Review Response

I. REVIEWER 3

This manuscript presents analytical expressions for steady-state general circulation characteristics in mid-latitudes, namely for zonal mean temperature and wind, secondary meridional overturning circulation (the Ferrel cell) and eddy momentum and heat fluxes. These analytical expressions are derived from the quasi-geostrophic vorticity and heat equations on the beta plane by parametrizing momentum and heat fluxes using flux-gradient relationships, respectively. Specifically, diffusive eddy heat fluxes are assumed, while momentum fluxes are assumed to be up-gradient. These relationships are also tested using idealized GCM simulations.

We thank the reviewer for the careful reading of the manuscript and the useful comments.

Overall I appreciate the attempt to come up with analytical insights into the relations between eddy fluxes and general circulation characteristics. However, similar to reviewer 1 I feel that the presented results do not seem to go beyond what is already well known. For example, Vallis' textbook summarizes what is known and presents theoretical insights into Ferrel cell dynamics based on eddy flux parameterizations.

Response:
The explanation of mid-latitude circulation in Vallis' textbook is based on the Kuo-Eliassen equation, which incorporates eddy fluxes in explicit form. This framework emphasizes the relationships among differential terms and relies heavily on numerical simulations for interpretation. However, such an approach does not readily reveal how the combination of key physical processes determines the overall structure of the circulation.

To illustrate the value of an analytical perspective, consider the classic spring–mass system governed by Hooke's law: $d^2x/dt^2 = -kx$. From this equation, one can interpret the presence of a restoring force proportional to displacement x. Numerical solutions yield oscillatory behavior, and by varying m and k, one can observe changes in frequency. However, deeper insight emerges from solving the equation analytically: the solution takes the form of a linear combination of $sin(\omega_0 t)$ and $cos(\omega_0 t)$, where $\omega_0^2 = k/m$. This clearly shows that the frequency of oscillation is controlled by $\omega_0 = \sqrt{k/m}$, providing a concise and physically meaningful summary of the system's behavior.

In a similar spirit, our analytical results introduce a new parameter—the structure number—which emerges directly from the solutions. Analogous to ω_0 in the spring system, the structure number encapsulates the dynamics of the idealized Ferrel cell. It integrates the effects of the baroclinic wave life cycle (through parameters M and D), the background static stability S, and the vertical structure q. This provides a compact and insightful way to understand how steady-state indirect circulation in the mid-latitudes is maintained.

Changes in the text:
We will mention the importance of our analytical framework in the revised introduction.

The presentation lacks focus (e.g., repetition of motivational statements such as referring to baroclinic life cycles, which is distracting given that instead of initial value problem, steady state solutions are considered)
Perhaps most importantly, I see some potential flaws in the presented theory:
The eddy fluxes giving rise to the Ferrel cell and the eddy-driven jet are ultimately forced by the equator-to-pole temperature contrast that is continuously reinforced by radiation. The strength of this radiatively forced temperature contrast would seem to be vital for the resulting jet and Ferrel cell (as also shown by GCM studies in the literature). Yet, the presented theory neglects this forcing, so the resulting eddy fluxes, jet and Ferrel cell are independent of it. Perhaps the authors envision their theory to only be applicable within a certain regime of forced meridional temperature contrast, but this is not stated nor discussed (I'm not referring to sensitivity to τ as in the final subsection 4.5, I'm rather referring to sensitivity to Θ_E and its meridional gradient).

Response:
In the absence of large-scale atmospheric dynamics, the equator-to-pole temperature gradient would follow the radiative-convective equilibrium temperature,Θ_E, as described in Equation (1). Large-scale atmospheric motions modify this equilibrium through the anomalous potential temperature η_0, which is maintained by eddy heat and momentum fluxes generated by baroclinic instability. As demonstrated in numerous previous studies—as well as

in the analytical solutions presented in this research—the maintenance of η_0 results from a balance between the eddy heat and momentum fluxes. This raises the question of the role played by the radiative-convective equilibrium temperature Θ_E in this dynamic system.

If the background temperature gradient defined by Θ_E is large (or small), it can enhance (or suppress) the strength of baroclinic instability, potentially resulting in stronger (or weaker) eddy heat and momentum fluxes. In our theoretical framework, the intensity or efficiency of these fluxes is governed by the parameters D and M, which are expected to be functions of Θ_E or its meridional difference $\Delta\Theta_E$. In other words, the generation and strength of eddies should be influenced by the radiative-convective equilibrium state. Furthermore, the background dry static stability S is also a function of Θ_E. The most important parameter controlling the structure of the Ferrel cell in this model is the structure number, which is proportional to D/SM. Therefore, the influence of the background radiative-convective equilibrium temperature is implicitly captured through the parameters D, M, and S in the analytic model. Determining how D, M, and S are derived from a given Θ_E represents a key research question for future investigation, building upon the analytical foundation established in this study.

Changes in the text:
We will incorporate this explanation in the revised manuscript.

The parametrization of up-gradient momentum fluxes seems new at first, but upon closer look is problematic for several reasons:

(a) A physical justification for this particular parametrization is lacking: it's not diffusion (for $M > 0$ as assumed throughout the manuscript), so you can't use the same arguments for this scaling as you would for (eddy) diffusion. Also, even if you can argue that momentum fluxes are qualitatively up-gradient you still lack arguments for why they should scale linearly with the background gradient. In that sense the parametrization seems ad hoc and is perhaps more akin to fitting, making it less of a theory. (BTW, I find it very confusing that M is being referred to as "momentum diffusivity" at places, when it is clearly not a diffusivity.)

Response:
It is well established that the horizontal propagation and subsequent breaking of Rossby waves near the center of the jet stream generate eddy momentum fluxes, which in turn reinforce the jet stream. This process leads to an enhancement of the meridional shear of the jet stream—an example of up-gradient transport.

The primary aim of this study is not to develop a detailed parameterization of the eddy momentum flux, but rather to examine its influence on the zonal-mean flow. Our central question is how to represent the dynamical balance between eddy heat and momentum fluxes. Although our formulation of the eddy momentum flux exhibits a simplified up-gradient structure, we argue that a linear parameterization is sufficient to capture the essential effects.

Furthermore, our goal is to derive explicit analytical expressions for the Ferrel cell. To achieve this, we deliberately adopt the simplest possible parameterizations that still yield meaningful analytic solutions. Despite the simplification, our model successfully captures the expected intensification of the upper-level jet stream driven by the convergence of eddy momentum fluxes.

Changes in the text:
We will add this argument in the revised manuscript.

(b) Making the momentum flux proportional to the meridional shear and the heat flux proportional to the meridional temperature gradient, in the way done here, basically builds in the answer: by thermal wind balance the heat flux will maximize in the core of the jet where the vertical shear is strongest, and the momentum flux will maximize (in amplitude) in the flanks of the jet where the meridional shear is strongest. The Ferrel cell and the jet then result automatically as a consequence and one can see this without an analytical solution. The presented "theory" then is not much more than a consistency check.

Response:
The structure of the Ferrel cell can be qualitatively inferred from the zonally averaged quasigeostrophic (QG) equations, especially when combined with insights from the life cycle of baroclinic waves. The interplay among the differential terms allows us to anticipate the general circulation pattern. Integrating the governing PDEs with appropriate boundary conditions not only validates these expectations but also yields quantitative insights into the

underlying physical processes. Among the various implications of the analytical solutions, two qualitative aspects represent key contributions:

First, the structure number γ determines the size and position of the idealized Ferrel cell. Although the solution is formulated on a beta plane, the length scale defined by γ closely aligns with the actual Ferrel cell dimensions observed in numerical simulations. The structure number depends on the eddy heat flux (D), momentum flux (M), background dry static stability (S), and the vertical structure eigenvalue (q). The analytical expression given in Equation (54) allows us to assess and predict midlatitude circulations under different climate conditions. This theoretical scaling framework offers a systematic approach to examining potential changes under global warming. The Rossby wave activity, captured by D and M, the dry static stability S, and the vertical structure q, all combine to shape the steady-state midlatitude circulation.

Second, the eigenvalue problem defined in Equation (42) illustrates the influence of the planetary boundary layer on interior midlatitude flow. In the absence of boundary layer effects, the first eigenvalue is π, placing the eddy heat flux maximum at the vertical midpoint. However, incorporating boundary layer dynamics through Equation (20) reduces the eigenvalue to approximately $\pi/2$, shifting the heat flux maximum just above the surface—a structure more consistent with reanalysis data and numerical simulations. Further investigation of this eigenvalue problem can enhance our understanding of how the boundary layer modulates large-scale atmospheric circulation.

Changes in the text:
In the revised paper, we will explain that while numerical simulations and reanalysis-based diagnostics remain indispensable, the analytical framework developed here provides valuable physical intuition and scaling relations. Together, these approaches enable more systematic and theoretically grounded investigations into midlatitude atmospheric dynamics.

(c) Flux-gradient relations are most appropriate for quasi-conserved quantities, which (relative) momentum does not belong to. Previous research has exploited this by parametrizing eddy PV fluxes this way (see, e.g., Vallis' textbook or some of the references that reviewer 1 lists). Near the vertical boundaries (surface, tropopause) it effectively works for buoyancy/potential temperature; since these boundaries may be considered to act as infinitesimal sheets of infinite PV gradient, these also effectively describe PV fluxes.

Response:
The central goal of this study is to investigate the steady-state mean circulation in the mid-latitudes under the condition of vanishing potential vorticity (PV) flux throughout the domain. Within the framework of the Transformed Eulerian Mean (TEM) formalism, the absence of external thermal or mechanical forcing requires that the meridional derivative of the Eliassen–Palm (EP) flux divergence vanishes everywhere—consistent with the non-acceleration state first described by Charney and Drazin.

Our analysis shows that the existence of such a steady circulation, maintained solely by turbulent eddies, demands a mechanical balance between the eddy momentum and heat fluxes. This balance gives rise to an indirect circulation pattern resembling the Ferrel cell around the jet stream. For this reason, a flux-gradient parameterization of the PV flux is not employed in this study.

Changes in the text:
We will explain in the revised text why a flux-gradient parameterization of the PV flux is not employed.

Speaking of conserved quantities: as far as I can tell the solutions presented (e.g., Fig. 5) violate global angular momentum conservation: the existence of surface westerlies within the Ferrel cell should demand surface easterlies somewhere outside that region so that the global mean zonal surface wind is zero (cf. Eq. 25). As far as I can tell the surface winds are westerly everywhere. So something seems to be wrong, either with the plot or with the solution.

Response:
The figures are designed to highlight the structure of the idealized Ferrel cell; therefore, they display only a portion of the full solution, focusing on the center of the Ferrel cell. As shown in equation (40), the meridional structure is represented by a sine function, implying that regions outside the Ferrel cell—such as those dominated by surface easterlies—exist but are not shown. Furthermore, as explained in the manuscript, the idealized model does not explicitly resolve the tropics. Instead, tropical influences are indirectly imposed through the boundary condition $f(z)$. In the tropics, surface easterlies are necessary to ensure global angular momentum balance. However, a detailed

treatment of tropical dynamics is beyond the scope of this study.

Changes in the text:
The issue of angular momentum conservation will be addressed in the revised manuscript.

Other comments:

abstract, lines 4-7: the indirect meridional circulation (Ferrel cell) is really just a byproduct of the eddy fluxes, necessary to achieve overall balance (e.g., see your statement on line 70), but because its part of the response to mechanical forcing (by the eddies) I think it should not be viewed as (by itself) "shaping regional weather" or "predictions of mid-latitude weather under global warming"

Suggestion will be followed in the updated manuscript.

line 46: this is not true, the Held-Hou theory, in fact, does predict expansion under global warming scenarios (e.g., due to increasing tropopause height); perhaps the authors mean to indicate that this Held-Hou expansion is insufficient in explaining to expansion seen in full-blown models?

In the revised paper, we will mention that this statement is based on Lu et al. (2007), who showed that the theory grounded in Held and Hou (1980) is insufficient to fully explain the expansion of the Hadley cell under global warming. We will revise the manuscript to incorporate this point more clearly, following your suggestion.

line 54: the jet is sustained by eddy fluxes, not the meridional overturning circulation (which, by itself, would act to weaken the jet)

We will update the manuscript based on your correction.

Fig. 2a, b: it's unclear to me to what degree these correlations are due to the spatial correlations shown in panels c, d? I.e., panels a, b combine two types of co-variations that I think should not be conflated: 1) spatially over the 15 points shown in panels c, d per simulation, 2) over the different simulations.

Response:
Thank you for your valuable suggestion. In our study, we calculated M and D using 15 surrounding grid points centered on a single reference point, across nine experiments with different baroclinicity. This choice was made to examine the correlation between M and D and each variable of interest from an average perspective.

Changes in the text:
We updated Fig. 2a and 2b by distinguishing each experiment with unique colors. The updated figures confirm that M and D exhibit behavior consistent with theoretical predictions within each experiment.

We expect that this visual modification will help reduce confusion for readers and enhance the clarity of the theoretical implications.

Fig. 6a: not sure about the precise settings of the Held-Suarez simulation (which are not provided), but the EP-flux divergence looks wrong in any case: 1) the magnitudes seem way too strong (off by at least an order of magnitude), 2) they should be predominantly negative in the interior (corresponding to wave dissipation, see panel b), so the large region of positive values seems wrong.

The dry dynamical model used in this study follows the benchmark setup of Held and Suarez (1994). As pointed out by the reviewer, the EP-flux divergence has been revised with area weighting, as shown below. This correction is generally accepted by the community. The following equation is used to calculation the divergence of the EP flux:

$$\frac{1}{a\cos\phi}\left[\frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}\left(-a\cos^2\phi\overline{u'v'}\right)+\frac{\partial}{\partial P}\left(fa\cos\phi\frac{\overline{v'\theta'}}{\overline{\theta_P}}\right)\right] \tag{1}$$

Again we appreciate the reviewer's insightful comment.

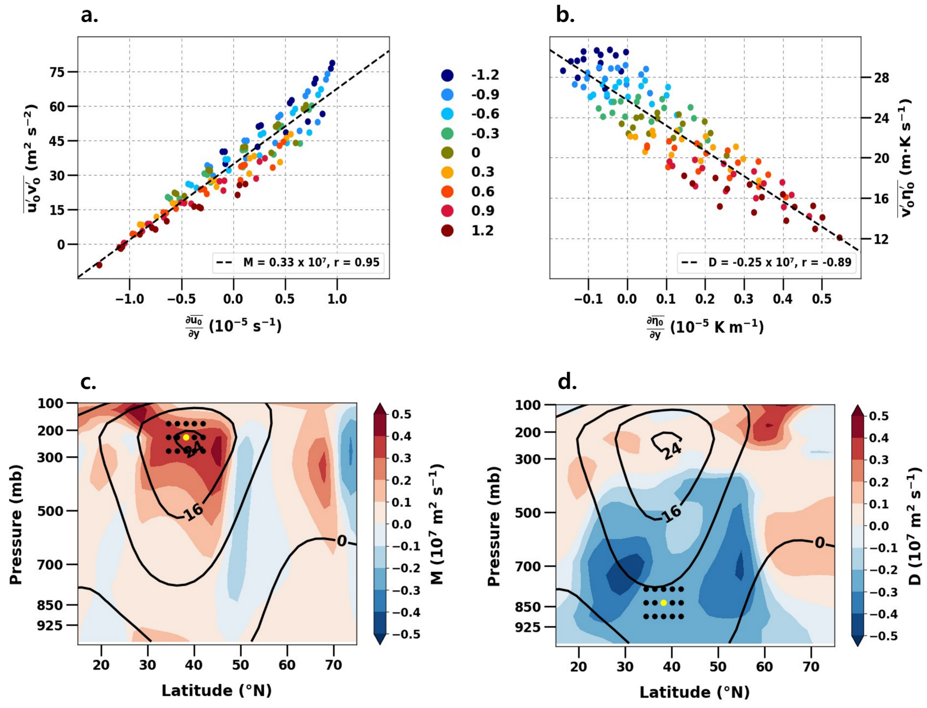

FIG. 1: Based on the control simulation, maps of D (panel c) and M (panel d) are constructed. Eddy momentum (M) and heat (D) diffusivities are calculated through linear regression using the results from the simulation at each location. The diffusivities at each location are determined by the turbulent fluxes and the shear of the zonal mean zonal wind within an area surrounding the location. Examples of these calculations are illustrated in panels (a) and (b) for M and D, respectively. The sampling locations for these examples are marked by yellow dots. The black dots surrounding the yellow dots represent the region for calculating M and D. Simulations with different thermal forcing in high latitudes are distinguished by color. In panels (c) and (d), the contours represent the zonal mean zonal wind, while the color shading indicates the diffusivity of the eddy momentum in (c) and the diffusivity of the eddy heat flux in (d).

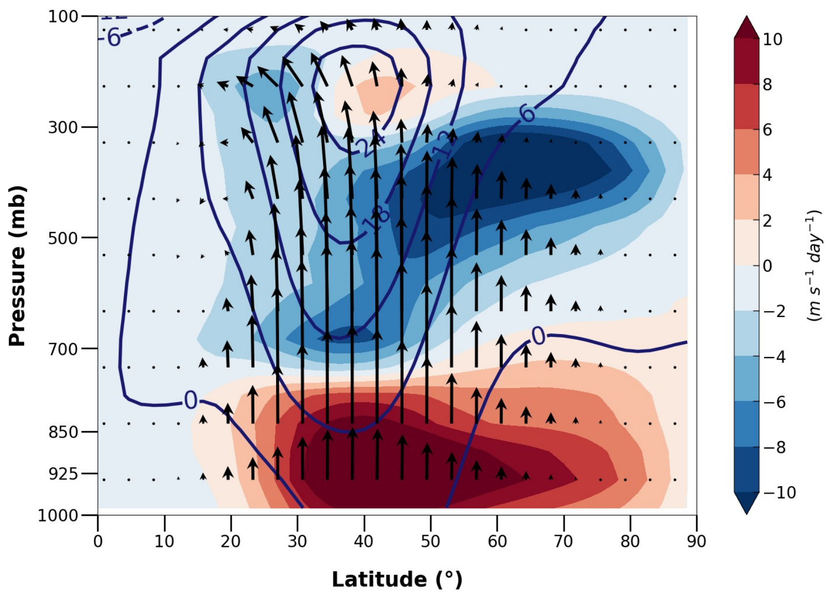

FIG. 2: The average Eliassen–Palm (EP) flux (arrows) and its divergence (shading) are shown. The simulation uses a dry general circulation model (GCM) with the idealized thermal forcing introduced by Held and Suarez (1994), integrated for 5000 days. After discarding the initial 1000 days as spin-up, the remaining 4000 days are used to compute the climatological mean EP flux and its divergence.