one of the possible future scenarios), this model can be written as:

$$\begin{cases} T_t^{\mathrm{SSP}_1} \sim \mathrm{GEV}(\mu_t^{\mathrm{SSP}_1}, \sigma_t^{\mathrm{SSP}_1}, \xi_t^{\mathrm{SSP}_1}) \\ \mu_t^{\mathrm{SSP}_1} = \mu_0 + X_t^{R,\mathrm{SSP}_1} \mu_1 \\ \log \sigma_t^{\mathrm{SSP}_1} = \sigma_0 + X_t^{R,\mathrm{SSP}_1} \sigma_1 \\ \xi_t^{\mathrm{SSP}_1} \equiv \xi_0 \\ X_t^{R,\mathrm{SSP}_1} = X^{R,0} + X_t^{R,\mathrm{N}} + X_t^{R,\mathrm{SSP}_1,\mathrm{A}} \\ X_t^{G,\mathrm{SSP}_1} = X^{G,0} + X_t^{G,\mathrm{N}} + X_t^{G,\mathrm{SSP}_1,\mathrm{A}} \\ \vdots = \vdots \\ T_t^{\mathrm{SSP}_{N_{\mathrm{SSP}}}} \sim \mathrm{GEV}(\mu_t^{\mathrm{SSP}_{N_{\mathrm{SSP}}}}, \sigma_t^{\mathrm{SSP}_{N_{\mathrm{SSP}}}}, \xi_t^{\mathrm{SSP}_{N_{\mathrm{SSP}}}}) \\ \mu_t^{\mathrm{SSP}_{N_{\mathrm{SSP}}}} = \mu_0 + X_t^{R,\mathrm{SSP}_{N_{\mathrm{SSP}}}} \mu_1 \\ \log \sigma_t^{\mathrm{SSP}_{N_{\mathrm{SSP}}}} = \sigma_0 + X_t^{R,\mathrm{SSP}_{N_{\mathrm{SSP}}}} \sigma_1 \\ \xi_t^{\mathrm{SSP}_{N_{\mathrm{SSP}}}} \equiv \xi_0 \\ X_t^{R,\mathrm{SSP}_{N_{\mathrm{SSP}}}} = X^{R,0} + X_t^{R,\mathrm{N}} + X_t^{R,\mathrm{SSP}_{N_{\mathrm{SSP}}},\mathrm{A}} \\ X_t^{G,\mathrm{SSP}_{N_{\mathrm{SSP}}}} = X^{G,0} + X_t^{G,\mathrm{N}} + X_t^{G,\mathrm{SSP}_{N_{\mathrm{SSP}}},\mathrm{A}} \end{cases} \tag{5}$$

This model appears to be an extension of the one defined by Eq. (1) for each of the SSP scenarios, while incorporating the response to external forcings on σ_t in addition to μ_t. The two variables $X_t^{R,\mathrm{SSP}}$ and $\hat{\Sigma}_{\theta^m}$ as the empirical covariance matrix.

Figure ?? shows the location parameter μ_t (in factual and counter-factual world), and the scale parameter is represented as $\mu_t \pm \sigma_t$. We can see that the factual and counterfactual parameters correspond until the 1970s, before diverging due to external forcing.To represent the shape parameters ξ (which is negative), we have added the upper bound, given by (see, e.g. Coles, 2001) :

$$B_t := \mu_t - \frac{\sigma_t}{\xi}.$$

$X_t^{G,\mathrm{SSP}}$ correspond respectively to the *Regional* and *Global* forcings of each SSP scenario. They are both decomposed as the sum of a constant ($X^{R,0}$ and $X^{G,0}$, respectively), a response to *Natural* forcings ($X_t^{R,\mathrm{N}}$ and $X_t^{G,\mathrm{N}}$, respectively) and a response to *Anthropogenic* forcings ($X_t^{R,\mathrm{SSP},\mathrm{A}}$ and $X_t^{G,\mathrm{SSP},\mathrm{A}}$, respectively), see Sect. **??** for the exact mathematical description and how the decomposition is performed. The vector θ, equivalent to Eq. (2), and obtained after including all these terms, is given by:

$$\begin{aligned} \theta := (&X^{R,0}, X_t^{R,\mathrm{N}}, X_t^{R,\mathrm{SSP}_1,\mathrm{A}}, \ldots, X_t^{R,\mathrm{SSP}_{N_{\mathrm{SSP}}},\mathrm{A}}, \\ &X^{G,0}, X_t^{G,\mathrm{N}}, X_t^{G,\mathrm{SSP}_1,\mathrm{A}}, \ldots, X_t^{G,\mathrm{SSP}_{N_{\mathrm{SSP}}},\mathrm{A}}, \\ &\mu_0, \mu_1, \sigma_0, \sigma_1, \xi_0). \end{aligned} \tag{6}$$

Note in particular that certain values for the second half of the 21st century are higher than the upper limit of the counterfactual world B_t^{C}. These values would therefore be *impossible* without the effect of the anthropic term $X_t^{*,\mathrm{A}}$.

3.2 Prior from a synthesis of climate models

At this stage, for each climate modelwe have a pair $(\hat{\theta}^m, \hat{\Sigma}_{\theta^m})$, the solution to the statistical model defined by In this statistical model, only anthropogenic terms depend on the SSP scenario. Compared to the model defined in Eq. (??). In order to construct *a prior* of the reality, we change our point of view and move from *frequentist* statistics – where θ is 1), the scale parameter σ_t^{SSP} depends on $X_t^{R,\mathrm{SSP}}$ and is therefore no longer constant over time. Note that the parameters μ_t^{SSP} and σ_t^{SSP} depend directly only on the regional forcings $X_t^{R,\mathrm{SSP}}$. They depend indirectly on the global forcings $X_t^{G,\mathrm{SSP}}$ through the knowledge of the estimate of the parameters , and Σ_θ is the uncertainty – to *Bayesian* statistics where dependence between the parameters in the vector θ is a random variable. We therefore assume that θ^m is a random variable with the following normal distribution:

$$\theta^m \sim \mathcal{N}(\hat{\theta}^m, \hat{\Sigma}_{\theta^m}).$$

A prior of the reality is constructed as a synthesis of different climate models using the following hypothesis: *models are statistically indistinguishable from reality*, developed by Ribes et al. (2017). Denote $\kappa \sim \mathcal{N}(\nu_\kappa, \Sigma_\kappa)$ the random variable of the multi-model synthesis we are looking for – to simplify, κ is a vector with the same components as (a vector that integrates all the information in the statistical model). This makes it possible to link global warming to local events. We also assume that the coefficients of the GEV distribution μ_0, μ_1, σ_0, σ_1 and ξ_0 are independent of external forcings. This model is flexible and allows the underlying distribution to be easily modified. For example, it is possible to replace the GEV law with a Gaussian law (this statistical model is also proposed in `ANKIALE`). Other statistical models (with possibly others probability distributions), not necessarily implemented immediately, are proposed in Sect. **??**.

In the following, in order to facilitate notation, we will break down θ –, and $\mathcal{N}(\nu_\eta, \Sigma_\eta)$ the "reality". Let's start by decomposing the θ^m of each model m as follows:

$$\theta^m \sim \mathcal{N}(\nu_\eta + \nu^m, \Sigma_u + \Sigma^m)$$
$$\nu^m \sim \mathcal{N}(0, \Sigma_u).$$

Here $\nu_\eta + \nu^m = \hat{\theta}^m$ is the mean of model m (and ν_m is the bias of the model with respect to the truth), Σ_u is a common part of the internal variability shared by all models and Σ^m is the additional internal variability of model m. The multi-model synthesis is itself a normal distribution that takes into account the modelling uncertainty, and the intra- and inter-model uncertainty. Intuitively, the uncertainty of the synthesis (defined by the covariance matrix of κ)"covers" the spread of the models . The multi-model synthesis is given by the followingequations, and the calculation method is described in Robin and Ribes (2020a, App. B) as follows:

$$\begin{cases} \nu_\kappa = \frac{1}{n}\sum_m \hat{\theta}^m = \frac{1}{n}\sum_m (\nu_\eta + \mu^m) \\ \Sigma_\kappa = \left(1+\frac{1}{n}\right)\hat{\Sigma}_u + \frac{1}{n^2}\sum_m \hat{\Sigma}^m. \end{cases}$$

$$\begin{cases} \theta := (\theta^R, \theta^G, \theta^{\mathrm{GEV}}), \\ \theta^R := (X^{R,0}, X_t^{R,\mathrm{N}}, X_t^{R,\mathrm{SSP}_1,\mathrm{A}}, \ldots, X_t^{R,\mathrm{SSP}_{N_{\mathrm{SSP}}},\mathrm{A}}), \\ \theta^G := (X^{G,0}, X_t^{G,\mathrm{N}}, X_t^{G,\mathrm{SSP}_1,\mathrm{A}}, \ldots, X_t^{G,\mathrm{SSP}_{N_{\mathrm{SSP}}},\mathrm{A}}), \\ \theta^{\mathrm{GEV}} := (\mu_0, \mu_1, \sigma_0, \sigma_1, \xi_0). \end{cases}$$

Figure ?? illustrates the effect of this synthesis for the grid point containing Paris. In

3.2 Estimation strategy

As mentioned at the beginning of the previous section, estimation in ANKIALE is carried out in three steps, which are summarised in Fig. ??a, we have represented the factual and counter-factual world (in red and blue) of the mean of each model (historical and SSP5-8.5) of the regional covariate Europe (the lines)??. This strategy can be summarised as follows:

1. Inference in climate models. For each climate model $m = 1, \ldots, N_M$, let θ_m be the value of θ for model m. We derive an estimate $\hat{\theta}_m$ of θ_m, as well as the 95% confidence interval covariance matrix describing the estimation error, denoted $\Sigma_{\hat{\theta}_m}$, using a standard frequentist approach. Details are given in App B1.

2. Construction of the multi-model synthesis(filled zone). This interval contains all the means of each model, showing that the prior is a representative of all the possibilities of these models for the covariate. Figures ??c-f show the level line at 95% of the covariance matrix between the parameter μ_0 and the other parameters μ_1, σ_0, σ_1 and ξ_0. The grey ellipses are each of the models, while the black ellipse is the . At this stage, we switch to a Bayesian approach. The vector θ_m is now considered a random variable for each climate model $m = 1, \ldots, N_M$, for which we seek to estimate the probability distribution $\mathbb{P}(\theta_m)$. Since we have an estimate $\hat{\theta}_m$ and a covariance matrix $\Sigma_{\hat{\theta}_m}$, the multivariate normal distribution is the natural choice:

$$\theta_m \sim \mathbb{P}(\theta_m) := \mathcal{N}(\hat{\theta}_m, \Sigma_{\hat{\theta}_m}).$$

The multi-model synthesis. This synthesis again covers most climate models, with the exception of one which is very different. As the weights are uniform across the models during the synthesis, this model has been excluded from the synthesis.

3.3 Posterior from observations

3.2.1 Bayesian method

At this stage, we have the random variable κ (the multi-model synthesis), and our aim is to use it as synthesis, denoted θ_*, follows also a multivariate normale distribution:

$$\theta_* \sim \mathbb{P}(\theta_*) := \mathcal{N}(\hat{\theta}_*, \Sigma_{\hat{\theta}_*}).$$

The mean $\hat{\theta}_*$ is just given by the prior of the observation parameters. Recall that we have observationsof the local extremes T_t^o, the global multi-model mean, but the covariance matrix $\Sigma_{\hat{\theta}_*}$ is more complex, and takes into account of intra and inter model uncertainy. Details are given in the App. B2. It is this random variable $X_t^{\mathrm{G,o}}$, that will serve as our *prior* in the following.

3. Derivation of the posterior with observations. Knowing the observations $X_t^{o,R}$, $X_t^{o,G}$ and the regional $X^{\mathrm{R,o}}$. Our goal T_t^o (regional, global and extreme average temperatures, defined in Sect. 2.1) and the prior θ_*, the aim here is to estimate the distribution $\mathbb{P}\left(\kappa \mid T_t^o \cap X_t^{*,\mathrm{o}}\right)$. Following the same calculations as Robin and Ribes (2020a, ,Sec. 3.5), we have the relationship:

$$\mathbb{P}\left[\kappa \mid \left(T_t^o \cap X_t^{*,\mathrm{o}}\right)\right] = \frac{\mathbb{P}\left[T_t^o \mid (\kappa | X_t^{*,\mathrm{o}})\right]\mathbb{P}(\kappa \mid X_t^{*,\mathrm{o}})}{\mathbb{P}(T_t^o)}.$$

In other words, we start from the prior κ, from which we derive the posterior $(\kappa \mid X_t^{*,\mathrm{o}})$ from the observations $X_t^{\mathrm{G,o}}$ and $X_t^{\mathrm{R,o}}$. For this first posterior, only the global and regional covariates are constrained. Equation (??)indicates that the posterior $(\kappa \mid X_t^{*,\mathrm{o}})$ can also be seen as the prior allowing the parameters of the GEV distribution to be constrained by the T_t^o observations.$\mathbb{P}[\theta_* \mid (X_t^{o,R}, X_t^{o,G}, T_t^o)]$, i.e. the distribution of θ_* *knowing* what has been observed. This conditional distribution thus incorporates information from climate models (through the prior) while being compatible with observations (via the conditioning). The derivation of the posterior from the prior is detailed in App. B3.

We will now look at how these two successive posteriors are constructedillustrate these different steps using temperatures in Paris.

3.2.1 Simultaneous constraint of external forcings

3.3 Illustration with the TX3x at Paris

To determine the posterior of global and regional temperatures , our aim is to apply the Gaussian conditioning theorem (see, e.g., Eaton, 200
. Starting from the prior κ and the observations $X^{*,\mathrm{o}}$, if we find a matrix A such that :

$$X_t^{\mathrm{GR,o}} := \begin{pmatrix} X_t^{\mathrm{G,o}} \\ X_t^{\mathrm{R,o}} \end{pmatrix} = A \cdot \kappa + \varepsilon^o.$$

Then as $\kappa \sim \mathcal{N}(\nu_\kappa, \Sigma_\kappa)$, according to the Gaussian conditioning theorem, the random variable $(\kappa \mid X_t^{*,\mathrm{o}})$ is also a normal distributionwith the following parameters:

$$\begin{cases} \nu_{(\kappa \mid X_t^{*,\mathrm{o}})} = \nu_\kappa + (\Sigma_\kappa A^T) \cdot (A \Sigma_\kappa A^T + \Sigma_o)^{-1} \cdot (X_t^{\mathrm{GR,o}} - A \nu_\kappa) \\ \Sigma_{(\kappa \mid X_t^{*,\mathrm{o}})} = \Sigma_\kappa - (\Sigma_\kappa A^T) \cdot (A \Sigma_\kappa A^T + \Sigma_o)^{-1} \cdot (A \Sigma_\kappa). \end{cases}$$

We propose to give the matrix A directly, and to explain briefly why it is of this form. The details are given in the App. **??**. The matrix A we are looking for is of the following form:

$$A := \begin{pmatrix} R^{\mathrm{G,o}} & R^{\mathrm{R,o}} \end{pmatrix} \cdot \begin{pmatrix} A^1 & A^0 & A^{\mathrm{GEV}} \\ A^0 & A^1 & A^{\mathrm{GEV}} \end{pmatrix},$$

with the following sub-matrix:

- A^1 is a matrix which transforms κ^{G} or κ^{R} into the mean covariate, taken along the SSPs. In other words for κ^{G}:

  $$A^1 \cdot \kappa^{\mathrm{G}} = \frac{1}{N^{\mathrm{SSP}}} \sum_{\mathrm{SSP}} X_t^{\mathrm{G,F,SSP}}$$

- A^0 is the same dimension as A^1, but with null values.
- A^{GEV} is a null matrix such that $A^{\mathrm{GEV}} \cdot \kappa^{\mathrm{GEV}} = 0$.
- $R^{\mathrm{G,o}}$ are $R^{\mathrm{R,o}}$ are matrices which restrict the time axis to that of the observations.

We then have:

$$\begin{aligned} A \cdot \kappa + \varepsilon^o &= \begin{pmatrix} R^{\mathrm{G,o}} & R^{\mathrm{R,o}} \end{pmatrix} \cdot \begin{pmatrix} A^1 & A^0 & A^{\mathrm{GEV}} \\ A^0 & A^1 & A^{\mathrm{GEV}} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} \kappa^{\mathrm{G}} \\ \kappa^{\mathrm{R}} \\ \kappa^{\mathrm{GEV}} \end{pmatrix}}_{=\kappa} + \underbrace{\begin{pmatrix} \varepsilon^{\mathrm{G,o}} \\ \varepsilon^{\mathrm{R,o}} \end{pmatrix}}_{\varepsilon^o} \\ &= \begin{pmatrix} R^{\mathrm{G,o}} & R^{\mathrm{R,o}} \end{pmatrix} \cdot \begin{pmatrix} A^1 \cdot \kappa^{\mathrm{G}} + A^0 \cdot \kappa^{\mathrm{R}} + A^{\mathrm{GEV}} \cdot \kappa^{\mathrm{GEV}} \\ A^0 \cdot \kappa^{\mathrm{G}} + A^1 \cdot \kappa^{\mathrm{R}} + A^{\mathrm{GEV}} \cdot \kappa^{\mathrm{GEV}} \end{pmatrix} + \varepsilon^o \\ &= \begin{pmatrix} R^{\mathrm{G,o}} \cdot A^1 \cdot \kappa^{\mathrm{G}} \\ R^{\mathrm{R,o}} \cdot A^1 \cdot \kappa^{\mathrm{R}} \end{pmatrix} + \varepsilon^o \\ &= \begin{pmatrix} R^{\mathrm{G,o}} \cdot \left(\frac{1}{N^{\mathrm{SSP}}} \sum_{\mathrm{SSP}} X_t^{\mathrm{G,F,SSP}} \right) \\ R^{\mathrm{R,o}} \cdot \left(\frac{1}{N^{\mathrm{SSP}}} \sum_{\mathrm{SSP}} X_t^{\mathrm{R,F,SSP}} \right) \end{pmatrix} + \varepsilon^o = \begin{pmatrix} X_t^{\mathrm{G,o}} \\ X_t^{\mathrm{R,o}} \end{pmatrix} = X_t^{\mathrm{GR,o}} \end{aligned}$$

In this section, we illustrate our procedure using the annual maximum daily temperatures over 3 days ($T_t :=$ TX3x, with a slight abuse of notation in the omission of time in TX3x). According to the block-maxima theorem (see, e.g., Coles, 2001), we can assume that the random variable TX3x follows a GEV distribution.

Note that other choices for A^1 exists.Here we have made the assumption that over the observed period, the average of the scenarios differs from the observations only by white noise. If we accept as an additional hypothesis that a particular scenario is similar to

3.3.1 Step 1: Estimations for the climate models

The first step is to estimate $\hat{\theta}_m$ and $\Sigma_{\hat{\theta}_m}$ for the different climate models m. We can see the result for the observations (such as SSP2-4.5), then we can take for A^1 the matrix that restricts to this scenario, i.e.:

$$A^1 \cdot \kappa^* = X_t^{*,\text{F,SSP2-4.5}}.$$

Here we prefer to remain more general and take for A^1 the matrix which uses the average of the scenarios. climate model `IPSL-CM6A-LR` (Boucher et al., 2020), over the historical period followed by the SSP5-8.5 scenario, in Fig. **??**.

A tricky point is the estimation of $\varepsilon^o \sim \mathcal{N}(0, \Sigma_o)$. To do this, we remove the trend of In Fig. **??**a, showing the regional covariate over Europe, the values of the 33 members of the IPSL model are shown in grey, and the estimate of $X_t^{R,m}$ in red, with the 95% confidence interval. The covariate appears to pass through the centre of the data set (as expected), with dips caused by volcanic activity. In Fig. **??**c, showing the variable TX3x, the prior from the observations, i.e. we calculate $X_t^{*,\text{o}} - A\nu_\kappa$, and Σ_o is the variance of the residuals. values of the 33 members of the IPSL model are shown in grey. The red line passing through the data is the median, with its 95% interval. The red line above the data set is the upper bound (see App. A), with its 95% interval.

We have also represented the estimates of $\hat{\theta}_m^{\text{GEV}}$ as pairs between μ_0 and the other parameters μ_1, σ_0, σ_1 and ξ_0, in figures **??**e-h. These estimates are represented with the ellipse defined by the covariance matrix $\Sigma_{\hat{\theta}_m}$ at the 95% level, in grey. We can see the effect of this constraint in Fig. **??**b: the confidence interval of the constrained variable is much tighter around the median (a reduction of 2K on each side). The covariate Europe passes through the observations (the black dots). We can also see a reduction in the confidence interval of the counterfactual covariate, which depends only on the natural terms that have also been constrained. These results are consistent with the work of Qasmi and Ribes (2022)that the parameter μ_1, which drives the average trend of extremes, ranges from a value of almost zero in some models to a factor of 4. The equivalent parameter for scale, σ_1, is centred at 0. The shape parameter ξ_0 is strictly negative regardless of the model, which is typical for temperatures. Note that the model with the lower μ_0, which differs from the other climate models, is the `NorESM2-LM` model (Norwegian Bentsen et al., 2019).

3.3.2 Metropolis-Hasting method for local extremesStep 2: construction of the prior with the multi-model synthesis

With our knowledge of the distribution $(\kappa \mid X_t^{*,o})$, we now want to obtain samples of the distribution $\mathbb{P}[(\kappa \mid X_t^o) \mid T_t^o]$. The whole problem is that T_t follows a GEV distribution, and there is no explicit expression for the posterior. Let us start again from the Eq. (**??**): The next step is the multi-model synthesis, which is shown in figures **??**b,d (for the covariable) and **??**e-h (for the GEV parameters).

$$\mathbb{P}\left[(\kappa \mid X_t^{*,o}) \mid T_t^o\right] = \frac{\mathbb{P}\left[T_t^o \mid (\kappa \,|X_t^{*,o})\right]\mathbb{P}(\kappa \mid X_t^{*,o})}{\mathbb{P}(T_t^o)}.$$

The distribution $\mathbb{P}(\kappa \mid X_t^{*,o})$ is known, it is our prior. The term $\mathbb{P}[T_t^o \mid (\kappa|X_t^o)]$ is directly calculable: the draws of $(\kappa|X_t^o)$ generate the parameters of the GEV law, which can thus be evaluated.When the denominator is analytically intractable, numerical methods are necessary to sample from the posterior distributionIn Fig. **??**b, the regional covariate over Europe is shown in light red. Compared to Fig. **??**a, the 95% confidence interval is much wider, encompassing all climate models. Since we are working with anomalies relative to the 1961–1990 period, the uncertainty is much lower in this period.

A common approach to perform this sampling is the Metropolis-Hasting algorithm (Metropolis et al., 1953; Hastings, 1970) This is the sampling algorithm originally used by Robin and Ribes (2020a). This Markov chain Monte Carlo algorithm relies on a random walk proposal: a new proposal is created by starting from an initial value κ_0 and adding a random noise to generate a κ_1. The new value is either accepted or rejected with a probability defined using the likelihood ratio of the proposal and the previous value.A key element of this procedure is the transition function between κ_i and κ_{i+1} that is used to sample successive possible values of the posterior.

In the Robin and Ribes (2020a) original implementation, Figure **??**d shows in light red the median and upper bound of the GEV distribution (see App. A for the definitions of the transition function was of the form $\kappa_{i+1} = \kappa_i + \varepsilon$ where ε follows a normal distribution with the same scale for all parameters. This can become an issue when the scale of quantile function and the upper bound), with their 95% confidence intervals. We can see that the intervals are particularly wide compared to those of the IPSL model in Fig. **??**c, as they encompass the uncertainty from all climate models. ERA5 observations have been added (grey dots), as well as the average of observations over the period 1961/1990 (black dotted line). The median of the target parameters is very different from one another. The transition also determines the rate of convergence and mixing, so this implementation can be computationally sub-optimal. Various diagnostics showed the algorithm suffered from slow-mixing chains (Gelman et al., 1997), high autocorrelation (Brooks et al., 2011), and low effective sample size (Gelman et al., 2015) synthesis seems strongly biased compared to the observations, as the confidence interval does not contain the 1961/1990 mean (the median and the mean are quite close for the GEV distribution).

To deal with these issues, we leverage the *No-U-Turn Sampler* algorithm (NUTS, Hoffman and Gelman, 2014), as implemented in STAN (Stan Development Team, 2024). This algorithm is based on the Hamiltonian Monte Carlo algorithm (Radford, 2011), a variant of the Metropolis-Hasting algorithm where the proposal is not generated using a random walk. Instead, the proposal is created through a series of gradient-informed steps (Betancourt, 2018). This allows for better parameter space exploration,especially in the multidimensional case. The NUTS variant relies on a specific criteria to select adaptively various hyper-parameters such as the steps length and stopping conditions. This adaptation makes the algorithm more robust against correlation in In figures **??**e-h, the posterior. The NUTS algorithm is particularly effective when the posterior dimensions are correlated or of different scales. It is very efficient to explore the parameter space and draw samples from the posterior.

The effect of the constraint can be seen in figures ??c/f. First of all, let's note that the posterior is itself Gaussian (in this case), so we have show green ellipses representing the multi-model synthesis is shown in blue. We can see that it encompasses all the parameters from each of the climate models, thus taking into account their uncertainties.

3.3.3 Step 3: derivation of the posterior with observations

The final step is a constraint by the observations, which is shown in figures **??**b,d (for the covariable) and **??**e-h (for the GEV parameters).

Fig. **??**b displays the observations in grey and the posterior in dark red for the European covariate. The posterior fits the observations well (modulo natural variability). The 95% confidence level of the parameters μ_0, μ_1, σ_0, σ_1 and ξ_0% confidence interval appears considerably reduced (from 2° at the beginning of the century to 4° at the end of the century). The posterior is shown in dark red in Fig. **??**d. The confidence interval (both for the median and the upper bound) has been considerably reduced compared to the prior (in light red). The median also appears unbiased, with a confidence interval close to the 1961/1990 average for the historical part.

In figures **??**e-h, the posterior is shown in green. For comparison, we have also added the direct inference of observations in orangewith a maximum likelihood estimate, the covariate being that of our posterior . The 95% confidence interval is constructed by bootstrap. We can see a direct estimate from ERA5 by maximum likelihood in orange: the statistical model of Eq. (1) is inferred (with the addition of σ_t which depends on X_t), and for the covariate X_t we use the posterior of X_t^R. Compared to the prior (in blue), the posterior shows a significant reduction in uncertainty. It should also be noted that the posterior is systematically within the uncertainty of both the prior and the observations. The median is the same for all the parameters, except for μ_1 which not centred on the prior, but may appear shifted. With the exception of μ_1, the direct estimate of ERA5 appears to be compatible with the prior, albeit with extremely high uncertainty: the parameter ξ_0 may even be positive in this case, allowing for extreme TX3x events with potentially colossal values. Two specific cases stand out:

1. σ_1 shows significant uncertainty in ERA5 (although centred at 0), which is strongly constrained to 0 by Bayesian inference methods.

2. μ_1 also shows significant uncertainty in ERA5, but also significant *bias*: the values can exceed +4°C (in the 95% confidence interval), whereas they do not exceed +2°C in the posterior (which implies a slope that is twice as large for the observations, with a large uncertainty.small). The explanation lies in the fact that μ_1 in ERA5 is estimated from a very small number of values (the climate change signal is almost imperceptible before the 1990s, as we can see in Fig. **??**3b). As our approach uses information from models, this parameter is estimated from future scenarios and becomes much less uncertain.

3.4 Contribution of the multi-scenario approach in an attribution context

4 ~~Comparison between the independent and dependent scenarios method~~

3.0.1 Analysis through the attribution of the 2019 French heatwave

~~We propose now to compare the result of our calculation between the case where the scenarios are estimated independently and the case where the scenarios are estimated simultaneously. In other words, we carry out the estimates either for each scenario separately as in Robin and Ribes (2020a), or as developed above. Based on the estimates of the laws,~~ A new feature of the ~~attribution of the~~ statistical model developed in equations (5) and (6) is the simultaneous consideration of several scenarios while requiring that the natural part of external forcings be common to the different scenarios. In order to measure the contribution of this approach, we attributed the 2019 ~~heatwave is performed for comparison purpose. First, let's recall that in the case of the GEV(μ_t, σ_t, ξ) law, the survival and quantile functions are given explicitly~~heatwave in Paris, which has a *Factual Intensity* of $I^{\mathrm{F}}_{2019} := 38.7$°C in ERA5. Our statistical model is inferred in five cases:

- A case where four SSP scenarios are used simultaneously, i.e. SSP $\in$ {SSP1-2.6, SSP2-4.5, SSP3-7.0, SSP5-8.5},
- A case where SSP is only the SSP1-2.6 scenario,
- A case where SSP is only the SSP2-4.5 scenario,
- A case where SSP is only the SSP3-7.0 scenario,
- A case where SSP is only the SSP5-8.5 scenario,

For each of these cases and each scenarios, we calculated the regional covariates *Factual* X^{F}_t (with human influence) and *Counterfactual* X^{C}_t (natural forcings only) as follows:

$$\begin{cases} X^{\mathrm{F}}_t = X^{R,0} + X^{R,\mathrm{N}}_t + X^{R,\mathrm{A}}_t, \\ X^{\mathrm{C}}_t = X^{R,0} + X^{R,\mathrm{N}}_t. \end{cases}$$

~~In an attribution context, an event occurs in year $t = \tau (= 2019)$ with an intensity I^{F}_τ, and we look for the probabilities that T_t exceeds this intensity~~ By inserting these two terms into the parameters of locations, scales and shapes of the GEV distribution, survival functions and quantile functions can be calculated in factual and counterfactual worlds. This makes it possible to calculate the probability of exceeding the threshold I^{F}_{2019} each year t, in factual and counterfactual worlds. Using the equations recalled in App. A, for a GEV distribution, the probability in the factual ~~(noted p^{F}_t) and counterfactual (noted p^{C}_t) world for all years t (including $t = \tau$). This is written:~~

$$\begin{cases} p^{\mathrm{F}}_t = \mathbb{P}^{\mathrm{F}}_t(T_t \geq I^{\mathrm{F}}_\tau), \\ p^{\mathrm{C}}_t = \mathbb{P}^{\mathrm{C}}_t(T_t \geq I^{\mathrm{F}}_\tau). \end{cases}$$

Each world p_t^{F} (resp. counterfactual p_t^{C}) that the TX3x exceeds I_{2019}^{F} in year t, we can also define the intensity and the intensity I_t^{F} (or I_t^{C}) of an event that has with the same probability as that of exceeding the intensity I_τ^{F} that was observed in year τ(as 2019) are given by:

$$\begin{cases} I_t^{\mathrm{F}} = \mathcal{Q}_t^{\mathrm{F}}(1-p_\tau^{\mathrm{F}}), \\ I_t^{\mathrm{C}} = \mathcal{Q}_t^{\mathrm{C}}(1-p_\tau^{\mathrm{F}}). \end{cases}$$

$$\begin{cases} p_t^{\mathrm{F}} := 1 - F_{\mathrm{GEV}}(I_{2019}^{\mathrm{F}}; \mu_t^{\mathrm{F}}, \sigma_t^{\mathrm{F}}, \xi_t^{\mathrm{F}}) \\ I_t^{\mathrm{F}} := \mathcal{Q}_{\mathrm{GEV}}(1 - p_{2019}^{\mathrm{F}}; \mu_t^{\mathrm{F}}, \sigma_t^{\mathrm{F}}, \xi_t^{\mathrm{F}}) \\ \mu_t^{\mathrm{F}} = \mu_0 + \mu_1 X_t^{\mathrm{F}} \\ \log \sigma_t^{\mathrm{F}} = \sigma_0 + \sigma_1 X_t^{\mathrm{F}} \\ \xi_t^{\mathrm{F}} \equiv \xi_0 \end{cases}, \quad \begin{cases} p_t^{\mathrm{C}} := 1 - F_{\mathrm{GEV}}(I_{2019}^{\mathrm{F}}; \mu_t^{\mathrm{C}}, \sigma_t^{\mathrm{C}}, \xi_t^{\mathrm{C}}) \\ I_t^{\mathrm{C}} := \mathcal{Q}_{\mathrm{GEV}}(1 - p_{2019}^{\mathrm{F}}; \mu_t^{\mathrm{C}}, \sigma_t^{\mathrm{C}}, \xi_t^{\mathrm{C}}) \\ \mu_t^{\mathrm{C}} = \mu_0 + \mu_1 X_t^{\mathrm{C}} \\ \log \sigma_t^{\mathrm{C}} = \sigma_0 + \sigma_1 X_t^{\mathrm{C}} \\ \xi_t^{\mathrm{C}} \equiv \xi_0 \end{cases}. \tag{7}$$

Note that this definition ensures that $I_{t=\tau}^{\mathrm{F}} = I_\tau^{\mathrm{F}}$. We can thus deduce the change in probability and intensity due to human influence PR_t and ΔI_t, as (see, e.g. Hannart et al., 2016):

$$\begin{cases} \mathrm{PR}_t = \dfrac{p_t^{\mathrm{F}}}{p_t^{\mathrm{C}}}, \\ \Delta I_t = I_t^{\mathrm{F}} - I_t^{\mathrm{C}}. \end{cases}$$

The probabilities in the factual and counterfactual world These formulas can be used to construct the classic indicators of change in intensity ($\Delta I_t := I_t^{\mathrm{F}} - I_t^{\mathrm{C}}$) and probability ratio ($\mathrm{PR}_t := p_t^{\mathrm{F}}/p_t^{\mathrm{C}}$). Note that the definition of I_t^{F} naturally implies that $I_{t=2019}^{\mathrm{F}} = I_{2019}^{\mathrm{F}}$.

We will use these six indicators p_t^{F} and , p_t^{C}, and the intensities I_t^{F} and , I_t^{C} for the two cases (dependent and independent) and four scenarios have been plotted with their 95% intervals in Fig. ??. For , X_t^{F} and X_t^{C}, calculated in cases where scenarios are inferred together or separately, in order to analyse the contribution of our approach. To do this, we performed 5000 samples of each of these indicators , we generated 5000 trajectories from the posterioraccording to the law defined by the posterior, and we constructed quantile-quantile diagrams between the different scenarios for the years 1850, 1884, 1950, i.e. 5000 realizations per year, from which we estimated the distributions. To verify the contribution of our methodology, we calculate for the independent case and the simultaneous case, for each pair of scenarios, 1992, 2050 and 2100. The last two years are only available for counterfactual variables, as the factual variables show divergences from the scenarios. The years 1884 and each year, the difference between the obtained distributions in the following way:

- For the factual world, these differences are calculated over the historical period (1850/2014);

- For the counter-factual world, the entire series is used;

– ~~The metric for calculating the difference between distributions is the Wasserstein distance, taken from optimal transport (see, e.g., San~~ ~~. This distance represents the average energy required to transform one distribution into another.~~

~~The calculated distances~~ 1992 correspond to two minima in natural forcings (see Fig **??**). In theory, if the scenarios are analysed separately, the counterfactual worlds associated with each scenario may be different, and the quantile-quantile plots will show deviations. Our approach is expected to greatly reduce these differences. The quantile-quantile plots are shown in Fig. ~~**??**.~~ ~~The first column corresponds to the independent case, the second to the simultaneous or dependent case~~ **??**. Let us begin with the probabilities p_t^{F} and p_t^{C}, lines a and b. In the counterfactual world in the case of joint scenarios (blue values), the quantile-quantile diagrams are almost perfectly aligned on the diagonal, showing that the data distributions of the four scenarios are indeed the same. In the case where the scenarios are handled separately (in red) in the counterfactual world, the values are much more dispersed, ~~the last being the relative difference representing the percentage reduction in inter-distances: the smaller the inter-distances, the more similar the different counter-factual and historical scenarios . In all cases, the distances decrease by 80% or even 100%,~~ showing a significant ~~improvement in consistency between the historical scenarios and between the counterfactuals. We note that pairs with a slight improvement or even deterioration already have low inter-distances. We can also see that in the independent case , inter-distances are altered by natural forcings (peaks), an alteration that has disappeared with our new methodology.~~ deviation between the different counterfactuals. In the factual world, the results are similar in 1850 and 1884, but the dispersion around the diagonal of the dependent case is greater in 1950 and 1992, due to the influence of the scenarios. The scenarios are not supposed to intervene at these points in time (in CMIP6, the SSPs begin in 2015), but inference on the complete series makes the smoothed values of the historical part partially dependent on the values of the part where the scenarios intervene.

~~Note also that we have compared the GEV coefficients on Fig. **??**. We can see that the SSP3-7.0, if estimated independently, is slightly ahead of the other scenarios, especially for the σ_1 parameter. Our approach therefore ensures consistency between the scenarios We also see that the posterior retains a Gaussian structure, allowing us to approximate it by a normal distribution.~~ Let us continue with the intensities in the factual and counterfactual worlds, lines c and d. The results are the same as for probabilities, with the dispersion appearing greater for probabilities due to the log scale.

~~To summarize, our new approach does a good job at ensuring historical and counterfactual consistency between the different climate scenarios~~ Let us conclude with the two factual and counterfactual regional covariates, lines e and f. The results are similar to the intensities and the probabilities.

3.0.2 Influence of the intensity of the event

During an attribution exercise, the analysed event may have a probability of zero (particularly in the counterfactual world), even within the entire confidence interval. This phenomenon may lead to the appearance of ceiling or floor values through the propagation of this 0. In order to quantify the significance of this phenomenon and how the multi-scenario behaves, we propose to resume the attribution where I_{2019}^{F} is defined as the median of the GEV distribution in 2019, and the 99.9% quantile (value that may pose a problem). We have reproduced Fig. **??** for these two events in figures **??** and **??**.

Figure ??, which is in a similar context to the attribution of a very strong event, is equivalent to Fig. ??, showing the same behavior. However, Fig. ??, constructed from a probable event (50%), shows a smaller difference between factual and counterfactual probabilities. The latter are now also distributed around the diagonal, showing equivalence between the multi-scenario and single-scenario approaches. We therefore conclude that the multi-scenario approach does indeed allow for a more consistent estimation of counterfactual probabilities between scenarios for the most extreme events, but that the contribution is weaker for more common events.

4 ANKIALE: ANalysis of Klimate with bayesian Inference: AppLication to extreme Events

The original method, proposed by Robin and Ribes (2020a), was accompanied by a package written in python (Van Rossum and Drake, 2009) or R (R Core Team, 2024): *Non-Stationary Statistics for Extreme Attribution* ~~(NSSEA, Robin and Ribes, 2020b)~~ (NSSEA Robin and Ribes, 2020b) to reproduce their results. Although this package can be used for attribution studies, the construction of its non-parallel code is not suitable for the simultaneous analysis of several thousand grid points, as is the case for a domain the size of Europe. Furthermore, its use requires in-depth knowledge of either the Python language or the R language.

We are therefore proposing a new package, which although written in Python, is presented as a command line tool that can be called in a bash script with the command ~~ank~~ ‘ank‘. The architecture of the package is described in ~~Sec.??~~Sect. 4.1. The various steps in ~~Sec. 3~~ Sect. 3.2 are broken down into sub-commands allowing them to be estimated, and are described in ~~Sec. ??~~Sect. 4.2. Examples are provided within the package, allowing ~~to reproduce the results presenting~~ reproduction of the results presented in this paper.

4.1 Architecture

The ANKIALE package contains two main classes: `ANKParams` which contains the computer parameters (temporary directories, number of CPUs, amount of memory, etc) and `Climatology` which describes the θ law. These two classes are instantiated when ANKIALE is launched. The first by the parameters of the user and the configuration of the system, the second either by a file passed by the user, or it is waiting to be built. The sub-module `ANKIALE.stats` then contains the classes and functions necessary for the estimations of θ:

- Class `ANKIALE.stats.MultiGAM`: inference of the covariates,
- Function `ANKIALE.stats.nslaw_fit`: maximum likelihood estimation, this function is generic and accepts the different laws grouped in the sub-module `ANKIALE.stats.models`. Note that minimisation calls the external package SDFC (Statistical Distribution Fit with Covariates Robin, 2020).
- Function `ANKIALE.stats.synthesis`: to build the multi-model synthesis.
- Function `ANKIALE.stats.gaussian_conditionning`: application of the Gaussian conditioning theorem.

- As explained in Sec ~~??~~3.2, the bayesian constraint uses the STAN (Stan Development Team, 2024) tool, which is used by default. It is possible to revert to the original algorithm with the ~~`--no-STAN`~~`-no-STAN` option.

~~Furthermor~~Furthermore, the display functions are grouped in the sub-module `ANKIALE.plot`, the commands in the sub-module `ANKIALE.cmd` and the data in the sub-module `ANKIALE.data`.

Parallelization and memory are controlled by several parameters:

- `-n-workers`: numbers of CPUs,
- `-memory-per-worker`: memory for each CPU, or,
- `-total-memory`: for the total available memory.

Parallelization occurs on several levels:

- The samples to construct covariance matrices or confidence intervals, which are independent;
- The grid, which can be unstructured, potentially allowing the analysis of several completely different events;
- The scenarios, in cases where there is independence (such as when constructing confidence intervals).

4.2 Package commands

`ank --help` Displays the documentation.

`ank fit` Starts the estimation of θ in the climate models simulations. The models ~~(see Sec. ??). The models~~ data should be netcdf files of dimension `(time,period,run)`, where `time` is the time axis, `period` the scenarios (historical and SSPs) and `run` the different members available. Additional dimensions can be added, representing spatial coordinates (e.g. latitude and longitude). The θ parameters are saved as a netcdf file containing the mean and covariance matrix for each estimated spatial dimensions.

`ank synthesize` Performs the multi-model synthesis calculation~~described in Sec. ??.~~. All the netcdf files produced by the previous command must be supplied. An important point at this stage is that each model is on its own grid, and they are interpolated onto the observation grid by nearest neighbor.

`ank constrain` Starts the observation-based constraint estimation~~described in Sec. ??~~, from the output file of the previous command.

`ank attribute` Starts an attribution by imposing either an event or a return time.

`ank draw` Draws θ parameters, and constructs the parameters of the statistical model given by Eq. ~~(??)~~5.

`ank show` Construct figures to analyze the different stages of the method.

`ank example` Places in a directory ready-to-use examples including data and scripts. Currently the following examples are supported:

- ~~GSMT,~~ **GSMT**: global warming estimation, allowing to reproduce the Fig. ~~??.~~ **??**. The values of global warming is in agreement with the work of Ribes et al. (2021).

– ~~Paris-Montsouris,~~ **Paris**: estimation and attribution of TX3x at ~~Paris-Montsouris French station~~Paris, allowing to reproduce the example used in the ~~See~~Sect. 3,

– **Ile-de-France**: this example reproduces the results of Sect. 5, except that the grid has been reduced to cover only the Ile-de-France ~~, estimation and attribution of TX3x over Ile-de-France, France, with ERA5, allowing to reproduce a sub-part of the results presented in the Sec. 5.~~ region (France) in order to reduce the size of the data and the computing time.

~~**Optional arguments**~~**Optional arguments** The optional arguments ~~--n-workers~~-n-workers and ~~--total-memory~~-tota allow to user to control the number of CPUs to be used, as well as the memory available. The parallelization and memory management tools are based on the package ~~dask (automatic parallelization, Dask Development Team, 2016)~~ dask (automatic parallelization D as well as ~~zarr (temporary files on disk to minimise memory usage, Miles et al., 2024)~~zarr (temporary files on disk to minimise memory .

4.3 Our example with ANKIALE

With ANKIALE, the entire procedure described in Sect. 3.3 can be performed in just a few lines of commands. For inference in climate models, this estimation can be done with two successive commands, one to estimate the parameters of the covariates $(\hat{\theta}_m^R, \hat{\theta}_m^G)$, and the other to estimate $\hat{\theta}_m$ (thus including the GEV part). Noting <file> as the input files and <climX>, <climY> as the files saving the estimates of θ_m, this gives:

```
ank fit X -input G,<file> R,<file> -save-clim <climX>
ank fit Y -input <file> -load-clim <climX> -save-clim <climY>
```

For the multi-model synthesis, noting <climY1>, <climYNM> the files of the inferred θ_m for each of the climate models, the command is:

```
ank synthesize -input <climY0> <climY1> ... <climYm> -save-clim <climS>
```

Constraints based on observations are applied using the two commands:

```
ank constrain X -input <obs> -load-clim <climS> -save-clim <climCX>
ank constrain Y -input <obs> -load-clim <climCX> -save-clim <climCY>
```

5 Highest temperatures in Europe

In order to study how the observed maxima behave (see Fig. **??**b) and could behave in the future, we propose to carry out their attribution. Classically, attributions, such as those carried out by the WWA ~~(see, e.g., Ciavarella et al., 2021; Philip et al., 2022; Zachariah e~~ (see, e.g. Ciavarella et al., 2021; Philip et al., 2022; Zachariah et al., 2023b), consider as a statistical variable the average of a climate variable (temperature, heat index, precipitation, etc.) over a domain (geographical area, country), and study an observed event and its impacts. For us, on the one hand, each ERA5 grid point in our domain will be a variable to be analyzed, and, on the other hand, we are not analyzing a specific event. No ~~form of~~ spatial dependency is ~~taken into account~~considered, so the existence occurrence of an event at one place location does not imply anything at another . about another location. For example, we cannot use these values to calculate the probability of an event occurring across the entire region.

We have plotted the maps of the parameters μ_0, μ_1, σ_0, σ_1 and ξ_0 in Fig. ??. The ERA5 bias for TX3x over the period 1961–1990 has also been added. None of these parameters show any particular spatial artifacts, leading us to believe that the inference was consistent between grid points. The parameter μ_1, which drives the trend of extremes, is positive (increase in extremes over time) and generally constant across the map. Its value close to 1 shows an increase in extremes parallel to regional warming. The parameter σ_1 is very close to 0, showing that variability remains constant until the end of the century. Finally, note that ξ_0 is systematically negative (which is consistent with the bounded nature of temperatures).

We start by looking at the current state of return durations times and intensity change, see Sec. ??defined as $\Delta I_t := I_t^{\mathrm{F}} - I_t^{\mathrm{C}}$ (from Eq. 7), see Sect. 5.1. We then continue with the near future in 2040, see Sec. ??Sect. 5.2. We finish with the end of the 21st century, see Sec. ??Sect. 5.3.

5.1 Current situation: 2024

Figure ?? shows the return times of the maximum observed in between 2024 since and 1940 in the counterfactual and factual world (Figs ??a,bFig. ??a-b), as well as the change in intensity (Fig. ??c). The 95% confidence intervals are given in figures ?? and ??. ?? and ??. It should be noted that in 2024 we are in the projection period (2015 to 2100), and therefore we potentially have several choices. At this point, we consider the influence of the choice of scenario to be too weak (compared to the internal variability), and we have therefore represented the average of the four scenarios.

We can see that the counterfactual world shows return times (Fig. ??a) greater than 1000 years over almost the whole of Europe, showing that the maxima currently recorded are almost impossible without anthropogenic climate change. The 95% confidence interval shows values down to 30 years over North Africa, Central Europe and Northern Europe, but almost the entire domain shows return periods of the order of at least 500 years.

In the factual world, North Africa shows return periods of 2 to 5 years (Fig. ??b), whereas in the counter-factual world they were in excess of 1000 years, showing that near-impossible events are currently becoming the new standard in this part of Europe. The same phenomenon can be seen over Western and Southern Asia, with equivalent values. The 95% confidence intervals show the same phenomenon. Generally speaking, with the exception of part of England, Belgium and Russia, the entire domain shows return periods for maximums of up to 100 years, and down to 10 years, which shows that maximums, which are supposed to be records (and therefore rare), are becoming the normthenorm.

The temperature increase from the conter-factual counter-factual to the factual world (Fig. ??c) is fairly uniform across the domain, with values around +2K+2K. The change is nevertheless marked in Northern Europe, with values of around +1.5+1.5K. The signal remains clearly positive, with the low value of the confidence interval around +1.5K, and falling to +0.6+0.6K in Northern Europe. The high end of the confidence interval is closer to +3K+3K, with peaks at +3.7+3.7K.

In line with all the studies on the attribution of extreme temperatures, it is clear that anthropogenic climate change implies a sharp increase in extreme temperatures. The sign of this change is unambiguous, as the low value of the confidence interval does not include a zero or negative change.

Let us finish with spatial variability, which may seem strange. Indeed, we can see that in Algeria, the return periods in Fig. **??**b can vary very rapidly from a value greater than 1000 years to less than 30 years. To understand this phenomenon, we have shown three series extracted from ERA5 in Fig. **??**: one in Paris and two in Algeria showing very different return periods. On these series (the black dots), we superimposed return periods of 2, 5, 10, 30, 50, 100, and 1000 years. In order to verify the quality of the fit for our three series, we have also displayed a histogram of the p-values of the Kolomogorov-Smirnov (KS) tests between 1000 draws of the GEV and ERA5 parameters. We can see that in at least 89% of cases, the KS test gives a p-value greater than 5%, showing that we cannot reject the hypothesis that the data are indeed derived from the inferred distribution. This therefore validates our fit. The difference between the series with a return period of $>$ 1000 years and the series with a return period of $<$ 30 years is the existence of an extreme event which does not change the adjustment but alters the value of the maximum observed. The spatial variability can therefore be explained by the existence or absence of an intense heatwave.

5.2 Mid-term: 2040

First and third rows of the Fig. **??** show the return times in 2040 in the counterfactual and factual world (Fig. **??**~~a/e~~a-e), as well as the change in intensity (Fig. **??**k-n). The 95% confidence intervals are given in figures ~~**??**a/e,k/n and **??**a/e,k/n~~**??**a-e,k-n and **??**a-e,k-n. Table ~~**??**~~ **??** gives a summary of the statistics by scenario and country. The 95% confidence interval is given in tables ~~**??** and **??**~~**??** and **??**.

For return times, the counterfactual world (Fig. **??**a) is the same as that in 2024 (Fig. **??**a), and shows return times of over 1000 years. The SSP2-4.5, SSP3-7.0 and SSP5-8.5 scenarios (Fig. **??**~~c/e~~c-e) are extremely close to each other, with values between 10 and 30 years over most of Europe, falling to 1 to 2 years over North Africa. Overall, the events are more likely than at present, reflecting the rise in temperatures over 16 years. However, these 3 scenarios have not yet been differentiated, unlike SSP1-2.6, which shows slightly longer return periods. Northern France, Belgium, Great Britain and Russia also show slightly longer return periods, between 50 and 500 years. The 95% confidence interval (figures ~~**??**a/e and **??**a/e~~**??**a-e and **??**a-e) shows a similar message: the three scenarios SSP2-4.5, SSP3-7.0 and SSP5-8.5 are extremely close, and SSP1-2.6 has slightly longer return times.

The change in intensity in 2040, visible in Fig. **??**k-n, shows, similarly to the return times, very close values – around ~~+3K to +3K~~ +3K to +3K – for the three scenarios SSP2-4.5, SSP3-7.0 and SSP5-8.5, and a scenario SSP1-2.6 with lower intensity changes of around ~~0.5~~0.5K. Northern Europe shows lower values, below ~~+2K~~+2K, while Eastern and Southern Europe reach almost ~~+4K~~+4K. The 95% confidence intervals (figures ~~**??**~~**??**k-n and ~~**??**~~**??**k-n) show a similar spatial dispersion of values, with ~~1K~~ 1K lower values for the lower bound, and ~~1K~~ 1K higher values for the upper bound. Some countries even show changes of more than ~~+5K~~+5K.

5.3 Long-term: 2100

Fig. **??**f shows the return times in 2100 of the maximums observed in the counterfactual world~~. Almost the entire domain shows return periods of over 1000 years, with the exception of central Eastern Europe, western North Africa, the far north of~~

~~Northern Europe and Ireland, where return periods of up to 100 years are rare.~~, as the values are the same as for Fig. **??**a, the conclusions of Sect. 5.2 apply.

Let us continue with the scenarios, represented on the Fig. **??**~~g/j~~g-j. Return times decrease with simulated climate change intensity. For SSP1-2.6, only 12 countries (on 55) show return times beyond 50 years, with 28 countries already having a return time of 10 years or less. From SSP2-4.5 onwards, the current maximums are almost commonplace, with only one country showing a return period in excess of 50 years. From SSP3-7.0 onwards, current maximums are the “normal” situation, with return times between 1 to 10 years and 1 to 2 years. The 95% confidence interval is shown in figures ~~**??**g/d and **??**g/d~~**??**g-d and **??**g-d, and shows that return periods can fall below 10 years over the whole of Europe~~, making annual events supposed to be centennial~~.

The scenarios for intensity change are represented on the Fig. **??**~~o/r~~o-r. For scenario SSP1-2.6, the change ranges from ~~+1.7~~+1.7K for Northern Europe to ~~+4.1~~+4.1K for Southern Europe, with the change relative to 2024 being almost the same everywhere, around ~~+1K~~+1K. For the SSP2-4.5 scenario, the change ranges from ~~+2.7~~+2.7K for Northern Europe to ~~+6.5~~+6.5K for Southern Europe. The different regions of Europe show different changes compared to today, ranging from ~~+1.5K to +3.8~~+1.5K to +3.8K. The SSP3-7.0 and SSP5-8.5 scenarios show increasing intensity increases, from ~~+5K to +9K~~+5K to +9K. The 95% confidence interval (figures ~~**??**o/r and **??**o/r~~**??**o-r and **??**o-r) even shows changes in intensity of up to ~~+18K~~+18K.

6 Conclusions and perspectives

6.1 Conclusions

In this paper, we have presented an extension of the Robin and Ribes (2020a) method for estimating probabilities of extremes following a GEV law. Our new method allows us, on the one hand, to treat several scenarios simultaneously, and on the other, to force a counter-factual world that is common to all scenarios. We first applied this method to temperature extremes over Paris (France), and demonstrated not only its validity, but also that it drastically reduces differences in the ~~historical part~~counter-factual probabilities, reinforcing the inter-scenario consistency of our estimates. We have also verified that our estimates of current and future global climate change are consistent with current literature.

We also offer an open-source software that can be ~~easily~~ used to reproduce our results and easily applied to other fields. This tool is natively parallelized, with particular attention paid to the memory used. It can be deployed just as easily on a personal computer, a computing cluster or a supercomputer. This software is also extensible, and other probability distributions – such as the Normal or Generalized Pareto Distribution – may be integrated in the future.

We have applied this new approach and tool to the attribution of observed maxima over Europe, enabling us to analyze these statistics up to the end of the 21st century for four climate scenarios. In the future, the observed maxima will become the new norm for scenarios greater than the SSP2-4.5 and SSP3-7.0, and will be 2K to 3K warmer even for a low-emission scenario like the SSP1-2.6. An increase in extreme temperatures of more than 10K is conceivable within the 95% confidence interval.

We have focused on Europe here, but an extension to the rest of the world and to temperature-like variables such as the heat-index would enable a global map of future heat ~~risks~~ hazards to be drawn up.

6.2 Perspectives

Even improvements to the GEV model are possible. For example, the GEV model tends to overestimate return times ~~(see, e.g., Diffenbaugh, , and recent~~ (see, e.g. Diffenbaugh, 2020; Zeder et al., 2023; Jewson et al., 2025). Recent work by Noyelle et al. (2025) proposed a new GEV model where the upper bound on temperatures is imposed by physics ~~(Zhang and Boos, 2023; Noyelle et al., 2023)~~ (Zhang and Boos, 2023; Noyelle et al., 2023). This approach would fit in naturally with the tools developed here. A similar method could be applied to precipitation by mixing an estimation of the upper bound (Martin et al., 2025) with a statistical model as the extended Generalized Pareto Distribution (Naveau et al., 2016). Another interesting possibility would be to use external forcings directly as covariates rather than their responses (global and regional temperatures), as their uncertainties are lower, even if they are not zero, particularly for the anthropogenic part.

Further work is also needed to extend our model to other variables such as wind and precipitation. The tools and statistical model developed here were developed in the context of attribution, particularly in relation to heat waves. Several studies use other types of covariates (such as CO2 or z500, see, e.g., Smith et al., 2021; Auld et al., 2023) or statistical models. Extending this to other variables would require work similar to that of Robin and Ribes (2020a), where the validity of the statistical model was verified in each climate model, as well as its quality after applying observational constraints.

It should also be noted that, on the one hand, we have remained in a univariate context, while the estimation of concurrent events increasing impacts appears increasingly necessary; and on the other hand, spatial structures are ignored. We also used only GCMs, which do not capture local specificities. The use of regional models (when those from CMIP6 become available) will allow us to refine the results obtained here, and ANKIALE will make this easy to do.

Finally, the analyses produced here provide local information on the worst possible future events, and show the need for rapid adaptation to extremes warming faster than global warming.

Code and data availability. GISTEMP data are available at data.giss.nasa.gov/gistemp (Lenssen et al., 2019). HadCRUT5 data were obtained from metoffice.gov.uk/hadobs/hadcrut5 (Morice et al., 2021; Osborn et al., 2021) on 2025 and are © British Crown Copyright, Met Office 2020, provided under an Open Government License, www.nationalarchives.gov.uk/doc/open-government-licence/version/3/ ERA5 data are available in the Climate Data Store at DOI:10.24381/cds.adbb2d47 (Hersbach et al., 2020). The CMIP6 model simulations can be downloaded through the Earth System Grid Federation portals. Instructions to access the data are available here.

The current version of ANKIALE is available from the project website: github.com/yrobink/ANKIALE under the GNU-GPL3 licence. The exact version of the model used to produce the results used in this paper is archived on Zenodo under **??**DOI:10.5281/zenodo.15038388 (Robin, 2025), as are input data and scripts to run the model and produce the plots for all the simulations presented in this paper.

Author contributions. YR had the initial idea of the study, which has been completed and enriched by all co-authors. YR developed the multi-scenarios methods, and OB developed the MCMC, both helped by MV, AR, and PN for the statistical modelling and inferential schemes. YR developed the ANKIALE package and applied it to Europe for the different experiments and wrote the codes for the analyses and to plot the figures. All authors contributed to the methodology and the analyses. YR wrote the first draft of the article with inputs from all the co-authors.

Competing interests. The authors declare no competing interests.

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Appendix A: Detailed mathematics of the methodologyGeneralized Extreme Value distribution

A1 Decomposition of the covariables

The maxima of a variable can be modelled using the GEV distribution (Generalised Extreme Value, see the book by Coles, 2001) . This distribution has three parameters:

We will then decompose the temperatures (global or regional) so as to isolate forcings of natural origin on the one hand, and anthropogenic forcings on the other hand. Recall that we have the following equation, already given in Eq. **??**, with the difference that the linear coefficient of $X^{*,\mathrm{N}}$ is explained

- The location parameter μ, similar to the mean;
- The scale parameter σ, similar to the standard deviation;
- The shape parameter ξ which controls the type of extreme. If $\xi < 0$, the extremes are bounded, if $\xi > 0$ the distribution is said to be *heavy-tailed*.

Noting $x_+ := \max(0, x)$, the cumulative distribution function F_{GEV} (and the survival function $1 - F_{\mathrm{GEV}}$) of a random variable $T \sim \mathrm{GEV}(\mu, \sigma, \xi)$ is given by an analytical equation (when $1 + \xi\frac{x-\mu}{\sigma} \geq 0$):

$$X_t^* F_{\mathrm{GEV}}(x; \mu, \sigma, \xi) := X^{*,0} \mathbb{P}(T < x) = \exp\left[-\left(1 + \alpha^{*,\mathrm{N}} X_t^{*,\mathrm{N}} + X_t^{*,\mathrm{A}} + \varepsilon \xi \frac{x-\mu}{\sigma}\right)_+^{-1/\xi}\right].$$

WhereThe quantile function $\mathcal{Q}_{\mathrm{GEV}}$ (inverse of the distribution function) is also given by:

$$\mathcal{Q}_{\mathrm{GEV}}(p; \mu, \sigma, \xi) := \mu + \frac{\sigma}{\xi}\left[(-\log p)^{-\xi} - 1\right] = F_{\mathrm{GEV}}^{-1}(p).$$

Note that if $\xi > 0$, the distribution is not bounded and is said *heavy tail*. If $\xi < 0$, the extremes are bounded, with the upper bound B given by:

$$B := \mu - \frac{\sigma}{\xi}.$$

In general, to use the GEV distribution, we use the block-maxima theorem, which states that if we divide a data set into blocks and take the maximum of each block, then asymptotically this random variable follows a GEV distribution. Here, annual maximums are used, and this works well overall with temperature.

Appendix B: Some details about the inference of θ

B1 Fit in the climate models

Our approach therefore begins by inferring the θ_m of the statistical model in Eq. 5 using data from N_M climate models.

For each climate model m, we have three time series (with possible repetitions of the time steps for each member) for each SSP scenario:

- $X^{*,0}$: a constant, $\tilde{X}^{R,\mathrm{SSP}}_{t,m}$, regional average temperature series, here for Europe;
- $X^{*,\mathrm{N}}_t$: either the response of an Energy Balance Model (Energy Balance Model, Held et al., 2010; Geoffroy et al., 2013) for CMIP5, or radiative forcings for CMIP6 (**?**); modeling natural forcings,
- $X^{*,\mathrm{A}}_t$: a smoothing spline, modelling the anthropogenic forcing (see below)$\tilde{X}^{G,\mathrm{SSP}}_{t,m}$, global average temperature series,
- ε: a white noise Gaussian error term describing natural variability$\tilde{T}^{\mathrm{SSP}}_{t,m}$, series of annual maximum temperatures over 3 days.

This equation is a Generalized Additive Model (GAM, see e.g., Hastie, 2017), with a parameter to be estimated for the constant (the constant itself in fact), a parameter $\alpha^{*,\mathrm{N}}$ for the natural term (the linear regression term). The term $X^{*,\mathrm{A}}_t$ is obtained from a smoothing spline.

Let dof denote the number of degrees of freedom of the (unbiased) spline basis. The inference begins by estimating the parameters of the covariates. Starting from the series $\tilde{X}^{R,\mathrm{SSP}}_{t,m}$ and N_T the number of time steps (here $N_T = 251$ years). The term $X^{*,\mathrm{A}}$ can then be reformulated as a matrix product between a matrix B^S (B for *Basis*) of size $N_T \times dof$, and a $\tilde{X}^{G,\mathrm{SSP}}_{t,m}$, the pair $(\hat{\theta}^R_m, \hat{\theta}^G_m)$ is estimated using the approaches described in Sect. **??**, as well as the covariance matrix $\Sigma_{(\theta^R_m,\theta^G_m)}$ describing the uncertainty in this estimation.

Next, we estimate $\hat{\theta}^{\mathrm{GEV}}_m$ from the series $\tilde{T}^{\mathrm{SSP}}_{t,m}$ of each model. To do this, we use the vector of coefficients of size dof $(\hat{\theta}^R_m, \hat{\theta}^G_m)$ to generate the forcings $X^{R,\mathrm{SSP}}_{t,m}$. The vector $\hat{\theta}^{\mathrm{GEV}}_m$ can thus be calculated by maximum likelihood, which allows us to estimate $\hat{\theta}_m = (\hat{\theta}^R_m, \hat{\theta}^G_m, \hat{\theta}^{\mathrm{GEV}}_m)$. The covariance matrix $\Sigma_{\hat{\theta}_m} = \Sigma_{(\theta^R_m,\theta^G_m,\theta^{\mathrm{GEV}}_m)}$ of $\hat{\theta}_m$ is estimated using a bootstrap on the series $\tilde{T}^{\mathrm{SSP}}_{t,m}$ and the forcings $X^{R,\mathrm{SSP}}_{t,m}$, forcings constructed from several samples according to the normal distribution $\mathcal{N}((\hat{\theta}^R_m, \hat{\theta}^G_m), \Sigma_{(\theta^R_m,\theta^G_m)})$.

Note that this approach is purely frequentist, in the sense that we estimate the value of θ_m and its covariance matrix, as opposed to the Bayesian view, where we want to determine the distribution of θ_m.

B2 Construction of the prior

The prior is then constructed as a synthesis of climate models. This is where we switch to a Bayesian view: θ is no longer seen as a value to be estimated but as a *random* vector. For each climate model, we then define the random variable θ_m, which follows the following multivariate normal distribution:

$$X^{*,\mathrm{A}}_t = B^S \cdot (s_0 \theta_m \sim \mathcal{N}\left(\hat{\theta}_m, , s_{dof-1})^T = \cdot (s_0, , s_{dof-1})^T . \Sigma_{\hat{\theta}_m}\right)$$

Following Robin and Ribes (2020a)the work of Ribes et al. (2017), we assume $dof = 6$ for the smoothing spline of the anthropogenic part. Temperatures (global or regional) can therefore be described by an 8-parameter θ vector of the following form (the $\oplus$ symbol designates the direct sum of vectors or matrices, i.e. concatenation along dimensions)that *reality is statistically indistinguishable from a set of climate models* (see also Annan and Hargreaves, 2010; Rougier et al., 2013). This allows us to construct a multi-model synthesis $\theta_* \sim \mathcal{N}\left(\hat{\theta}_*, \Sigma_{\hat{\theta}_*}\right)$ which also follows a normal distribution with parameters:

$$\begin{cases} \hat{\theta}_* = \dfrac{1}{N_M}\sum_m \hat{\theta}_m, \\ \Sigma_{\hat{\theta}_*} = \left(1 + \dfrac{1}{N_M}\right)\hat{\Sigma}_u + 1/N_M^2 \sum_m \Sigma_{\hat{\theta}_m}. \end{cases}$$

In this last equation, the matrix $\hat{\Sigma}_u$ describes the internal variability of the models. The mathematics describing this approach can be found in Sect. **??**.

B3 Derivation of the posterior

For the construction of the posterior, we have the observations $X_t^{o,R}$, $X_t^{o,G}$ and T_t^o (regional average temperature, global average temperature and temperature extremes). Let us start again from the calculation in Robin and Ribes (2020a, Sect. 3.5), which allows us to separate the conditioning by $X_t^{o,R}$ and $X_t^{o,G}$ from that by T_t^o. We then have:

$$\theta := \left(X^{*,0}, \alpha^{*,\mathrm{N}}\right) \oplus \left(s_0^{*,A}, \ldots, s_5^{*,A}\right) = \left(X^{*,0}, \alpha^{*,\mathrm{N}}, s_0^{*,A}, \ldots, s_5^{*,A}\right).$$

$$\mathbb{P}\left[\theta_* | (X_t^{o,R}, X_t^{o,G}, T_t^o)\right] = \frac{\mathbb{P}\left[T_t^o | (\theta_* | (X_t^{o,R}, X_t^{o,G}))\right] \mathbb{P}\left[\theta_* | (X_t^{o,R}, X_t^{o,G})\right]}{\mathbb{P}(T_t^o)} \tag{B1}$$

In a similar way to B^S, we will note B^N the design matrix of the natural part, The important point is that, starting from the prior θ_*, we can construct the posterior $(\theta_* | (X_t^{o,R}, X_t^{o,G}))$, thus defining a new random variable. This latter variable can itself be considered as a prior to be constrained by T_t^o, which allows us to write: derive the complete posterior. This constraint is therefore applied in two steps.

$$X_t^* = \left[\underbrace{\begin{pmatrix} 1 & X_0^{*,\mathrm{N}} \\ 1 & X_1^{*,\mathrm{N}} \\ \vdots & \vdots \\ 1 & X_{N_T-1}^{*,\mathrm{N}} \end{pmatrix}}_{B^N} \oplus \underbrace{\begin{pmatrix} B_0^{S,0} & \cdots & B_0^{S,5} \\ B_1^{S,0} & \cdots & B_1^{S,5} \\ \vdots & \ddots & \vdots \\ B_{N_T-1}^{S,0} & \cdots & B_{N_T-1}^{S,5} \end{pmatrix}}_{B^S}\right] \cdot \theta^T + \varepsilon = \begin{pmatrix} 1 & X_0^{*,\mathrm{N}} & B_0^{S,0} & \cdots & B_0^{S,5} \\ 1 & X_1^{*,\mathrm{N}} & B_1^{S,0} & \cdots & B_1^{S,5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{N_T-1}^{*,\mathrm{N}} & B_{N_T-1}^{S,0} & \cdots & B_{N_T-1}^{S,5} \end{pmatrix} \cdot \theta^T + \varepsilon.$$

B3.1 Covariables constraint

~~From this equation we can then construct the detailed version of Eq. (??)~~The estimate of $(\theta_*|(X_t^{o,R}, X_t^{o,G}))$ is in fact analytical, and the Gaussian conditioning theorem (Eaton, 2007) applies. Let X^o be a global vector of observations that concatenates $X_t^{o,R}$ and $X_t^{o,G}$ over time, and $\varepsilon^o \sim \mathcal{N}(0, \Sigma^o)$ be white noise of the same dimension as X^o. If we find a matrix A such that $X^o = A \cdot \theta_* + \varepsilon^o$, and since θ_* follows a normal distribution, then $(\theta_*|X^o) = (\theta_*|(X_t^{o,R}, X_t^{o,G}))$ also follows a normal distribution, i.e. $(\theta_*|X^o) \sim \mathcal{N}(\mu_{(\theta_*|X^o)}, \Sigma_{(\theta_*|X^o)})$, with value:

$$
\begin{aligned}
\theta &= \theta^{\mathrm{G}} \\
&\oplus \theta^{\mathrm{R}}. \\
&= \left(X^{\mathrm{G},0}, \alpha^{\mathrm{G,N}}\right) \oplus \left(s_0^{\mathrm{G,A,SSP1\text{-}2.6}}, \ldots, s_5^{\mathrm{G,A,SSP1\text{-}2.6}}\right) \\
&\qquad \oplus \left(s_0^{\mathrm{G,A,SSP2\text{-}4.5}}, \ldots, s_5^{\mathrm{G,A,SSP2\text{-}4.5}}\right) \\
&\qquad \oplus \left(s_0^{\mathrm{G,A,SSP3\text{-}7.0}}, \ldots, s_5^{\mathrm{G,A,SSP3\text{-}7.0}}\right) \\
&\qquad \oplus \left(s_0^{\mathrm{G,A,SSP5\text{-}8.5}}, \ldots, s_5^{\mathrm{G,A,SSP5\text{-}8.5}}\right) \\
&\oplus \left(X^{\mathrm{R},0}, \alpha^{\mathrm{R,N}}\right) \oplus \left(s_0^{\mathrm{R,A,SSP1\text{-}2.6}}, \ldots, s_5^{\mathrm{R,A,SSP1\text{-}2.6}}\right) \\
&\qquad \oplus \left(s_0^{\mathrm{R,A,SSP2\text{-}4.5}}, \ldots, s_5^{\mathrm{R,A,SSP2\text{-}4.5}}\right) \\
&\qquad \oplus \left(s_0^{\mathrm{R,A,SSP3\text{-}7.0}}, \ldots, s_5^{\mathrm{R,A,SSP3\text{-}7.0}}\right) \\
&\qquad \oplus \left(s_0^{\mathrm{R,A,SSP5\text{-}8.5}}, \ldots, s_5^{\mathrm{R,A,SSP5\text{-}8.5}}\right) \\
&= \left(X^{\mathrm{G},0}, \alpha^{\mathrm{G,N}}\right) \bigoplus_{\mathrm{SSP}} \left(s_0^{\mathrm{G,A,SSP}}, \ldots, s_5^{\mathrm{G,A,SSP}}\right) \\
&\oplus \left(X^{\mathrm{R},0}, \alpha^{\mathrm{R,N}}\right) \bigoplus_{\mathrm{SSP}} \left(s_0^{\mathrm{R,A,SSP}}, \ldots, s_5^{\mathrm{R,A,SSP}}\right)
\end{aligned}
$$

$$
\begin{cases}
\mu_{(\theta_*|X^o)} = \theta_* + (\Sigma_{\theta_*} A^T) \cdot (A \Sigma_{\theta_*} A^T + \Sigma^o)^{-1} \cdot (X^o - A\theta_*), \\
\Sigma_{(\theta_*|X^o)} = \Sigma_{\theta_*} - (\Sigma_{\theta_*} A^T) \cdot (A \Sigma_{\theta_*} A^T + \Sigma^o)^{-1} \cdot (A \Sigma_{\theta_*}).
\end{cases}
$$

~~We therefore add the GEV parameters to the statistical model in Eq. (??):~~ The difficulties here are constructing the matrix A, which models how our parameters are transformed into the observation signal, and estimating Σ^o, which models the internal variability of the observations.

$$
\begin{aligned}
\theta &= \theta^{\mathrm{G}} \oplus \theta^{\mathrm{R}} \oplus (\mu_0, \mu_1, \sigma_0, \sigma_1, \xi_0) \\
&= \theta^{\mathrm{G}} \oplus \theta^{\mathrm{R}} \oplus \theta^{\mathrm{GEV}}
\end{aligned}
$$

For matrix A, two approaches are proposed in Sect. **??**. One is based on the idea that, over the observed period, the scenarios are sufficiently similar that their mean can be projected onto the observations. The second requires choosing a scenario. In this article, we will use the first approach.

For the covariance matrix Σ^o, two approaches are also proposed in Sect. **??**. The first is simply to consider it as white noise, estimates of observations from which a trend has been removed. The second approach was developed by Ribes et al. (2021) and Qasmi and Ribes (2022), and assumes that ε^o takes the form of a sum of two first-order autoregressive processes. One is *fast* to model inter-annual variability, while the second is *slow* to model decadal variability. In this article, we will use the first approach.

B4 Simultaneous constraint of external forcings: construction of the A^1 matrix

B3.1 Variable constraint

Recall the Eq. (**??**): With our knowledge of the distribution $(\theta_*|(X_t^{o,R}, X_t^{o,G}))$, we now want to obtain samples of the distribution $([\theta_*|(X_t^{o,R}, X_t^{o,G})]|T_t^o)$. The whole problem is that T_t^o follows a GEV distribution, and there is no explicit expression for the posterior. Let us start again from the Eq. B1.

$$X_t^{\mathrm{GR,o}} := \begin{pmatrix} X_t^{\mathrm{G,o}} \\ X_t^{\mathrm{R,o}} \end{pmatrix} = A \cdot \kappa + \varepsilon^o$$

The A matrix is the following form:

$$A := \begin{pmatrix} R^{\mathrm{G,o}} & R^{\mathrm{R,o}} \end{pmatrix} \cdot \begin{pmatrix} A^1 & A^0 & A^{\mathrm{GEV}} \\ A^0 & A^1 & A^{\mathrm{GEV}} \end{pmatrix}$$

where:

- A^1 is a matrix which transforms κ^{G} or κ^{R} into the mean covariate, taken along the SSPs. In other words for κ^{G}:

  $$A^1 \cdot \kappa^{\mathrm{G}} = \frac{1}{N^{\mathrm{SSP}}} \sum_{\mathrm{SSP}} X_t^{\mathrm{G,F,SSP}}$$

  The term $\mathbb{P}[\theta_*|(X_t^{o,R}, X_t^{o,G})]$ is known, it is our prior.

- A^0 is the same dimension as A^1, but with null values. The term $\mathbb{P}[T_t^o|(\theta_*|(X_t^{o,R}, X_t^{o,G}))]$ is directly calculable: the draws generate the parameters of the GEV law, which can thus be evaluated.

- A^{GEV} is a null matrix such that $A^{\mathrm{GEV}} \cdot \kappa^{\mathrm{GEV}} = 0$.

- $R^{\mathrm{G,o}}$ are $R^{\mathrm{R,o}}$ are matrices which restrict the time axis to that of the observations. When the denominator is analytically intractable, numerical methods are necessary to sample from the posterior distribution.

We propose here to detail theconstruction of A^1. Let's start with Eq. (??) and (??), we can write the following relation for κ^{G}: A common approach to perform this sampling is the Metropolis-Hasting algorithm (Metropolis et al., 1953), (Hastings, 1970). This is the sampling algorithm originally used by Robin and Ribes (2020a). This Markov chain Monte Carlo algorithm relies on a random walk proposal: a new proposal is created by starting from an initial value θ_0 and adding a random noise to generate a θ_1. The new value is either accepted or rejected with a probability defined using the likelihood ratio of the proposal and the previous value. A key element of this procedure is the transition function between θ_i and $\theta_(i+1)$ that is used to sample successive possible values of the posterior.

$$\left[B^N \oplus \bigoplus_{\mathrm{SSP}} B^S\right] \cdot \kappa^{\mathrm{G}} = X^{\mathrm{G},0} + X_t^{\mathrm{G,N}} + \sum_{\mathrm{SSP}} X_t^{\mathrm{G,A,SSP}}$$

In the Robin and Ribes (2020a) original implementation, the transition function was of the form $\theta_{i+1} = \theta_i + \varepsilon$ where ε follows a normal distribution with the same scale for all parameters. This can become an issue when the scale of the target parameters is very different from one another. The transition also determines the rate of convergence and mixing, so this implementation can be computationally sub-optimal. Various diagnostics showed the algorithm suffered from slow-mixing chains (Gelman et al., 1997), high autocorrelation (Brooks et al., 2011), and low effective sample size (Gelman et al., 2015).

By multiplying the matrix B^N by the number of SSP scenarios N^{SSP}, we have: To deal with these issues, we leverage the *No-U-Turn Sampler* algorithm NUTS, (Hoffman and Gelman, 2014), as implemented in STAN (Stan Development Team, 2024). This algorithm is based on the Hamiltonian Monte Carlo algorithm (Radford, 2011), a variant of the Metropolis-Hasting algorithm where the proposal is not generated using a random walk. Instead, the proposal is created through a series of gradient-informed steps (Betancourt, 2018). This allows for better parameter space exploration, especially in the multidimensional case. The NUTS variant relies on a specific criteria to select adaptively various hyper-parameters such as the steps length and stopping conditions. This adaptation makes the algorithm more robust against correlation in the posterior. The NUTS algorithm is particularly effective when the posterior dimensions are correlated or of different scales. It is very efficient to explore the parameter space and draw samples from the posterior.

$$\frac{1}{N^{\mathrm{SSP}}}\left[\left(N^{\mathrm{SSP}} B^N\right) \oplus \bigoplus_{\mathrm{SSP}} B^S\right] \cdot \kappa^{\mathrm{G}} = \frac{1}{N^{\mathrm{SSP}}} \sum_{\mathrm{SSP}} \left[X^{\mathrm{G},0} + X_t^{\mathrm{G,N}} + X_t^{\mathrm{G,A,SSP}}\right] = \frac{1}{N^{\mathrm{SSP}}} \sum_{\mathrm{SSP}} X_t^{\mathrm{G,F,SSP}}$$

The matrix A^1 is then given by:

$$A^1 := \frac{1}{N^{\mathrm{SSP}}}\left[\left(N^{\mathrm{SSP}} B^N\right) \oplus \bigoplus_{\mathrm{SSP}} B^S\right]$$