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An idealized model for the spatial structure of the eddy-driven Ferrel cell in mid-latitudes

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Abstract. As global warming intensifies, mid-latitude regions increasingly experience unusual and disruptive weather phenomena, such as extreme heatwaves and devastating floods, posing significant threats to societies. The dynamics of the jet stream largely govern mid-latitude weather patterns, with fluid-dynamical instabilities generating baroclinic waves that propagate through the atmosphere before breaking. These waves play a critical role in shaping regional weather by influencing the jet stream's maintenance and inducing an indirect meridional circulation. Understanding how the life cycle of baroclinic waves maintains the zonal-mean zonal wind and the indirect circulation is essential for improving predictions of mid-latitude weather under global warming. This study introduces a simplified theoretical framework that provides analytic solutions for the steady-state zonal-mean zonal wind and the indirect circulation. By incorporating a parameterization of turbulent eddies into the zonal-mean quasi-geostrophic potential vorticity equation, the model establishes a balance that drives eddy-induced circulations in mid-latitudes, analogous to the Ferrel cell. The meridional temperature structure reveals two key features: (1) a linear decrease in anomalous potential temperature, producing westerly winds, and (2) jet streams generated by eddy momentum fluxes. These jet streams are accompanied by downward (upward) motions on their southern (northern) flanks, further characterizing the eddy-driven circulations.The intensity and extent of the Ferrel cell are found to be governed by the baroclinic wave life cycle, constrained by the meridional temperature gradient, static stability, and boundary conditions influenced by tropical forcing. Under global warming, projected changes in these factors will alter mid-latitude circulation patterns. The theoretical framework proposed in this study offers a robust basis for analyzing and predicting the evolving dynamics of mid-latitude circulations in a warming climate.

1 Introduction

The Earth's large-scale atmosphere serves as a medium for redistributing excess energy from tropical regions to higher latitudes (Held and Hoskins, 1985). As a result, the general circulation, acting as a mean field, controls the behavior of transient weather patterns in the atmosphere (Starr, 1956). This circulation encompasses various features, including large-scale tropical convection that gives rise to rainforests in the tropics, subsidence areas near the edges of the tropical regions where major deserts are located (Webster, 2004), and mid-latitudes where turbulent synoptic eddies influence weather and climate (Schneider, 2006).

Despite the prevalence of turbulent eddies throughout the entire atmospheric domain due to convection and instability, the general circulation exhibits an organized structure comprising three meridional cells on Earth (Held and Hoskins, 1985). A fundamental question arises: How do these turbulent eddies contribute to the formation of organized circulations while adhering to the constraints imposed by large-scale atmospheric conditions, such as geostrophic and hydrostatic balance, planetary size, and radiative transfer involving various gases? This question is a long standing one in the field of atmospheric science.

The Hadley circulation, spanning from the tropics to the subtropics, plays a crucial role in shaping the climates of tropical rainforests and subtropical deserts (Webster, 2004). In this circulation, air near the surface of tropical regions is lifted through convection and then moves poleward. However, it is unable to reach the poles and then the air descends in subtropical areas. A significant theoretical advancement in understanding the structure of the Hadley circulation stems from the application of a dynamical constraint: angular momentum conservation. This constraint allows for the simplification of the nonlinear dynamics involved. In a zonally symmetric situation, the conservation of angular momentum enables the calculation of the zonal mean zonal wind along with the meridional gradient of potential temperature (Held and Hou, 1980).

Assuming that the Hadley circulation is bounded by non-turbulent flows, two thermodynamic constraints come into play: the continuity of potential temperature and the energy flux balance within the cell. These constraints facilitate the construction of a model (Held and Hou, 1980; Lindzen and Hou, 1988) that calculates the potential temperature at the tropical region and the latitude of the edge of the Hadley cell. The model successfully reproduces key features of the real-world system, including the zonal mean zonal wind and surface winds such as the trade winds in the tropics.

Despite its structural similarities to the Hadley cell constructed from reanalysis data, the original theory has limitations when applied to current issues related to global warming (Lu et al., 2007). Evidence from reanalysis data (Nguyen et al., 2013; Davis and Davis, 2018; Choi et al., 2014) and climate model simulations (Grise and Davis, 2020; Kang and Lu, 2012; Seo et al., 2014) suggests that the Hadley circulation is expanding due to global warming. However, the original theory, which deals with an isolated circulation in the tropics, cannot explain this expansion as it requires interactions with mid-latitudes (Lu et al., 2007; Seo et al., 2014). It has become evident that the dynamics of the Hadley cell are influenced by the life cycle of synoptic eddies generated by the baroclinic instability of the mid-latitude jet (Levine and Schneider, 2015).

In contrast to the Hadley circulation, which is directly influenced by thermal forcing, leading to the generation of tropical convection, the circulation in mid-latitudes exhibits distinct fluid-dynamical characteristics. Rather than being subject to direct forcing, an indirect circulation known as the Ferrell cell is generated by turbulent synoptic eddies. Originating from the downward motion near the edge of the Hadley cell, the Ferrell cell is characterized by poleward (equatorward) meridional winds in the lower (upper) levels. The most noteworthy feature, in addition to the meridional structure, is the jet stream located near the center of the cell in the upper levels. It can be inferred that the persistent state of the jet stream is sustained by the meridional circulation (Robinson, 2000). The temporal evolution of the jet stream around its steady state is pivotal in understanding the low-frequency variability of the mid-latitude atmosphere.

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Describing the circulation in mid-latitudes involves addressing two fundamental tasks. First, it is imperative to elucidate how turbulent synoptic eddies are generated, propagate, and eventually decay. Second, understanding the life cycle of turbulent eddies and their contribution to the formation of organized circulation in mid-latitudes is crucial. These tasks appear to be approached through the lens of wave-mean interactions.

The nonlinear characteristics of large-scale atmospheric dynamics in mid-latitudes have been extensively studied through model simulations (Thorncroft et al., 1993; Simmons and Hoskins, 1978) and reanalysis data (Randel and Stanford, 1985; Moon and Feldstein, 2009). One of the primary drivers of the flow in this region is the life cycle of synoptic eddies (Edmon Jr et al., 1980), which can be viewed as an engine for transferring energy to higher latitudes. Synoptic eddies grow by extracting energy from the baroclinic mean fields, leading to a weakening of the meridional temperature gradient through turbulent poleward heat flux (Charney, 1947; Eady, 1949; Phillips, 1951). This weakening of the temperature gradient is equivalent to a decrease in the zonal mean zonal wind in upper levels through thermal wind balance. As saturated synoptic eddies reach the jet stream, they break and release their energy back to the mean field through eddy momentum flux, resulting in the strengthening of the zonal mean zonal wind (Simmons and Hoskins, 1978). The life cycle of synoptic eddies in mid-latitudes gives rise to the Ferrell cell (Held and Hoskins, 1985). The formation of the Ferrell cell results from a dynamic balance between the turbulent heat and momentum fluxes generated by synoptic eddies.

Numerous studies utilizing numerical models (Maher et al., 2019) and reanalysis data (Lorenz and Hartmann, 2001) have shed light on various aspects of the Ferrell cell, including its interactions with the tropics and polar regions, as well as its response to ongoing global warming. However, a simple theoretical model that can explain the overall structure of the eddy-driven circulation, encompassing jet formation, the vertical motion at the flanks of jet streams, and the formation of multiple jets depending on planetary size and the characteristics of turbulent eddies, is still lacking. Development of a theoretical frame to include the accumulated effects of turbulent eddies in a mean structure of hemispheric atmospheric circulation would help interpreting models and observations.

The description of large-scale atmospheric dynamics primarily relies on the quasi-geostrophic potential vorticity equation (Pedlosky, 2013), which serves as an approximation of the primitive equations for large-scale atmospheric motion satisfying geostrophic and hydrostatic balances. This equation is particularly designed to capture the characteristics of synoptic eddies. One can study the life cycle of synoptic eddies (Simmons and Hoskins, 1978), ranging from their generation through baroclinic instability to their breaking in the upper levels, based on the quasi-geostrophic potential vorticity equation. This equation is, hence, highly valuable for investigating the detailed dynamical features of synoptic eddies in the presence of various mean fields.

In this paper, the focus is on theory for zonally averaged geostrophic motions for providing a steady-state solution similar to the Ferrell cell. The main challenge lies in parameterizing the turbulent heat and momentum fluxes using zonally averaged variables (Whitaker and Sardeshmukh, 1998; Jin et al., 2006). The goal is to develop analytic solutions for a parameterized zonal average geostrophic motion that describe the eddy-driven circulation in mid-latitudes.

To achieve this, we propose two simple parametrizations. Their simplicity neglects the complicated dynamics and interactions of eddies, however the same simplicity allows us to qualitatively understand the contrasting role of heat and momentum

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fluxes on the PV-distribution in the mid-latitude atmosphere and how this distribution gives rise to a thermally indirect circulation cell; The Ferrell cell. Section 2 outlines the steps involved in constructing a parameterized zonal average quasi-geostrophic equation for mid-latitudes. The construction of solutions of the main equation and an interpretation of these solutions is provided in Section 3. Finally, implications are discussed in Section 4.

2 Parameterized quasi-geostrophic motion in a zonally symmetric atmosphere

2.1 Zonally-averaged quasi-geostrophic equation

Consider the steady-state quasi-geostrophic motion in a zonally symmetric atmosphere (Pedlosky, 2013), described by the following two equations:

$$(U_E+u_0)\frac{\partial \zeta_0}{\partial x}+v_0\frac{\partial \zeta_0}{\partial y}+\beta v_0=\frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\rho_S w_1\right), \tag{1}$$

$$(U_E+u_0)\frac{\partial \eta_0}{\partial x}+v_0\frac{\partial}{\partial y}(\Theta_E+\eta_0)+Sw_1=-\frac{1}{\tau}\eta_0, \tag{2}$$

where U_E is the zonal mean zonal wind derived from the radiative equilibrium temperature Θ_E via thermal wind balance, u_0 and v_0 are the anomalous zonal and meridional geostrophic winds, ζ_0 is the horizontal vorticity defined as $\partial u_0/\partial y - \partial v_0/\partial x$, ρ_S is a reference density profile, w_1 is the ageostrophic vertical velocity, S is the non-dimensionalized dry static stability, η_0 is the anomalous potential temperature, and τ is the response timescale of the radiative-convective equilibrium. We assume that the radiative-convective equilibrium temperature Θ_E is already known.

Here, the first equation represents the vorticity balance, while the second equation represents the heat flux balance. These equations are non-dimensionalized. The subscripts 0 and 1 represent leading-order variables satisfying geostrophic and hydrostatic balances and first-order variables describing ageostrophic motion, respectively. The leading-order equation is derived based on the Rossby number defined by $\epsilon \equiv L/f_0U$. The geostrophic winds u_0 and v_0 are scaled by a characteristic horizontal velocity scale $U \simeq 10$ m/s and horizontal and vertical length scales are chosen as the internal Rossby deformation radius and the tropopause height, respectively.

The dimensional pressure P^* is written as $P_S(z)+\rho_S U f_0 L p$, where P_S and ρ_S are the horizontally averaged vertical profiles of pressure and density. Here, p is a non-dimensional pressure, and f_0 is the Coriolis parameter at a reference latitude. The term $\rho_S U f_0 L$ originates from the geostrophic balance.

The dimensional potential temperature Θ^* is written as $\Theta_S(1+\epsilon F\Theta)$, where ϵ is the Rossby number, and F is the Froude number defined as L^2/L_D^2. Here, L is the internal Rossby deformation radius, and L_D is the external Rossby deformation radius defined as $f_0/\sqrt{gH}$.

The radiative-convective equilibrium potential temperature Θ_E originates from local energy flux balance. The quantities u_0, v_0, and η_0 satisfy the geostrophic and hydrostatic balances, given by $u_0=-\partial\phi_0/\partial y$, $v_0=\partial\phi_0/\partial x$, and $\eta_0=\partial\phi_0/\partial z$, where ϕ_0 is the anomalous pressure acting as a stream function. The radiative-convective equilibrium temperature Θ_E provides a

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balanced field based on the thermal wind balance, expressed as $\partial U_E/\partial z = -\partial\Theta_E/\partial y$. For a detailed derivation, refer to Pedlosky (2013); Moon and Cho (2020).

The radiative-convective equilibrium temperature Θ_E is also incorporated into the heat equation as follows:

$$u_0\frac{\partial\Theta_0}{\partial x} + v_0\frac{\partial\Theta_0}{\partial y} + Sw_1 = Q(\Theta_0), \tag{3}$$

where u_0 and v_0 are the geostrophic zonal and meridional velocities and Θ_0 is the potential temperature. Here, Q represents thermal forcing, interpreted as a consequence of local energy flux balance. It is assumed that there exists a Θ_E such that $Q(\Theta_E) = 0$. We approximate Θ_0 as $\Theta_E + \frac{\partial Q}{\partial\Theta_0}|_{\Theta_0=\Theta_E}\eta_0$, with $\frac{\partial Q}{\partial\Theta_0}|_{\Theta_0=\Theta_E} = -1/\tau$.

Through this procedure involving the radiative-convective equilibrium temperature Θ_E, equation (2) is constructed. The anomalous potential temperature η_0 is generated by large-scale atmospheric dynamics. Considering the role of large-scale eddies in transferring energy surplus from low latitudes to high latitudes, it is expected that η_0 is negative in tropical regions and positive in higher latitudes.

The equations (1) and (2) are zonally-averaged, leading to the following equations:

$$\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\overline{w}_1 = -\frac{\partial^2}{\partial y^2}\overline{u_0'v_0'} \tag{4}$$

$$-\frac{\partial}{\partial y}\overline{v_0'\eta_0'} - S\overline{w}_1 - \frac{1}{\tau}\overline{\eta}_0 = 0. \tag{5}$$

Here, $\overline{X}$ implies the zonal average of the variable X, X' represents the anomaly of X from $\overline{X}$, and $-\frac{1}{H} \equiv \frac{1}{\rho_S}\frac{\partial\rho_S}{\partial z}$. We assume that H is a constant for simplicity. Equation (5) expresses the balance between the heat flux convergence, adiabatic warming/cooling, and the local thermal forcing. The relationship between the eddy momentum flux and the ageostrophic vertical wind is depicted in the vorticity equation (4).

In a steady-state, there is no temporal evolution of the zonal mean zonal wind ($\partial\overline{u}_0/\partial t = 0$), which implies from the zonally averaged zonal momentum balance that

$$-\overline{v}_1 = -\frac{\partial}{\partial y}\overline{u_0'v_0'} - r\overline{u}_s, \tag{6}$$

where u_s is the surface zonal wind, and r is a constant measuring the surface stress induced by the Ekman pumping. Equation (6) describes a dynamical balance among the ageostrophic meridional wind, the eddy momentum convergence, and the surface friction. In upper levels where surface friction is negligible, the ageostrophic meridional wind is balanced by the eddy momentum flux convergence. This balance is included in the equation (4), obtained by combining the equation (6) and the continuity equation. However, the eddy momentum flux convergence in lower levels is negligible, thus the surface friction is necessary to balance with the acceleration of the zonal mean zonal wind induced by the poleward meridional wind. Therefore, it is necessary to include the influence of the atmospheric boundary layer on the interior solutions. In the later chapter, the solution of the Ekman boundary layer is considered to include the surface friction $-r\overline{u}_s$.

To account for the influence of eddy heat and momentum fluxes using zonally-averaged fields, parameterization becomes essential. The objective is to represent the contributions of these fluxes in relation to zonally-averaged variables. By employing

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suitable parameterization techniques, it is possible to express the eddy heat and momentum fluxes in a simplified form utilizing zonally-averaged fields.

2.2 Parameterization of the eddy heat and momentum fluxes

In equation (5), the overall structure of the general circulation is determined by the interactions between the eddy heat and momentum fluxes and the zonal mean field, which is a solution of the zonally-averaged heat equation. The zonal average anomalous potential temperature, denoted as $\overline{\eta}_0$, is influenced by several factors, including the meridional heat flux convergence $-\frac{\partial}{\partial y}\overline{v_0'\eta_0'}$ generated by growing synoptic waves, the synoptic-scale vertical velocity $\overline{w}_1$ induced by the eddy momentum flux, and the stabilizing process represented by the Newtonian cooling term $-\frac{1}{\tau}\overline{\eta}_0$.

To construct solutions of equation (5), it is necessary to parameterize the eddy heat and momentum fluxes using the zonal averaged variables such as $\overline{\eta}_0$ and $\overline{u}_0$. The parameterization of the eddy heat and momentum fluxes should be based on the life cycle of baroclinic waves, which play a significant role in the dynamics of synoptic-scale systems. Baroclinic waves grow through baroclinic instability, during which the eddy heat flux reaches its maximum intensity. As the waves saturate, they propagate upward and towards the equator and eventually break near critical latitudes. The process of wave breaking generates momentum flux, which contributes to the increase of the zonal mean zonal wind.

The poleward heat flux is closely associated with the baroclinic instability, while the eddy momentum flux is linked to wave breaking in the upper levels. Baroclinic instability in the synoptic-scale system requires a stronger vertical shear of the zonal mean zonal wind, which is proportional to the meridional temperature gradient of the mean field. As baroclinic instability leads to a decrease in the meridional temperature gradient, we can approximate the eddy heat flux as $\overline{v_0'\eta_0'} \approx -D\frac{\partial \overline{\eta}_0}{\partial y}$, where D represents the eddy diffusivity, quantifying the ability of synoptic eddies to transport residual heat from low latitudes to higher latitudes (Barry et al., 2002).

The eddy momentum flux plays a crucial role in strengthening the zonal mean zonal wind and maintaining the jet stream in the mid-latitudes (Edmon Jr et al., 1980). While the eddy heat flux weakens the vertical shear of the zonal mean zonal wind, leading to the decrease of the jet stream, the eddy momentum flux acts to increase the horizontal shear of the jet stream resulting in the increase of the jet stream. In a steady state, the existence and maintenance of the zonal jet stream can be attributed to the interplay between these two opposing eddy fluxes. The eddy momentum flux is expected to be upgradient with respect to the horizontal shear of the zonal mean zonal wind (Held and Andrews, 1983; Dijkstra and van der Vaart, 1998). This relationship can be expressed as $\overline{u_0'v_0'} \approx M\frac{\partial \overline{u}_0}{\partial y}$, where M represents a constant controlling the sensitivity of the eddy momentum flux to the meridional shear of the zonal mean zonal wind.

The validity of the aforementioned simple parameterization can be substantiated through numerical simulations conducted using the dry spectral dynamic core of the GFDL Flexible Modeling System (FMS) (Held and Suarez, 1994). The model employed in this study uses a radiative-convective equilibrium temperature and the thermal forcing is given as a Newtonian cooling to make the temperature field relaxed toward the radiative-convective equilibrium with a specific response time-scale. To test the simple parameterizations, eddy heat and momentum fluxes are investigated with various shapes of the zonal mean zonal wind with different horizontal and vertical shears. To vary the shape of the zonal mean zonal wind in mid-latitudes, we

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impose additional high-latitude thermal forcing with varying intensities, $q_0\cos(\phi-\phi_0)\exp[6(P-P_0)]$, where q_0 represents the thermal forcing intensity ranging from -1.2 to $+1.2$ K/day at 0.3 K/day intervals and the central latitude (ϕ_0) and pressure (P_0) are set to $90N$ and $1000hPa$, respectively. Thus, 9 simulations including the control simulation ($q_0 = 0$ K/day) are conducted.

For each increment in the thermal heating, the model is run until it reaches a statistically steady state. The simulations give us various eddy fluxes and zonal mean states in a certain location, which can be used to determine M and D. To use the simple linear parameterizations, it is required to verify linear relationships between the horizontal and vertical shear of the zonal mean zonal wind and the eddy momentum and heat fluxes.

Figure 1 illustrates the variation in mid-latitude atmospheric structure as a function of the magnitude of additional surface thermal forcing. For high-latitude cooling ($q_0 = -1.2\,\mathrm{K/day}$), the overall atmospheric circulation, represented by the stream function, is intensified (panel a). Furthermore, the magnitudes of the zonal mean zonal wind and the eddy momentum and heat fluxes increase (panels c and e). In contrast, when high-latitude warming is applied ($q_0 = +1.2\,\mathrm{K/day}$), all dynamic quantities, including the stream function as well as the eddy momentum and heat fluxes, weaken. Based on these results and previous studies on the baroclinic wave life cycle, it can be inferred that the horizontal and vertical shear of the zonal mean zonal wind are closely linked to the eddy momentum and heat fluxes.

Linear regressions are performed between the eddy momentum flux and the horizontal shear of the zonal mean zonal wind, as well as between the eddy heat flux and the meridional gradient of the zonal mean temperature at each location. The results, shown in Fig. 2 (a) and (b), reveal strong correlations for both fluxes. As expected, the eddy heat flux is negatively correlated with the meridional temperature gradient (down-gradient), while the eddy momentum flux is positively correlated with the horizontal shear of the zonal mean zonal wind (up-gradient) in mid-latitudes. The spatial maps of M and D in Fig. 2 (c) and (d), respectively, demonstrate the spatial dependence of both fluxes. The up-gradient characteristics of the eddy momentum flux are observed in the equatorward region of mid-latitudes. This is consistent with the upward and equatorward propagation of Rossby waves, which generate poleward momentum flux during their propagation and breaking. The positive M highlights the dynamic role of Rossby waves in shaping the spatial distribution of turbulent eddy momentum in mid-latitudes. Outside the mid-latitude region, the momentum diffusivity M exhibits down-gradient behavior. The eddy heat flux is characterized by a reduction in the meridional temperature gradient from the tropics to the polar regions, represented by negative Ds. Despite latitudinal variations in dominant processes, turbulent eddies consistently transport thermal energy from low to high latitudes.

In mid-latitudes, distinctive characteristics of the two fluxes become apparent. The eddy heat flux transports heat from the tropics to polar regions, leading to a decrease in the jet stream at upper levels. Conversely, the eddy momentum flux intensifies the jet stream at upper levels and induces adiabatic warming at the equator flank of the jet stream, resulting in an increased meridional temperature gradient. Thus, a steady-state mid-latitude circulation emerges as a delicate balance between these two opposing fluxes.

Realistic representations of mid-latitude circulations based on eddy parameterizations may be achieved by considering the spatial variation of M and D or designing nonlinear relationships between the gradient of the mean wind or temperature and the eddy fluxes (Green, 1970; Stone, 1978; Stone and Yao, 1987). Furthermore, it is possible to scale the eddy fluxes based on

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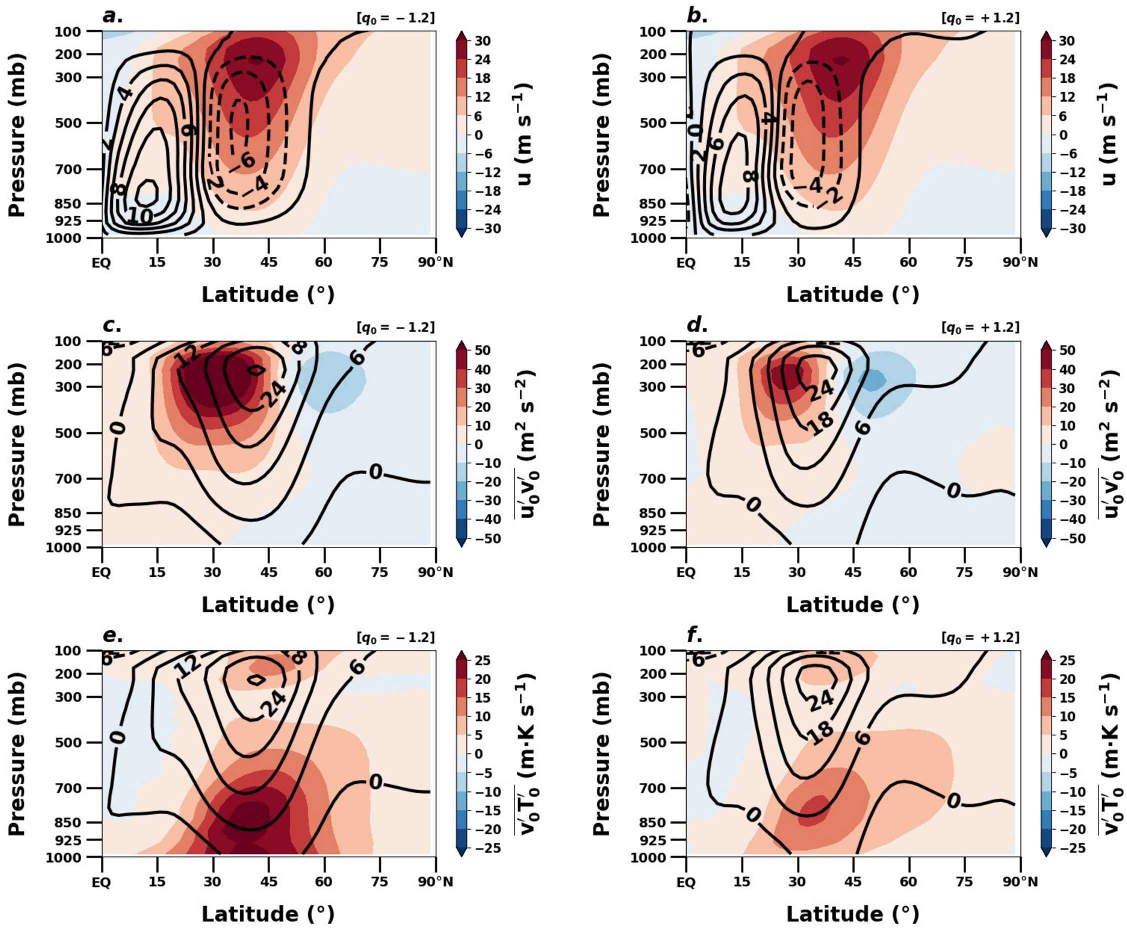

Figure 1. The zonal mean atmospheric structures under different high-latitude thermal forcings are depicted. Additional surface thermal forcing is applied in high latitudes, with results for the cooling case ($q_0 = -1.2\,\mathrm{K/day}$) shown in panels (a), (c), and (e), and the warming case ($q_0 = +1.2\,\mathrm{K/day}$) in panels (b), (d), and (f). The zonal mean structures are compared in terms of stream functions (panels a, b), eddy momentum fluxes (panels c, d), and eddy heat fluxes (panels e, f) for the two contrasting cases. In panels (a) and (b), the color shading represents the zonal mean zonal winds, while the contour lines depict the mass stream function. Similarly, in panels (c), (d), (e), and (f), the color shading indicates the eddy fluxes, and the contour lines represent the zonal mean zonal winds.

the energetics describing the energy exchange between eddies and mean fields (Schneider and Walker, 2008). In particular, the two parameterizations must be considered together to obtain a dynamic equilibrium. For example, in a purely barotropic flow, there is no generation of eddies by baroclinic instability. Hence, the acceleration of the jet by eddy momentum flux would lead to singular results. As such, a useful momentum flux parameterization is only obtained under the assumption that baroclinic eddies can extract energy from the zonal mean state. Under this assumption, where singularity is avoided, the first order effects of eddy momentum fluxes are a cascading of energy to larger scales (or anti-diffusivity).

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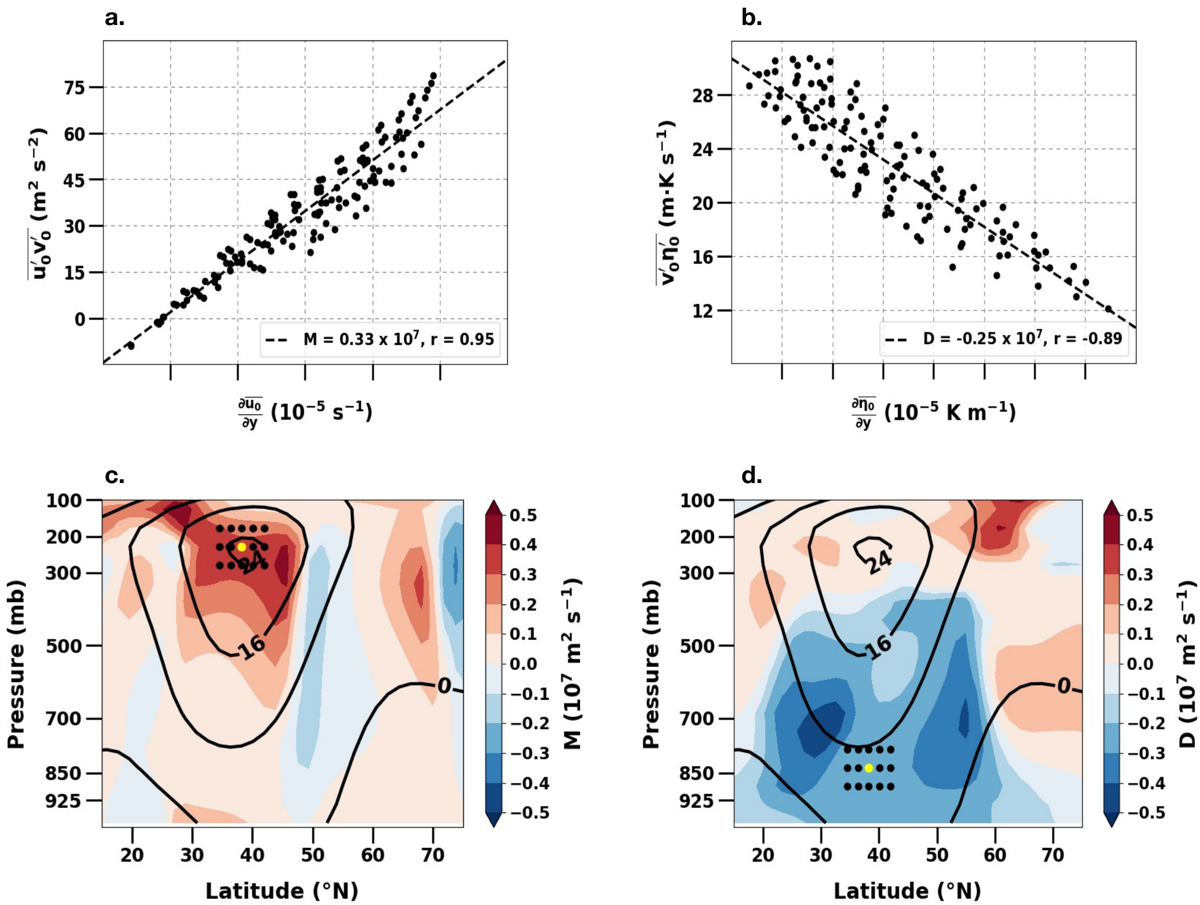

Figure 2. Based on the control simulation, maps of D (panel c) and M (panel d) are constructed. Eddy momentum (M) and heat (D) diffusivities are calculated through linear regression using the results from the simulation at each location. The diffusivities at each location are determined by the turbulent fluxes and the shear of the zonal mean zonal wind within an area surrounding the location. Examples of these calculations are illustrated in panels (a) and (b) for M and D, respectively. The sampling locations for these examples are marked by yellow dots. The black dots surrounding the yellow dots represent the region for calculating M and D. In panels (c) and (d), the contours represent the zonal mean zonal wind, while the color shading indicates the diffusivity of the eddy momentum in (c) and the diffusivity of the eddy heat flux in (d).

In analytical approaches, constants D and M are often employed, taking into account parameterizations such as $\overline{v_0'\eta_0'} \approx -D\frac{\partial \overline{\eta}_0}{\partial y}$ and $\overline{u_0'v_0'} \approx M\frac{\partial \overline{u}_0}{\partial y}$. Compared with the realistic parameterizations, it could be interpreted as being too much simplified, but we show it contains the essence of forming the eddy-driven meridional circulations in midlatitudes. These constants, D and M, underscore the importance of achieving a balance involving eddy heat and momentum fluxes in mid-latitudes to sustain the jet stream. We can understand the *first*-order effect of eddy heat and momentum fluxes as the driver of thermally indirect (eddy-driven) circulation cell.

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After applying the proposed parameterizations to the equations (5) and (4), we obtain

$$D\frac{\partial^2\overline{\eta}_0}{\partial y^2} - S\overline{w}_1 - \frac{1}{\tau}\overline{\eta}_0 = 0 \tag{7}$$

$$\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\overline{w}_1 = -M\frac{\partial^3\overline{u}_0}{\partial y^3}. \tag{8}$$

We proceed by taking $\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)$ of (7) and using (8), resulting in

$$D\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\frac{\partial^2\overline{\eta}_0}{\partial y^2} + MS\frac{\partial^3\overline{u}_0}{\partial y^3} - \frac{1}{\tau}\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\overline{\eta}_0 = 0. \tag{9}$$

Further, taking $\frac{\partial}{\partial z}$ and utilizing the thermal wind balance $\frac{\partial \overline{u}_0}{\partial z} = -\frac{\partial \overline{\eta}_0}{\partial y}$, we arrive at the following expression:

$$D\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\frac{\partial^2\overline{\eta}_0}{\partial y^2} - MS\frac{\partial^4\overline{\eta}_0}{\partial y^4} - \frac{1}{\tau}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\overline{\eta}_0 = 0. \tag{10}$$

By simplifying the equation to focus on the influence of the poleward heat flux and the eddy momentum flux on the general circulation, and disregarding the negative feedback provided by the local radiative-convective equilibrium ($\tau \to \infty$), we arrive at the simplified equation:

$$D\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\frac{\partial^2\overline{\eta}_0}{\partial y^2} - MS\frac{\partial^4\overline{\eta}_0}{\partial y^4} = 0, \tag{11}$$

This simplified equation captures the essential dynamics involved in shaping the overall structure of the general circulation in mid-latitudes. Solving this equation with appropriate boundary conditions will allow for a better understanding of the role of synoptic eddies on the formation of general circulation in mid-latitudes within the framework of the planetary geostrophic motion. Finding solutions to equation (11) with appropriate boundary conditions is the remaining task at hand.

2.3 Boundary conditions

While a β-plane is a crude approximation when considering the entire hemisphere from tropics to polar areas, it offers valuable analytic simplifications when compared to using a full spherical coordinate system. The advantage of utilizing a cartesian coordinate system lies in the ability to obtain analytical solutions and insights into the dynamics of the system. The simplification provided by the β-plane approximation allows for a clearer examination of how synoptic eddies interact with the large-scale circulation, particularly in terms of the transfer of heat and momentum.

For the vertical domain, it is reasonable to choose $0 \leq z \leq 1$ to represent the troposphere, where $z = 0$ corresponds to the surface and $z = 1$ represents the tropopause. This choice allows us to focus on the dynamics of the general circulation within the tropospheric region. In the vertical domain, the boundary conditions are imposed on the vertical velocity $\overline{w}_1$. At the top of atmosphere ($z = 1$), it is assumed that $\overline{w}_1$ is vanished. However, at the surface $z = 0$, the vertical velocity is determined by the influence of planetary boundary layer and assumed to be a consequence of the Ekman pumping proportional to the surface vorticity.

In terms of the latitudinal domain, we can consider a range from $y = 0$ to $y = L_A$, where $y = 0$ indicates the end of the tropical areas and $y = L_A$ represents the polar areas. It is important to note that the Hadley circulation, driven by large-scale

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tropical convection, operates near the center of the tropics. The dynamics in the tropical regions differ from those in the mid-latitudes, where the life cycle of synoptic eddies interacting with the zonal mean zonal wind plays a major role. As a result, the latitudinal domain for the study of the general circulation cannot encompass the entire hemisphere but should focus on the relevant regions of interest.

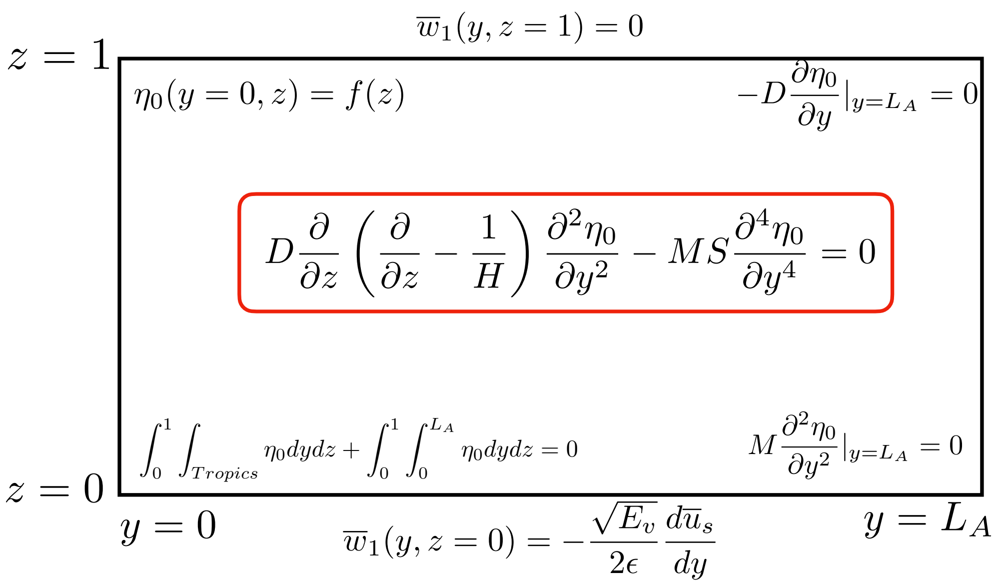

Figure 3. A schematic illustrating the primary differential equation along with the boundary conditions. The equation (11) is provided, including both latitudinal and vertical boundary conditions. The vertical velocity at the lowest level ($z = 0$) is determined by the Ekman pumping. The non-zero boundary conditions $\overline{\eta}_0(y = 0, z) = f_L(z)$ and the energy-conserving condition emphasize the impact of tropical convections. The "Tropics" is referred to a region from tropics to the starting latitude of mid-latitudes ($y = 0$).

At $y = L_A$, where we approach the polar areas, it is reasonable to assume that the heat and momentum fluxes diminish, leading to $\frac{\partial \overline{\eta}_0}{\partial y} = 0$ and $\frac{\partial^2 \overline{\eta}_0}{\partial y^2} = 0$. The primary role of synoptic eddies is to transport excess heat and momentum from the tropics to the polar regions. This boundary condition highlights that the heat and zonal momentum transported from the tropics is redistributed and deposited in the mid-latitudes. At $y = 0$, representing the edge of the tropical region, we can impose a boundary condition that captures the energy transfer and dynamics associated with the Hadley circulation. Based on previous research, a negative anomaly in temperature $\overline{\eta}_0$ near the tropics indicates the poleward heat transfer (Held and Hou, 1980). As we move towards the end of the tropical region, this anomaly transitions to positive values. We can utilize this information to impose a boundary condition on the anomalous temperature $\overline{\eta}_0(y = 0, z)$, which reflects the presence of the Hadley circulation.

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Furthermore, we can consider the conservation of energy as an additional boundary condition (Held and Hou, 1980). This implies that the integral of the anomalous temperature $\overline{\eta}_0$ over the entire domain, from tropics to the pole, should be zero, i.e.

$$\int\limits_{hemisphere} \overline{\eta}_0 dA = \int\limits_{Tropics} \overline{\eta}_0 dA + \int\limits_{extra-tropics} \overline{\eta}_0 dA = 0, \tag{12}$$

where $dA = dydz$ representing an infinitesimal area in a zonally-averaged domain. This condition ensures the overall energy balance within the system and aligns with the purpose of the general circulation to transport energy from the tropics to high latitudes. By incorporating these boundary conditions, we can develop a well-posed problem, which is summarized as a schematic in figure 3, and provide a suitable framework for analyzing the behavior and properties of the general circulation in mid-latitudes.

2.4 Boundary layer

The main equation (11) is derived from inviscid interior dynamics under the assumptions of geostrophic and hydrostatic balance. The zonal momentum balance incorporates the meridional ageostrophic wind (v_1) and eddy momentum convergence. In the upper levels, strong eddy momentum convergence arises due to the propagation and breaking of Rossby waves, establishing a dynamic balance with the meridional wind. However, near the surface, the eddy momentum convergence becomes negligible. The near-surface poleward meridional velocity, induced by strong downward motion on the equatorward flank of the jet stream, accelerates the zonal mean zonal wind via the Coriolis force. To maintain a steady state near the surface, surface friction balances the surface meridional wind.

To account for the effect of surface friction in the zonal momentum balance, a boundary layer solution must be included. The main equation (11), along with the specified boundary conditions, determines the spatial variation of the anomalous potential temperature, $\overline{\eta}_0$. Using the thermal wind balance,

$$\frac{\partial \overline{u}_0}{\partial z} = -\frac{\partial \overline{\eta}_0}{\partial y},$$

we derive the total wind as:

$$\overline{u}_T = \overline{u}_0 + \overline{u}_s = -\int\limits_0^z \frac{\partial \overline{\eta}_0}{\partial y} dz' + \overline{u}_s, \tag{13}$$

where $\overline{u}_T$ represents the total zonal wind, which is the sum of the geostrophic interior zonal mean zonal wind, $\overline{u}_0$, and the surface zonal mean zonal wind, $\overline{u}_s$. The geostrophic wind satisfies the boundary condition $\overline{u}_0(z=0) = 0$. Subsequently, the near-surface geostrophic zonal wind, $\overline{u}_s$, will serve as a boundary condition for the Ekman solution within the boundary layer.

The momentum balance in the boundary layer is given by:

$$\begin{aligned} -\overline{v}_b &= -\frac{\partial \overline{P}_0}{\partial x} + \frac{E_v}{2}\frac{\partial^2 \overline{u}_b}{\partial z^2}, \\ \overline{u}_b &= -\frac{\partial \overline{P}_0}{\partial y} + \frac{E_v}{2}\frac{\partial^2 \overline{v}_b}{\partial z^2}, \end{aligned} \tag{14}$$

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where E_v is the non-dimensionalized Ekman number, defined as $E_v \equiv 2\frac{A_v}{f_0 D^2}$, with A_v representing the vertical turbulent viscosity coefficient. The boundary conditions are:

$$\overline{u}_b(y,z=0) = \overline{v}_b(y,z=0) = 0, \quad \lim_{z\to\infty} \overline{u}_b = \overline{u}_s, \quad \text{and} \quad \lim_{z\to\infty} \overline{v}_b = 0.$$

From the hydrostatic balance, it can be shown that:

$$\frac{\partial}{\partial z}\left(\frac{\partial \overline{P}_0}{\partial x}\right) = \frac{\partial}{\partial z}\left(\frac{\partial \overline{P}_0}{\partial y}\right) = 0, \tag{15}$$

which implies that $\frac{\partial \overline{P}_0}{\partial x} = 0$ and $-\frac{\partial \overline{P}_0}{\partial y} = \overline{u}_s$. Substituting these into equation (14), the momentum balance simplifies to:

$$\begin{aligned} -\overline{v}_b &= \frac{E_v}{2}\frac{\partial^2 \overline{u}_b}{\partial z^2}, \\ \overline{u}_b &= \overline{u}_s + \frac{E_v}{2}\frac{\partial^2 \overline{v}_b}{\partial z^2}. \end{aligned} \tag{16}$$

The solutions for the boundary layer are:

$$\begin{aligned} \overline{u}_b &= \overline{u}_s\left(1 - e^{-\frac{z}{\sqrt{E_v}}}\cos\left(\frac{z}{\sqrt{E_v}}\right)\right), \\ \overline{v}_b &= \overline{u}_s e^{-\frac{z}{\sqrt{E_v}}}\sin\left(\frac{z}{\sqrt{E_v}}\right). \end{aligned} \tag{17}$$

By the continuity equation, we can calculate the vertical velocity $\overline{w}_b$:

$$\overline{w}_b = -\frac{d\overline{u}_s}{dy}\int_0^z e^{-\frac{z'}{\sqrt{E_v}}}\sin\left(\frac{z'}{\sqrt{E_v}}\right)dz', \tag{18}$$

where we used $\partial \overline{u}_b/\partial x = 0$. This results in

$$\overline{w}_b = -\frac{1}{2}\sqrt{E_v}\frac{\partial \overline{u}_s}{\partial y}\left(1 - e^{-\frac{z}{\sqrt{E_v}}}\cos\left(\frac{z}{\sqrt{E_v}}\right) - e^{-\frac{z}{\sqrt{E_v}}}\sin\left(\frac{z}{\sqrt{E_v}}\right)\right). \tag{19}$$

For asymptotic matching for an uniformly continuous solution, $\lim_{z\to 0}\overline{w}_1 = \lim_{z/\sqrt{E_v}\to\infty}\overline{w}_b/\epsilon$, which results in

$$\lim_{z\to 0}\overline{w}_1 = -\frac{\sqrt{E_v}}{2\epsilon}\frac{\partial \overline{u}_s}{\partial y}, \tag{20}$$

where ϵ is the Rossby number and $\overline{w}_b$ is divided by the Rossby number because $\overline{w}_1$ is a $O(1)$ variable in the $O(\epsilon)$ balance.

The boundary condition imposed on $\overline{w}_1$ at the surface can be interpreted in the zonal momentum balance. From the continuity equation, we have:

$$\frac{\partial}{\partial y}\int_0^1 \rho_s \overline{v}_1\,dz + \int_0^1 \frac{\partial}{\partial z}\rho_s \overline{w}_1\,dz = 0. \tag{21}$$

Considering the boundary condition (20), we get:

$$\int_0^1 \rho_s \overline{v}_1\,dz = \frac{\sqrt{E_v}}{2\epsilon}\overline{u}_s. \tag{22}$$

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The zonal momentum balance in the free atmosphere is expressed as:

$$-\overline{v}_1 = -\frac{\partial}{\partial y}\overline{u_0' v_0'}. \tag{23}$$

Multiplying by ρ_s and integrating vertically on both sides leads to:

$$-\frac{\partial}{\partial y}\int_0^1 \rho_s \overline{u_0' v_0'}\, dz = -\frac{\sqrt{E_v}}{2\epsilon}\overline{u}_s, \tag{24}$$

which shows that the upper-level eddy momentum convergence is balanced by the surface friction.

Finally, we obtain the expression for $\overline{u}_s$:

$$\overline{u}_s = \frac{2\epsilon}{\sqrt{E_v}}\frac{\partial}{\partial y}\int_0^1 \rho_s \overline{u_0' v_0'}\, dz. \tag{25}$$

The addition of the surface wind, $\overline{u}_s$, allows us to establish the momentum balance near the surface.

2.5 TEM formalism and potential vorticity flux

The main equations in the quasi-geostrophic approximation are

$$\begin{aligned}
\frac{\partial \overline{u}_0}{\partial t} &= \overline{v}_1 - \frac{\partial}{\partial y}\overline{u_0' v_0'} + \overline{\varkappa}^*,\\
\frac{\partial \overline{\eta}_0}{\partial t} &= -S\overline{w}_1 - \frac{\partial}{\partial y}\overline{\eta_0' v_0'} + \overline{\Xi}^*,\\
\frac{1}{\rho_S}\frac{\partial}{\partial z}(\rho_S \overline{w}_1) &+ \frac{\partial}{\partial y}\overline{v}_1 = 0,
\end{aligned} \tag{26}$$

where $\overline{\varkappa}^*$ and $\overline{\Xi}^*$ represent external mechanical and thermal forcing. The quasi-geostrophic equation can be interpreted using the Transformed Eulerian Mean (TEM) theory. A key step in the TEM formalism involves introducing new zonally-averaged ageostrophic velocity components (Pedlosky, 2013),

$$\begin{aligned}
\overline{w}_1^* &= \overline{w}_1 + \frac{\partial}{\partial y}\left(\frac{\overline{\eta_0' v_0'}}{S}\right),\\
\overline{v}_1^* &= \overline{v}_1 - \frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\frac{\rho_S \overline{\eta_0' v_0'}}{S}\right),
\end{aligned} \tag{27}$$

which leads to

$$\begin{aligned}
\frac{\partial \overline{u}_0}{\partial t} &= \overline{v}_1^* + \frac{1}{\rho_S}\nabla\cdot(\rho_S \boldsymbol{F}) + \overline{\varkappa}^*,\\
\frac{\partial \overline{\eta}_0}{\partial t} &= -S\overline{w}_1^* + \overline{\Xi}^*,\\
\frac{1}{\rho_S}\frac{\partial}{\partial z}(\rho_S \overline{w}_1^*) &+ \frac{\partial}{\partial y}\overline{v}_1^* = 0.
\end{aligned} \tag{28}$$

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The Eliassen-Palm (EP) flux $\boldsymbol{F} = (F_y, F_z)^T$ is defined in terms of its components $F_y = -\overline{u_0' v_0'}$ and $F_z = \overline{\eta_0' v_0'}/S$. Moreover,

$$\frac{1}{\rho_S}\nabla\cdot(\rho_S \boldsymbol{F}) = -\frac{\partial}{\partial y}\overline{u_0' v_0'} + \frac{1}{\rho_S}\frac{\partial}{\partial z}\left(\frac{\rho_S \overline{\eta_0' v_0'}}{S}\right). \tag{29}$$

350 Consider now a steady-state without any external thermal forcing or surface stress. The TEM equations simplify to

$$\begin{aligned}
&\overline{v}_1^* + \frac{1}{\rho_S}\nabla\cdot(\rho_S \boldsymbol{F}) = 0,\\
&\overline{w}_1^* = 0,\\
&\frac{1}{\rho_S}\frac{\partial}{\partial z}(\rho_S \overline{w}_1^*) + \frac{\partial}{\partial y}\overline{v}_1^* = 0.
\end{aligned} \tag{30}$$

Since $\overline{w}_1^* = 0$, the continuity equation reduces to $\frac{\partial}{\partial y}\overline{v}_1^* = 0$. Combining this with the zonal momentum equation results in

355 $$\frac{\partial}{\partial y}\left(\frac{1}{\rho_S}\nabla\cdot(\rho_S \boldsymbol{F})\right) = 0. \tag{31}$$

This indicates that, in a steady-state without dissipation and thermal forcing, the EP flux divergence is a function of z only, with no meridional variation. Within the quasi-geostrophic formalism, it is known that $\frac{1}{\rho_S}\nabla\cdot(\rho_S \boldsymbol{F}) \simeq \overline{v_0' q_0'}$, where q is the potential vorticity. Assuming that $\frac{\partial}{\partial y}\overline{q}_0$ is positive and considering that the potential vorticity flux $\overline{v_0' q_0'}$ is down-gradient, we should have $\frac{1}{\rho_S}\nabla\cdot(\rho_S \boldsymbol{F}) = c(z) < 0$.

360 Applying simple eddy parameterizations, $\overline{u_0' v_0'} \simeq M\frac{\partial \overline{u}_0}{\partial y}$ and $\overline{\eta_0' v_0'} \simeq -D\frac{\partial \overline{\eta}_0}{\partial y}$, to the equation (31), we obtain

$$M\frac{\partial^3 \overline{u}_0}{\partial y^3} + \frac{D}{S}\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\frac{\partial^2 \overline{\eta}_0}{\partial y^2} = 0. \tag{32}$$

Taking the derivative with respect to z and applying the thermal wind balance $\frac{\partial \overline{u}_0}{\partial z} = -\frac{\partial \overline{\eta}_0}{\partial y}$, we get

$$-M\frac{\partial^4 \overline{\eta}_0}{\partial y^4} + \frac{D}{S}\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z} - \frac{1}{H}\right)\frac{\partial^2 \overline{\eta}_0}{\partial y^2} = 0, \tag{33}$$

which is the same as equation (11).

365 In particular, there exists a special case where the Eliassen-Palm (EP) flux is zero everywhere, i.e., $c(z) = 0$. This state is considered as a result of the non-acceleration theorem (Charney and Drazin, 1961; Plumb, 1990), which implies that turbulent eddies do not alter the zonal mean state. The EP flux comprises the eddy momentum and heat fluxes, and its divergence is the sum of the meridional convergence of the eddy momentum flux and the vertical divergence of the eddy heat flux. The non-acceleration state indicates that these two eddy fluxes balance each other, thereby maintaining a steady state of the zonal 370 mean zonal wind. It should be emphasized that the simple parameterizations for the eddy momentum and heat fluxes enable us to obtain solutions satisfying the non-acceleration theorem.

The Eliassen-Palm (EP) flux convergence is not zero in realistic situations, implying the existence of non-zero potential vorticity fluxes within the domain. Previous studies have sought to parameterize the potential vorticity flux $\overline{v_0' q_0'}$ using the meridional gradient of the zonal mean potential vorticity $\frac{\partial}{\partial y}\overline{q}_0$, i.e.,

375 $$\overline{v_0' q_0'} = -K_q\frac{\partial}{\partial y}\overline{q}_0, \tag{34}$$

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where the diffusivity K_q may be a function of state variables such as the thermal wind scale and the length scale of the most energetic baroclinic wave (Held and Larichev, 1996), or a spatially and temporally varying function (Wang and Lee, 2016; Nakamura, 1996).

The parameterization of potential vorticity fluxes, which drives the temporal change of the zonal mean zonal wind, is useful for investigating the changes in meridional circulation associated with the zonal mean zonal wind due to synoptic eddies or external forcing. However, we here focus on idealized situations where the potential vorticity flux is either constant across the latitudinal domain or zero. Even in the absence of potential vorticity flux, meridional circulations exist to maintain the zonal mean zonal wind, in accordance with the non-acceleration theorem. We next aim to find solutions that represent the meridional circulation and the zonal mean zonal wind without the influence of meridional potential vorticity flux.

3 Results

3.1 An analytic solution and its structure

Let $\overline{\eta}_0 = e^{z/2H}Q(z)R(y)$, then (11) gives

$$SMQ(z)\frac{d^4R}{dy^4} - D\left(\frac{d^2}{dz^2} - \frac{1}{4H^2}\right)Q\frac{d^2R}{dy^2} = 0. \tag{35}$$

Separation results in

$$\frac{SM\frac{d^4R}{dy^4}}{D\frac{d^2R}{dy^2}} = \frac{\left(\frac{d^2}{dz^2} - \frac{1}{4H^2}\right)Q}{Q} = -\lambda^2, \tag{36}$$

where we choose a negative constant $-\lambda^2$ for the separation constant. To satisfy the boundary condition at $y = 0$, the eigenfunctions Q_n must be complete, thus it is required to have trigonometric functions as eigenfunctions. This leads to the choice of the negative eigenvalues $-\lambda^2$. The above equations imply that

$$\frac{d^4R}{dy^4} + \frac{D\lambda^2}{SM}\frac{d^2R}{dy^2} = 0 \tag{37}$$

$$\frac{d^2Q}{dz^2} + \left(\lambda^2 - \frac{1}{4H^2}\right)Q = 0. \tag{38}$$

The boundary conditions at $z = 1$ and $y = L_A$ are represented by $Q(1) = 0$ and $R'(L_A) = R''(L_A) = 0$. To satisfy the boundary conditions $Q(1) = 0$, $Q(z)$ can be represented as $Q(z) = \sin q(1-z)$, where $q^2 = \lambda^2 - 1/4H^2$. Then, the eigenfunction R satisfies $R'(L_A) = R''(L_A) = 0$, which results in

$$R(y) = A - \gamma Ey + E\sin\gamma(y - L_A), \tag{39}$$

where A and E are undetermined coefficients and $\gamma^2 = \frac{D}{SM}\lambda^2$. The general solution is

$$\overline{\eta}_0(y,z) = C\sin q(1-z)(A - \gamma Ey + E\sin\gamma(y - L_A)), \tag{40}$$

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which leads to

$$\overline{u}_0 = -\frac{\partial}{\partial y}\int^z \overline{\eta}_0 dz' = C\gamma E(1-\cos\gamma(y-L_A))\frac{e^{z/2H}}{q^2+1/4H^2}\left(\frac{1}{2H}\sin q(1-z)+q\cos q(1-z)\right)$$

$$\overline{u_0'v_0'} = M\frac{\partial \overline{u}_0}{\partial y} = CME\gamma^2\sin\gamma(y-L_A)\frac{e^{z/2H}}{q^2+1/4H^2}\left(\frac{1}{2H}\sin q(1-z)+q\cos q(1-z)\right)$$

405 $$\overline{v_0'\eta_0'} = -D\frac{\partial\overline{\eta}_0}{\partial y} = C\gamma DE(1-\cos\gamma(y-L_A))e^{z/2H}\sin q(1-z)$$

$$\overline{w}_1 = \frac{D}{S}\frac{\partial^2\overline{\eta}_0}{\partial y^2} = -C\frac{D}{S}E\gamma^2\sin\gamma(y-L_A)e^{z/2H}\sin q(1-z)$$

$$\overline{v}_1 = \frac{\partial}{\partial y}\overline{u_0'v_0'} = C\gamma^3 E\cos\gamma(y-L_A)\frac{e^{z/2H}}{q^2+1/4H^2}\left(\frac{1}{2H}\sin q(1-z)+q\cos q(1-z)\right). \tag{41}$$

The boundary condition at $z=0$, $\overline{w}_1(y,z=0) = -\frac{\sqrt{E_v}}{2\epsilon}\frac{\partial\overline{u}_s}{\partial y}$, implies

$$\sin q\left(q^2+\frac{1}{4H^2}\right) = \frac{\sqrt{E_v}}{2\epsilon}\frac{S}{D}\left(\frac{1}{2H}\sin q + q\cos q\right). \tag{42}$$

410 If we let $h(q)=\sin q$ and $g(q)=\frac{\sqrt{E_v}}{2\epsilon}\frac{S}{D}\frac{\sin q/2H+q\cos q}{q^2+1/4H^2}$, $f(q)=g(q)$ has infinite number of solutions. Figure 4 shows $h(q)$ (red curve) and $g(q)$ (blue curve) when $E_v=0.01$, $\epsilon=0.1$, $S=1.0$, $D=0.22$, and $H=1.0$. The first positive solution is slightly smaller than $\pi/2$ and the other solutions are located near $q=n\pi$. In particular, as q becomes larger, $g(q)$ converges to zero such that the solutions approach to $n\pi$.

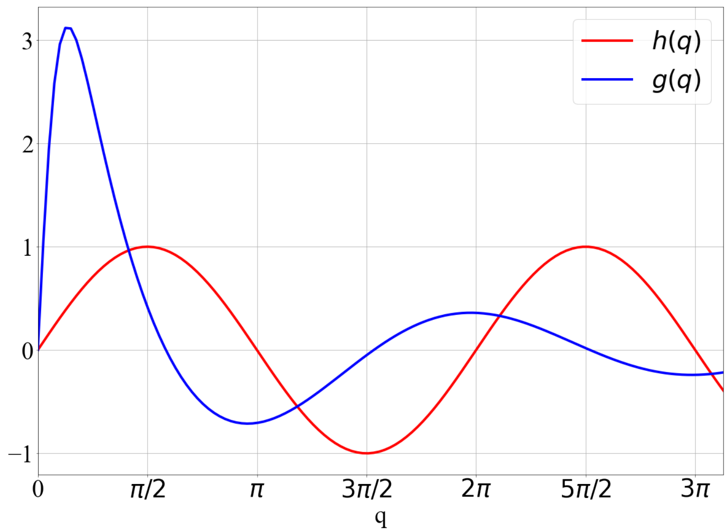

Figure 4. The characteristic equation is represented by $f(q)=g(q)$, where $h(q)=\sin q$ and $g(q)=\frac{\sqrt{E_v}}{2\epsilon}\frac{S}{D}\frac{\sin q/2H+q\cos q}{q^2+1/4H^2}$. Here, we use $E_v=0.01$, $\epsilon=0.1$, $S=1.0$, $D=0.22$, and $H=1.0$.

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The solution can be represented as the sum of infinite eigenfunctions defined by the eigenvalues q_n with positive integer n. Thus,

$$\overline{\eta}_0 = \sum_{n=1}^{\infty} (A_n - \gamma_n E_n y + E_n \sin\gamma_n (y - L_A)) \sin q_n (1-z), \tag{43}$$

where the coefficients E_n and A_n are determined by the boundary condition $y = 0$.

The boundary conditions at $y = 0$ is $f(z)$, which implies that

$$\overline{\eta}_0(y=0,z) = \sum_{n=1}^{\infty} (A_n - E_n \sin(\gamma_n L_A)) e^{\frac{z}{2H}} \sin n\pi z = f(z). \tag{44}$$

Recall that $\overline{\eta}_0$ in mid-latitudes represents the anomalous potential temperature induced by the life cycle of synoptic eddies summarized by the generation of heat and momentum fluxes. Therefore, $f(z)$ is the vertical structure of the anomalous potential temperature induced by tropical atmospheric dynamics. The coefficients for the eigenfunctions are determined by the boundary condition at $y = 0$. Physically, it implies that the thermal influence coming from tropics has a large impact upon the structure of the circulation in mid-latitudes.

The first vertical model $\sin q_1(1-z)$ implies that the maximum of the eddy heat flux lies near the surface, which is similar to real atmosphere. Thus, investigating the structure of the first vertical mode is the first step to see how the tropics can influence the general circulation in mid-latitudes. Let

$$f(z) = \alpha e^{\frac{z}{2H}} \sin q_1 (1-z), \tag{45}$$

then it follows that $A_1 - E_1 \sin\gamma_1 L_A = \alpha$. Finally, the energy conserving condition means that

$$\int_0^{L_A} R_1(y) dy = S_T, \tag{46}$$

where

$$S_T = -\frac{\int_{\text{Tropics}} \overline{\eta}_0 dy dz}{\int_0^1 Q_1 dz}. \tag{47}$$

This implies that S_T is the loss of thermal energy in the tropics due to large-scale atmospheric dynamics. Hence, the equation (46) represents the conservation of thermal energy in the hemisphere. Large-scale atmospheric dynamics redistributes the thermal energy from tropics to mid-latitudes. Equation (46) gives

$$(\alpha + E_1 \sin\gamma_1 L_A) L_A - \frac{1}{2}\gamma_1 E_1 L_A^2 - \frac{E_1}{\gamma_1}(1 - \cos\gamma_1 L_A) = S_T. \tag{48}$$

Thus,

$$E_1 = \frac{\alpha L_A - S_T}{\frac{1}{2}\gamma_1 L_A^2 + \frac{1}{\gamma_1}(1 - \cos\gamma_1 L_A) - L_A \sin\gamma_1 L_A} \tag{49}$$

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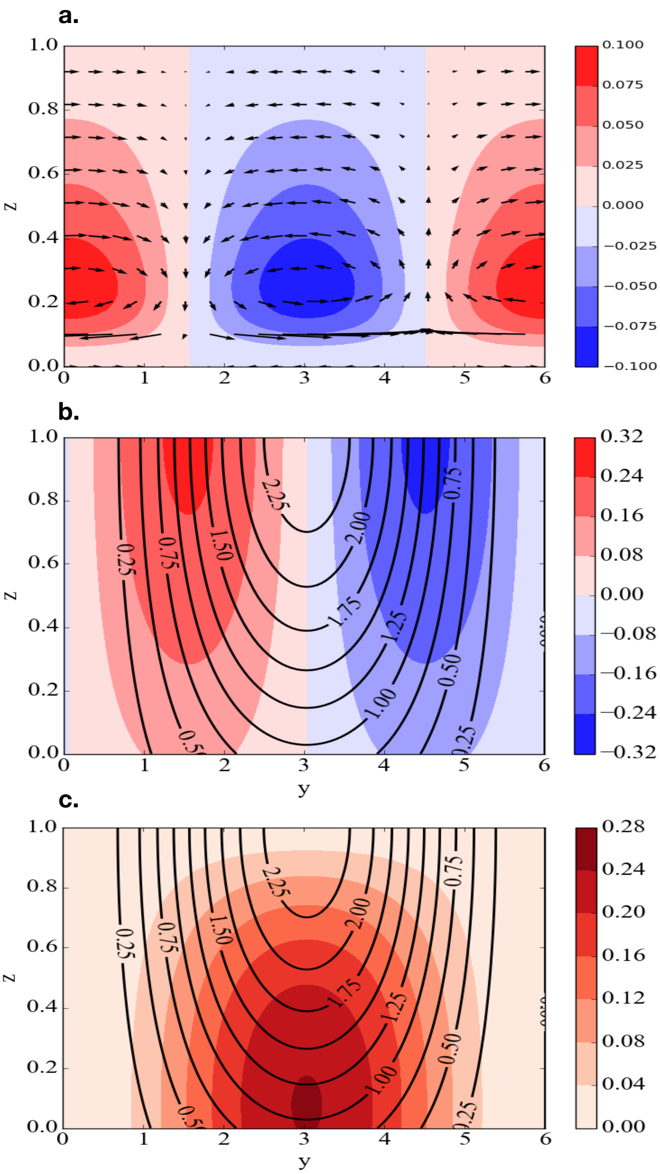

Figure 5. Eddy driven circulation in mid-latitudes when γ_1 is equal to 0.3. The meridional domain size is set up as $L_A = 2.0$ and α_L and E_T are equal to 1.0. $H = 1.0$ is used as the density height. The mass stream function (contour) and the following vertical and meridional velocities (arrows) are shown in (a). Zonal mean zonal wind (contour) is drawn with eddy momentum flux (shading) (b) and with poleward heat flux (shading) (c).

and $A_1 = \alpha + E_1 \sin\gamma_1 L_A$. We obtain the anomalous temperature $\overline{\eta}_0$ as

440 $$\overline{\eta}_0(y,z) = (A_1 - \gamma_1 E_1 y + E_1 \sin\gamma_1 (y - L_A))\, e^{\frac{z}{2H}} \sin q_1 (1 - z). \tag{50}$$

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The meridional structure of $\overline{\eta}_0$ is a key factor in determining the overall zonal mean zonal wind in mid-latitudes. It consists of two main components: a linear part and a sinusoidal part. The linear part, represented by $A_1 - \gamma_1 E_1 y$, captures the decline of the anomalous potential temperature from subtropics to the pole. It represents the background westerly flow associated with the thermal wind balance. The sinusoidal part, given by $E_1 \sin(\gamma_1(y - L_A))$, is associated with the eddy momentum flux and plays a crucial role in intensifying the regional westerlies. It represents the eddy-driven jet stream in mid-latitudes, which is a result of the interaction between synoptic eddies and the zonal mean zonal wind. The eddy momentum flux transfers momentum from the synoptic eddies to the zonal mean flow, enhancing the vertical shear of the zonal wind and leading to the formation of the jet stream. Combining the linear part and the sinusoidal part of $\overline{\eta}_0$, we obtain the overall meridional structure of the zonal mean zonal wind in mid-latitudes.

The constant $\gamma_1 = \sqrt{\frac{D}{SM}}\lambda_1$ plays a crucial role in determining the overall structure of the circulation in mid-latitudes. It governs the spatial distribution and characteristics of important features such as the jet stream, the locations of upward and downward motions, and the patterns of eddy momentum and heat fluxes. The magnitude of γ_1 influences the steepness and width of the meridional structure. A larger value of γ_1 leads to a sharper transition between different regions and a narrower jet stream, while a smaller value results in a more gradual transition and a broader jet stream.

The value of γ_1 is determined by the interplay between various factors related to the dynamics of synoptic eddies in mid-latitudes and the vertical structure determined by the Ekman pumping which governs the momentum balance near the surface. It is proportional to the eddy diffusivity D, which characterizes the ability of synoptic eddies to transport heat and momentum. A higher eddy diffusivity leads to a larger γ_1 and a more pronounced meridional structure. Conversely, γ_1 is inversely proportional to the dry static stability S, which reflects the stability of the atmosphere against vertical motion. A higher stability leads to a smaller γ_1 and a smoother meridional structure. The momentum diffusivity M also influences γ_1. A higher momentum diffusivity leads to a smaller γ_1 and a stronger eddy-driven jet stream. Therefore, the values of D, S, and M, which are all related to the life cycle of synoptic eddies, are crucial in determining the structure of the meridional circulation in mid-latitudes. The characteristics of geostrophic turbulence shaped by synoptic eddies determine the value of γ_1 and, in turn, the overall spatial distribution and behavior of the circulation.

The vertical structure is characterized by $\lambda_1 = \sqrt{q_1^2 + 1/4H^2}$. It is important to note that the eigenvalue q_1 is determined through asymptotic matching between the vertical wind and the surface vorticity, which arises from Ekman pumping. This process is equivalent to the momentum balance in the vertical column, established by the convergence of eddy momentum flux in the free atmosphere and surface friction.

All factors influencing the baroclinic wave life cycle collectively determine the vertical and meridional structure of the mid-latitude circulation. Surface friction is directly linked to the vertical shear of the zonal-mean zonal wind near the surface, which serves as the source of baroclinic instability. Unstable baroclinic waves break and generate eddy momentum flux convergence, thereby accelerating the zonal-mean zonal wind. This acceleration is counteracted by surface friction, resulting in a balanced system. This entire process can be formulated as an eigenvalue problem.

Figure 5 demonstrates the circulation in mid-latitudes resulting from the solution of the parameterized planetary heat equation. With a chosen value of $D = 0.22$, $M = 0.30$, $S = 1.0$, $H = 1.0$, $E_v = 0.01$ and $L_A = 6.0$, along with the specified values

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of $\alpha = 1$ and $S_T = 1$, the characteristics of the circulation are found. In Figure 5a, the major circulation pattern is displayed, resembling the structure of the Ferrell cell. The circulation features downward motion near $y = 1.5$ and upward motion near $y = 4.5$, which is indicative of the large-scale overturning circulation. The zonal wind speed, shown in contour plots in Figure 5b, exhibits a single jet stream located near the top of the atmosphere, positioned in the middle of the meridional domain. Moreover, due to the surface friction, surface westerlies are clearly seen. The presence of the jet stream is consistent with the dynamics of the eddy-driven circulation in mid-latitudes.

Figure 5b and c provide additional insights into the circulation, particularly the eddy momentum fluxes (b) and the poleward heat fluxes (c). The extrema of the eddy momentum fluxes are observed on either side of the jet stream, with positive values in the equatorward flank and negative values in the poleward flank. These extrema align with the local extrema of the vertical velocity, indicating the interaction between synoptic eddies and the zonal mean flow. Furthermore, the poleward heat flux is maximized near the surface. This distribution is consistent with the poleward transport of heat associated with the eddy-driven circulation. Although the model is simplified, the solution of the parameterized planetary heat equation successfully generates a circulation in mid-latitudes that exhibits similarities to the Ferrell cell, capturing key features such as the zonal jet stream, eddy momentum fluxes, and poleward heat fluxes.

3.2 EP flux and non-acceleration theorem

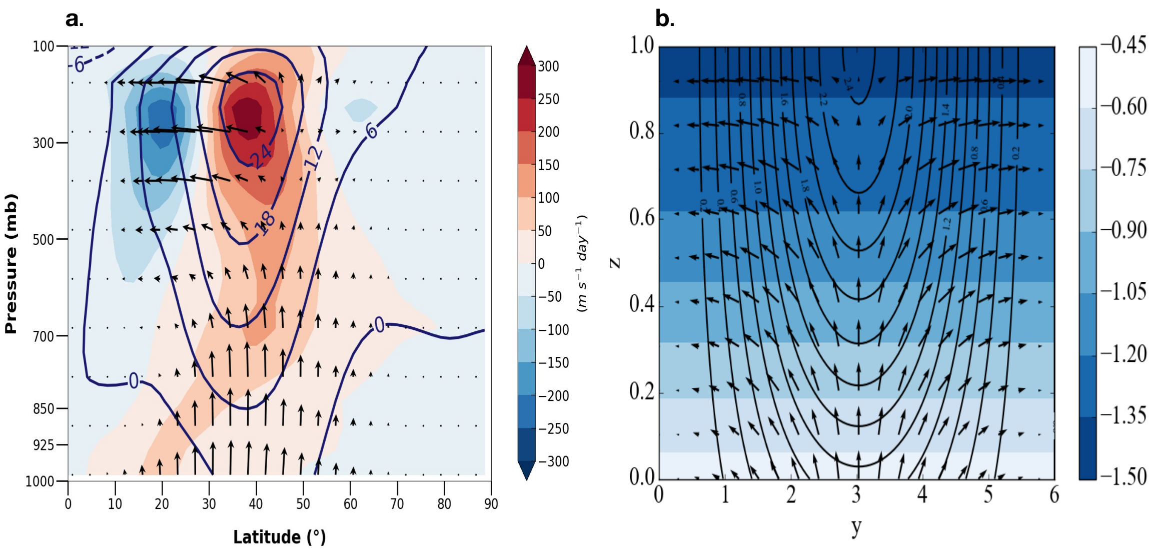

Figure 6. The Elissen-Palm flux (arrows) and its divergence (color) constructed from the idealized GCM dynamic core (a) and the analytic solution (b).

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The analytically constructed solution can be analyzed using the Transformed Eulerian Mean (TEM) formalism. In the absence of surface stress and external thermal forcing, the EP flux and its divergence become key factors in the meridional circulation. By utilizing the analytic representations of the eddy fluxes, the EP flux divergence is calculated as

$$\frac{1}{\rho_S}\nabla\cdot(\rho_S\boldsymbol{F}) = -M\gamma_1^3 E_1 \int\limits_0^z e^{\frac{z'}{2H}} \sin q_1(1-z')dz', \tag{51}$$

which indicates that the EP flux divergence is negative throughout the entire domain and its magnitude increases with height. Figure 6 shows the EP flux and its divergence obtained from the idealized General Circulation Model (GCM) dynamic core simulation (a) and the analytical solution (b). The analytical solution demonstrates that the Rossby waves propagate upward from the surface and then travel toward the flanks of the jet stream. In contrast, the idealized GCM dynamic core simulation exhibits upward and equatorward propagation.

The analytical solution $\overline{\eta}_0$ consists of two parts in the meridional domain. The first part is represented by the linear decline of the anomalous potential temperature, $A_1 - \gamma_1 E_1 y$, while the second part is a periodic function, $\sin(\gamma_1(y - L_A))$, which induces eddy-driven circulations in midlatitudes. The non-zero negative values of the EP flux divergence originate from the first part, resulting from the contribution of tropical areas included as a boundary condition. The meridional circulation consists of $\overline{v}_1$ and $\overline{w}_1$. Here, the ageostrophic vertical velocity is calculated by $\overline{w}_1 = \frac{D}{S}\frac{\partial^2\overline{\eta}_0}{\partial y^2}$. Due to the second derivatives with respect to y, the linear part does not contribute to the ageostrophic vertical velocity $\overline{w}_1$. Therefore, the meridional circulation is generated by the periodic function ($\sin(\gamma_1(y - L_A))$, satisfying

$$\frac{1}{\rho_S}\nabla\cdot(\rho_S\boldsymbol{F}) = 0,$$

which implies the non-acceleration theorem. The meridional circulation shown in Figure 5a can be interpreted as a dynamic manifestation of the non-acceleration theorem.

3.3 Multiple jets

In Figure 7, we show the effects of further increasing the structural constant γ_1 by updating M from $M = 0.29$ to $M = 0.15$, which results in almost doubling γ_1. As a result, two jet streams appear in the same domain, and three meridional circulations can be observed in mid-latitudes. The increase in γ_1 alters the number and arrangement of jet streams, which in turn affects the overall structure of the meridional circulations. The relationship between the structural constant D/SM and the size of the meridional domain L_A plays a crucial role in determining the number of meridional circulations in mid-latitudes. This suggests that the interplay between the dynamics and thermodynamics of turbulent eddies, as characterized by D/SM, and the size of the domain, as represented by L_A, influences the overall structure of the general circulation.

Previous research has primarily focused on understanding the characteristics of turbulent eddies, including their life cycle, wave-mean interactions, and the formation of jet streams (Edmon Jr et al., 1980; Simmons and Hoskins, 1978). However, connecting these dynamic and thermodynamic features of eddies to the broader structure of the general circulation has remained a challenge. The presented parameterized planetary-scale heat equation, along with the analysis of the structural constant γ_1,

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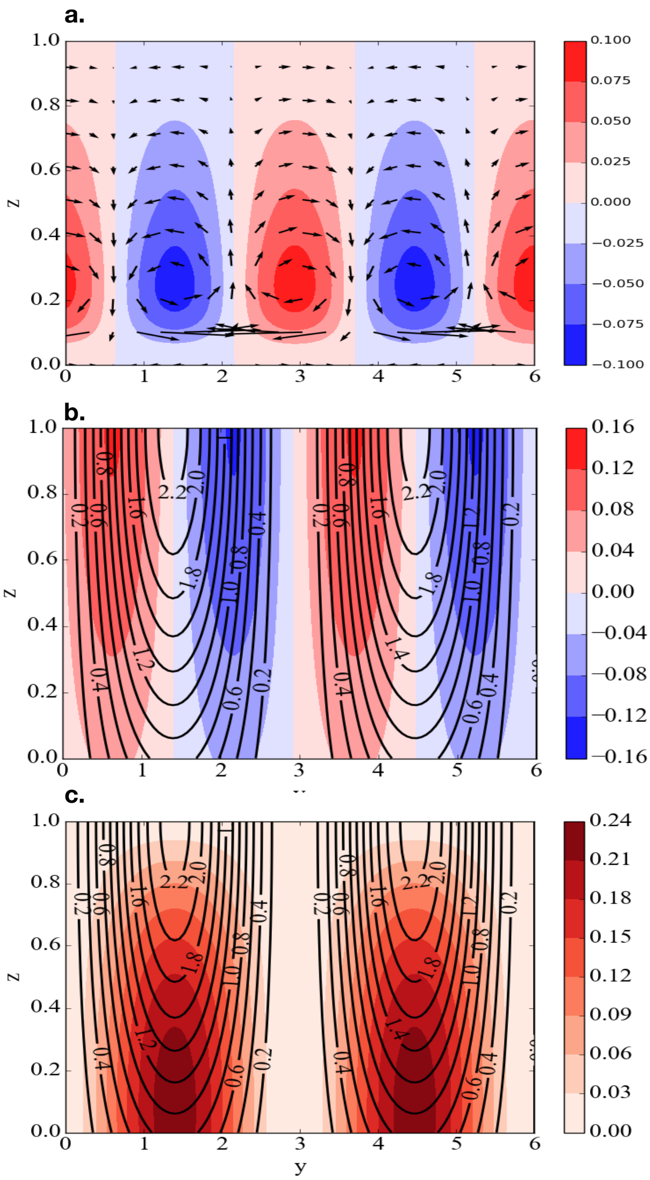

Figure 7. Eddy driven circulation in mid-latitudes when γ_1 is equal to 0.6. All of the parameters are similar to Fig. 5

provides a step towards bridging the gap between the dynamics and thermodynamics of eddies and the overall structure of the meridional circulations in mid-latitudes.

It is crucial to assess the role of the eigenvalue λ_1. In this example, M is controlled to double γ_1 while keeping λ_1 fixed. However, it is possible to increase λ_1 to modify the meridional structure of the circulation.

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The meridional boundary condition $f(z)$, which represents the influence of the Hadley circulation, can exhibit various vertical modes defined by λ_1. Similarly, surface thermal forcing can also excite different vertical modes, thereby influencing the meridional structure. This indicates that localized surface thermal forcing has the potential to alter the zonal-mean zonal wind in multiple ways.

Thermal forcing can modify the meridional temperature gradient, updating the eddy heat flux and generating various vertical modes, which in turn affect the meridional structure. Updates to the boundary conditions can impact the overall solution structure. Consequently, changes in the jet stream or the meridional structure of potential temperature under specific conditions must be understood as a result of the interaction among various eigenfunctions.

It might be possible to apply the insights to other planets with various background fields. Different planets may exhibit diverse vertical structures (Showman and Polvani, 2011), characterized by various combinations of eigenvalues and eigenfunctions. For instance, in atmospheres with larger values of the vertical eigenvalue λ, it is conceivable to observe multiple vertical layers and multiple jet streams in the meridional domain. Understanding the connection between the vertical structure of the anomalous potential temperature and the meridional structure of the large-scale atmosphere is crucial. This knowledge can provide insights into the atmospheric structure not only on Earth but also on other planets, where different physical processes and conditions may shape their atmospheric circulation patterns.

3.4 The size of the Ferrel cell and the eddy-driven length-scale

According to the theory, the size of the mid-latitude circulation is determined by the structure number γ_1. The solution shows that $\sin\gamma_1(y-L_A)$ controls the size of the circulation. The structure number γ_1 with dimension is

$$\gamma_1 = \sqrt{\frac{f^2 D}{N^2 M}}\sqrt{q_1^2 + \frac{1}{4H^2}}, \tag{52}$$

where f is a representative Coriolis parameter in mid-latitudes, N^2 is the Brunt-Väisälä frequency, and D and M have dimensions of m^2/s. The eigenvalue q_1 is the smallest positive solution of the characteristic equation,

$$\frac{\delta_E N^2 z_T}{2Df}\left(\frac{z_T}{2H}\sin(qz_T) + (qz_T)\cos(qz_T)\right) = \sin(qz_T)\left((qz_T)^2 + \frac{z_T^2}{4H^2}\right), \tag{53}$$

where z_T is the mean tropopause height and δ_E is the atmospheric boundary layer height defined as $\delta_E = \sqrt{2A_v/f}$. Here, A_v is the vertical viscosity coefficient. The meridional eigenvalue γ_1 can be transformed for further analysis to

$$\gamma_1 = \sqrt{\frac{f^2 D\partial\overline{\eta}_g/\partial y}{N^2 M\partial\overline{u}_g/\partial y}\frac{\partial\overline{u}_g/\partial y}{\partial\overline{\eta}_g/\partial y}}\sqrt{q^2 + \frac{1}{4H^2}} = \sqrt{\frac{f^2\overline{v'_g\eta'_g}}{N^2\overline{u'_g v'_g}}\frac{\partial\overline{u}_g/\partial y}{\partial\overline{\eta}_g/\partial y}}\sqrt{q^2 + \frac{1}{4H^2}} \tag{54}$$

Thus, the meridional length-scale L_1 for the mid-latitude circulation can be estimated by π/γ_1. The length-scale L_1 is determined by the eddy fluxes, dry static stability, and the shear of the jet stream. It implies that the length-scale associated with the size of the mid-latitude circulation is determined by the baroclinic wave life cycle and its overall influence on the zonal mean zonal wind. As emphasized above, the eddy heat flux is originated from the baroclinic growth of synoptic waves near the surface and the eddy momentum flux from the propagation and breaking of the saturated synoptic waves in upper levels. The

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dry static stability is also influenced by the activity of synoptic eddies. Eventually, the consequence of the baroclinic wave life cycle is the shape of the jet stream represented by the horizontal and vertical shear.

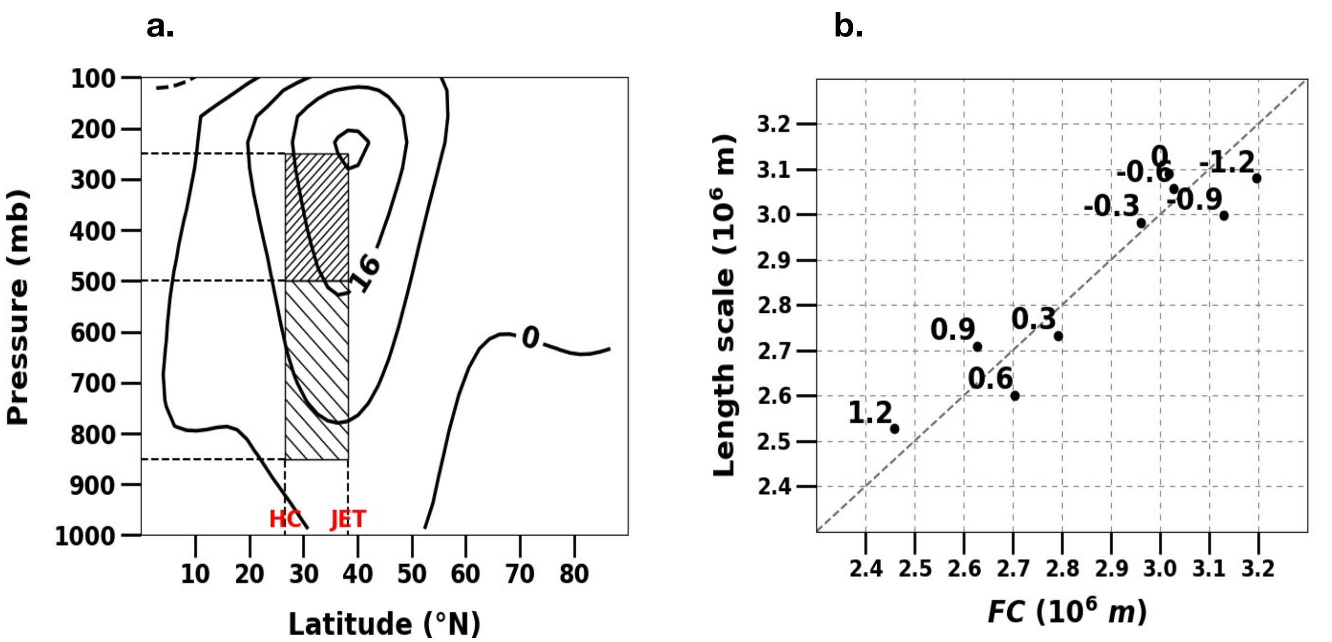

Figure 8. The eddy fluxes, jet stream shear, and dry static stability are computed around the jet stream (contour). The eddy momentum flux is evaluated equatorward of the jet stream at upper levels, from 500 mb to the jet core, while the eddy heat flux is measured at lower levels between 850 mb and 500 mb. Black dots indicate the specific locations where eddy fluxes are sampled (panel a). Panel (b) compares the mid-latitude circulation length scale, L_1 (y-axis), with the size of the Ferrel cell (x-axis), based on results from nine idealized GCM dynamic core simulations.

The applicability of the length-scale L_1 can be evaluated by the comparison with the size of the Ferrel cell in numerical simulations. Using the GFDL Flexible Modeling System (FMS), 9 simulations are prepared with various high latitude surface thermal forcing which controls the baroclinicity in mid-latitudes. Depending on the baroclinicity, the growth of synoptic eddies in mid-latitudes differs, which also influences the degree of the eddy momentum flux in upper levels. Figure 1 shows two examples among the 9 simulations. When high-latitude cooling (warming) is applied with the magnitude of the thermal forcing $-1.2\ K/day$ (a) ($+1.2\ K/day$ (b)), it is observed that the size of the Ferrel cell is larger with the cooling in high latitudes. The size of the Ferrel cell is calculated using the mass stream function from the equator edge to the polar one defined by the latitudes when the sign of the mass stream function changes.

The length-scale L_1 can also be calculated in the same simulations. The eddy heat fluxes are deduced in the lower levels between 850 mb and 500 mb, while the eddy momentum fluxes are calculated in the equator flank of the jet stream in the upper levels between 500 mb and 250 mb. The horizontal and vertical shear of the jet stream are deduced around the center of the jet stream and the dry static stability is calculated in the vertical column below the maximum of the jet stream.

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According to Figure 9b, the length-scale L_1 (y-axis) is closely related to the size of the Farrel cell (x-axis). Two lengths are concentrated around $y = x$, suggesting that the length-scale L_1 can be a proper meridional scale of the Ferrel cell in mid-latitudes. With various baroclinicity controlled by high-latitude surface thermal forcing, the change of the size of the Ferrel cell is nonlinear as shown by the abrupt change of the size from $q_0 = 0\ K/day$ to $q_0 = +0.3\ K/day$. The length-scale L_1 reflects the nonlinear change consistently. The change of the length-scale L_1 could tell us how the synoptic eddies shapes the Ferrel cell.

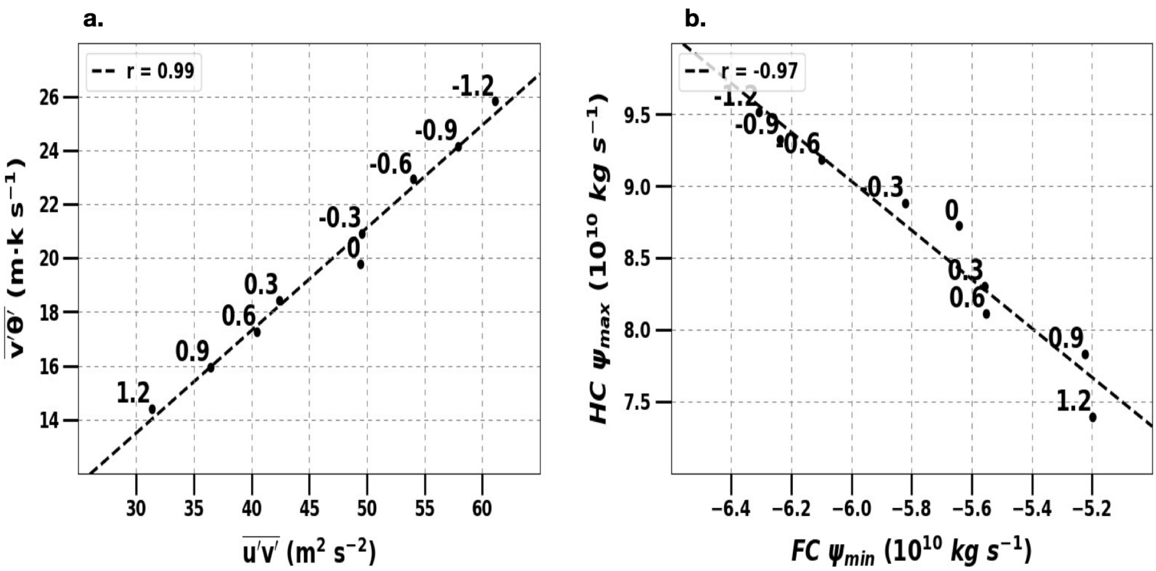

Figure 9. The relationship between the eddy momentum (x axis) and the eddy heat flux (y axis) (a) and that between the Ferrel cell and the Hadley cell intensity (b) are shown with various thermal forcing.

The length-scale L_1 differs depending on the average baroclinicity controlled by the thermal forcing controlling the overall meridional temperature gradient. Especially, L_1 has a linear relationship with the degree of the thermal forcing. Even though the theory treats the eddy heat and momentum fluxes independently with two distinct diffusivities D and M, these are not independently determined in large-scale atmospheric dynamics. The major sources of the two fluxes are unstable baroclinic waves growing and propagating from the lower levels, hence the two fluxes should be correlated. The linear relationship shown in figure 9 represents the dynamical relationship of the two fluxes, which might imply that the structure number γ_1 do not change dramatically due to the update of thermal forcing in the planet.

The theory deals with the influence of the Hadley circulation by the equatorward boundary condition $f(z)$ and the residual heat measured by S_T. Because the theory treats the isolated mid-latitude, the equatorward boundary conditions represent the influence of the Hadley circulation on mid-latitudes. Figure 9b shows the relationship between the magnitude of the Hadley

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cell and that of the Ferrel cell. The magnitude of the two cells measured by the mass stream functions is strongly correlated. As the intensity of the Hadley cell increases, the Ferrel cell is also intensified, implying that the two cells are closely correlated dynamically. In the analytic solutions, the magnitude of $f(z)$, α, and the adiabatic cooling E_T determines the intensity of the circulation in mid-latitudes.

4 Summary & discussion

In the context of a zonally-symmetric atmosphere, the evolution of planetary-scale potential temperature is influenced by the residual effects of local radiative-convective equilibrium, the poleward heat flux generated by synoptic eddies, and the adiabatic cooling or heating resulting from eddy momentum flux. The radiative-convective equilibrium potential temperature, denoted as Θ_E, is determined by the external heat flux. Concurrently, the anomalous potential temperature $\overline{\eta}_0$ is primarily shaped by turbulent synoptic eddies that induce heat and momentum flux.

To construct solutions for the zonally-averaged heat equation, simplified parameterizations were here introduced based on prior research and idealized GCM dynamic core experiments. These parameterizations capture the impact of the poleward heat flux, which modulates the meridional gradient of the zonal averaged anomalous potential temperature $\overline{\eta}_0$, as well as the eddy momentum flux, which enhances the horizontal shear of the zonal mean zonal wind $\overline{u}_0$. The constants D and M represent the linear relationships between the fluxes and the corresponding gradients.

The simple parameterizations are insufficient to capture the complexity of the life cycle of unstable baroclinic waves. The construction of M and D from the idealized GCM dynamic core reveals that both M and D vary spatially, which is related to the characteristics of Rossby wave propagation and breaking. Furthermore, the parameterizations do not adequately represent the relationship between M and D. The eddy heat and momentum fluxes are dynamically connected through the baroclinic wave life cycle, implying that M and D are not independent of each other. Despite these limitations, the conceptual model with these parameterizations describes a sustained indirect meridional circulation shaped by baroclinic eddies. The down-gradient characteristic of the eddy heat flux and the up-gradient characteristic of the eddy momentum flux are in a state of dynamical balance. Notably, the solutions include an example that satisfies the non-zero acceleration theorem, emphasizing the existence of an indirect meridional circulation necessary to maintain a steady-state zonal mean field.

In our model, the overall structure of the general circulation in mid-latitudes is determined by several key factors. The structural number D/SM reflects the characteristics of turbulent synoptic eddies. The eigenvalue λ is associated with the vertical structure of the large-scale atmosphere, influencing the presence of multiple vertical layers or jet streams. Additionally, L_A represents the size of the planet relative to the Rossby deformation radius of the large-scale atmosphere. Through the consideration of these factors and the wave-mean interaction, our model offers valuable insights into the dynamics and thermodynamics of synoptic eddies and their impact on the general circulation in mid-latitudes.

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4.1 Life cycle of synoptic eddies and turbulent fluxes

Synoptic eddies play a crucial role in shaping the climate of mid-latitudes. The presence of a radiative imbalance in the atmosphere leads to a baroclinic instability, causing synoptic eddies to grow by extracting potential energy from the mean field. As these eddies develop, they generate a poleward heat flux, resulting in a decrease in the meridional temperature gradient. When the synoptic eddies reach saturation, they break on the flanks of jet streams, producing turbulent momentum fluxes that strengthen the jet stream. In a steady state, the poleward heat flux is balanced by the momentum flux, thereby maintaining the jet stream in mid-latitudes.

The life cycle of synoptic eddies also influences the dry static stability, which is a critical factor in determining the overall atmospheric circulation. The current research highlights that the processes governing the life cycle of synoptic eddies have a direct impact on the structural number D/SM, which in turn determines the characteristics of the eddy-driven circulations in mid-latitudes, including the position and number of jet streams.

An important question arises concerning the potential changes in atmospheric circulation due to ongoing global warming. It becomes essential to investigate how the life cycle of synoptic eddies may be affected and to understand the corresponding changes in the structural number D/SM. Such research can provide insights into the altered dynamics and structure of eddy-driven circulations in response to global warming.

4.2 Parameterizations of the eddy heat and momentum fluxes

The main balance in the Ferrell cell is represented in the heat equation (5). The meridional eddy heat flux is balanced by adiabatic warming or cooling induced by the eddy momentum flux and the diabatic thermal forcing to adjust to a radiative-convective equilibrium temperature. In this research, it is assumed that the response time of the radiative-convective process τ is large. The surface energy flux balance over the ocean has a longer time-scale due to the large heat capacity of the ocean mixed layer. Under this assumption, the diabatic process is ignored. The major balance is between the eddy heat flux generated by baroclinic instability and the adiabatic warming induced by the eddy momentum flux. The main issue for the development of the simple theory is to parameterize the eddy heat and momentum fluxes using the zonal mean anomalous temperature or zonal wind.

According to previous research, the poleward eddy heat flux generated by unstable baroclinic waves induces warming (cooling) in the poleward (equatorward) side of the jet stream, resulting in the decrease of the meridional temperature gradient. Thus, the parameterization $\overline{\eta_0' v_0'} \simeq -D\frac{\partial \overline{\eta}_0}{\partial y}$ represents the role of the eddy heat flux in decreasing the vertical shear of the jet. On the other hand, the eddy momentum flux maximized at the flank of the jet strengthens the jet stream by increasing its horizontal shear, which is represented by the simple parameterization $\overline{u_0' v_0'} \simeq M\frac{\partial \overline{u}_0}{\partial y}$.

The poleward heat flux is maximized at the meridional center of the jet stream where the meridional gradient of the anomalous potential temperature is largest, inducing anomalous cooling (warming) at the equatorward (poleward) side of the jet. Simultaneously, the maximized eddy momentum fluxes at the sides of the jet induce adiabatic warming (cooling) at the equator-

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ward (poleward) side of the jet stream, establishing a thermodynamic balance with the eddy heat flux. This feature encapsulates the characteristics of the Ferrel cell observed in mid-latitudes.

4.3 Surface friction, the momentum balance, and the vertical structure of large-scale atmosphere

The eigenvalue problem is ultimately formulated by the momentum balance in the vertical column, which involves the eddy momentum flux convergence and surface friction. This is equivalent to the surface boundary condition, which states that the surface vertical wind $\overline{w}_1(y, z=0)$ is proportional to the vertical vorticity $-\frac{d\overline{u}_s}{dy}$. The eigenvalue q is influenced by the dry static stability S, the turbulent heat diffusivity D, and the Ekman number E_v. This implies that q emerges as a result of the interaction between the baroclinic wave life cycle in the free atmosphere and processes in the planetary boundary layer. The generation of surface westerlies, driven by momentum balance, is inherently linked to the determination of q.

The eigenvalue q is also used to compute γ, highlighting the strong connection between the meridional structure of the general circulation and the vertical structure of the large-scale atmosphere. The eigenvalue q plays a critical role in defining the vertical characteristics of the atmosphere. As a result, the magnitude of γ, which is proportional to $\sqrt{q^2 + \frac{1}{4H^2}}$, is crucial in shaping the meridional structure of the circulation. The structural parameter γ directly determines the location and number of eddy-driven circulations observed in mid-latitudes.

The average vertical structure of the large-scale atmosphere in mid-latitudes is influenced by various factors, including local radiative-convective equilibrium. This average structure may also be linked to planetary composition, which governs radiative transfer processes and, consequently, affects the meridional structure of the general circulation. Further investigation is necessary to validate these theoretical calculations and gain deeper insight into the underlying dynamics.

4.4 Size of planets

An additional influential factor that governs the location and number of circulations in mid-latitudes is the size of the planet relative to the internal Rossby deformation radius. Consequently, the size of the meridional domain, denoted as L_A, is measured in relation to this horizontal length scale. Consequently, the ratio L/R, where L signifies the horizontal length scale and R denotes the planet's radius, emerges as a critical determinant of the atmospheric circulation's structure.

4.5 Interaction with local radiative-convective equilibrium

The current research intentionally omits the inclusion of the Newtonian cooling term, which arises from deviations from radiative-convective equilibrium. The damping effect on the anomalous potential temperature, represented by the time-scale τ, is not considered in this study. However, it is important to acknowledge that the incorporation of the damping term would certainly have an impact on the solutions, as it would change the characteristic equation for the meridional domain. This inclusion becomes particularly relevant as it highlights the interaction between local radiative-convective equilibrium and large-scale atmospheric dynamics.

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The local stability of the radiative-convective equilibrium is denoted by the term $-1/\tau$. As the process of global warming persists, there are changes in the local energy flux balance, including alterations in stability. Within the framework employed in this research, it appears feasible to investigate how changes in local stability can influence the behavior of the large-scale atmosphere.

Appendix A: Two Layer Model

A simple two-layer model can be considered to derive an equation analogous to the main equation (11). The non-dimensionalized two-layer model is expressed as follows:

$$\left(\frac{\partial}{\partial t}+u_1\frac{\partial}{\partial x}+v_1\frac{\partial}{\partial y}\right)\left(\nabla^2\psi_1-F(\psi_1-\psi_2)+\beta y\right)=0,$$
$$\left(\frac{\partial}{\partial t}+u_2\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}\right)\left(\nabla^2\psi_2+F(\psi_1-\psi_2)+\beta y\right)=0, \quad \text{(A1)}$$

where $u_i=-\partial\psi_i/\partial y$, $v_i=\partial\psi_i/\partial x$, and $F=f_0^2L^2/g'D$. The subscript i denotes the layer, with $i=1$ referring to the upper layer and $i=2$ to the lower layer. Here, F represents the Froude number, D is the height of the layer, and g' is the reduced gravity (Pedlosky, 2013).

After taking the zonal average, equation (A1) becomes

$$-\frac{\partial^2}{\partial y^2}\overline{u_1'v_1'}-F\frac{\partial}{\partial y}\overline{v_1'(\psi_1'-\psi_2')}=0,$$
$$-\frac{\partial^2}{\partial y^2}\overline{u_2'v_2'}+F\frac{\partial}{\partial y}\overline{v_2'(\psi_1'-\psi_2')}=0, \quad \text{(A2)}$$

where $\overline{X}$ and X' represent the zonal mean of X and the deviation from the zonal mean, respectively. Applying a simple parameterization for the eddy momentum flux, $\overline{u_1'v_1'}=M\frac{\partial\overline{u}_1}{\partial y}$ and $\overline{u_2'v_2'}=M\frac{\partial\overline{u}_2}{\partial y}$, transforms equation (A2) into

$$-M\frac{\partial^3\overline{u}_1}{\partial y^3}-F\frac{\partial}{\partial y}\overline{v_1'(\psi_1'-\psi_2')}=0,$$
$$-M\frac{\partial^3\overline{u}_2}{\partial y^3}+F\frac{\partial}{\partial y}\overline{v_2'(\psi_1'-\psi_2')}=0, \quad \text{(A3)}$$

Summing these two equations yields

$$-M\frac{\partial^3}{\partial y^3}(\overline{u}_1-\overline{u}_2)-2F\frac{\partial}{\partial y}\overline{\frac{v_1'+v_2'}{2}(\psi_1'-\psi_2')}=0. \quad \text{(A4)}$$

In the middle of the two layers, we can apply the thermal wind balance, $\overline{u}_1-\overline{u}_2=-\frac{\partial}{\partial y}(\overline{\psi}_1-\overline{\psi}_2)$, and the eddy heat parameterization, $\frac{\partial}{\partial y}\overline{\frac{v_1'+v_2'}{2}(\psi_1'-\psi_2')}=-D\frac{\partial}{\partial y}\Delta\psi$, where $\Delta\psi=\overline{\psi}_1-\overline{\psi}_2$. This leads to

$$M\frac{\partial^4}{\partial y^4}\Delta\psi+2DF\frac{\partial^2}{\partial y^2}\Delta\psi=0. \quad \text{(A5)}$$

This equation is very similar to equation (11) in terms of the meridional domain. However, the detailed vertical structure cannot be accurately represented in the two-layer model.

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