Supplement S1: Alternative analytical form of hypergeometric functions for EBM rheology

This supplement describes a simple procedure to be implemented for the hypergeometric function, $\Upsilon =_2 F_1[1,1+\alpha;2+\alpha;z]$; $z=-s\tau_{H,L}$ for the extended Burgers Material model with $0<\alpha<1$ without requiring a function call to software for the general case of the hypergeometric function. The main purpose of avoiding such a function call is to gain some computational efficiency by tailoring the replacement to the detailed requirements of the EBM/GIA ensemble for Bayesian analysis. While past experience suggests that the α-dependency is weak, further testing is required. A method relied upon here is to employ Mathematica software.

For example, Mathematica returns the specific case of $\alpha = \frac{1}{2}$:

$$ {}_2F_1[1,\frac{3}{2};\frac{5}{2};z] \;=\; \frac{3(-z+\sqrt{2}\mathrm{ArcTan}\sqrt{z})}{z^2}. \tag{S11}$$

We remind the reader that all actions on the variable, z, for transcendental equations assume complex computation, and similarly for the hypergeometric function. There are no straightforward simplifications for ${}_2F_1$ offered by Mathematica 13.1, or earlier versions. Furthermore no useful reductions may be derived from Abramowitz and Stegun (1970) in any obvious way. The functions ${}_2F_1[a,b;c;z]$ may be recovered from the following definition

$$ {}_2F_1[a,b;c;z] \;\equiv\; \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{p=0}^{\infty}\frac{\Gamma(a+p)\Gamma(b+p)}{\Gamma(c+p)}\frac{z^p}{p!}. \tag{S12}$$

where Γ represents the Gamma function, p is an integer and the mark ! indicating factorial. However, the functions are also a special solution of the equation:

$$ z(1-z)\frac{d^2\Upsilon}{d^2z} + [c-(a+b+1)z]\frac{d\Upsilon}{dz} - ab\Upsilon \;=\; 0. \tag{S13}$$

This equation is sometimes called Gauss' differential equation. Solutions have three regular points [e.g.,Wylie 1966, p. 653, Mathews and Walker 1970, p. 187]; $z=0,1,\infty$. The solution ${}_2F_1[a,b;c;z]$ behaves as a constant near $z=0$. If this constant is equal to unity, we formally arrive at the hypergeometric function ${}_2F_1$.

We may treat α as a ratio of two integers, m and n where $m<n$, and both n and m are positive. In our case $a=1$, $b=(n+m)/n$ and $c=(2n+m)/n$. There is no advantage to this substitution other than it provides a way to solve special cases of the one point boundary value problem

$$ nz(1-z)\frac{d^2\Upsilon}{d^2z} + [2n+m-(3n+m)z]\frac{d\Upsilon}{dz} - (n+m)\Upsilon \;=\; 0, \tag{S14}$$

with the condition

$$ \Upsilon(0) \;=\; 1. \tag{S15}$$

For example, when $m=1$, $n=4$ ($\alpha=\frac{1}{4}$), then the solution for $\Upsilon(z)$ is

$$ \Upsilon(z) \;=\; -\frac{5}{2z^{\frac{5}{4}}}\cdot\{2z^{\frac{1}{4}} - \mathrm{ArcTan}[z^{\frac{1}{4}}] - \mathrm{ArcTanh}[z^{\frac{1}{4}}]\}. \tag{S16}$$

Possibly a more illustrative example is when $m = 1, n = 3$ ($\alpha = \frac{1}{3}$). A more complicated expression is generated. The solution is

$$\Upsilon(z) = -\frac{2}{9z^{\frac{4}{3}}} \cdot \{(\sqrt{3} - 6i)\pi + 3(6z^{\frac{1}{3}} - 2\sqrt{3}ArcTan[\frac{2z^{\frac{1}{3}}+1}{\sqrt{3}}] + 2ln[z^{\frac{1}{3}} - 1] - ln[z^{\frac{1}{3}} + z^{\frac{2}{3}} + 1])\} \tag{S17}$$

where i is the imaginary unit $\sqrt{-1}$. Clearly, complex arithmetic should be fully implemented when using the simplifications owing to a Gauss' differential equation solution method for generating the special cases that arise in the power law distribution function of EBM theory. Figure S1 shows a comparison of the two methods, one which we term 'exact' is obtained in a function call in Mathematica computing language and the Gauss' differential equation method for the case $\alpha = \frac{1}{3}$. Some of the expressions generated by this method are more complicated than in our two examples, but these will always be more efficient than general call to functions generating ${}_2F_1[1, 1+\alpha; 2+\alpha; -\tau_j s/\tau_M]$. This is especially important due to the necessity of inverting the Laplace transforms numerically in order to generate time-dependent GIA solutions.

While transcendental equations and hypergeometric functions assume complex argument, computation of the imaginary parts reveal they are numerically near zero. EBM cases for $\alpha = \frac{1}{3}$, for example, reveal the following results using the Gauss method;

$$\Upsilon(-10001.0) \ = \ 0.000377533 \ + \ 1.6263 \times 10^{-19} i, \tag{S18}$$

$$\Upsilon(-.0001) \ = \ 0.999943 \ - \ 2.63911 \times 10^{-11} i, \tag{S19}$$

with the default precision in Mathematica on a MacBook Pro with Apple M1 Max chip.

In addition to the cases of α at values $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{2}$, as discussed here, we also give three additional solutions using the Gauss differential equation method. Case $\alpha = \frac{1}{8}$ has the solution

$$\begin{aligned} \Upsilon(z) \quad &= \quad -\frac{9}{z} + \frac{9}{16z^{\frac{9}{8}}}\{\sqrt{2}(\pi + 2\,\mathrm{ArcCoth}[\frac{1+z^{\frac{1}{4}}}{\sqrt{2}z^{\frac{1}{8}}}]) \\ &\quad + 4\,(\mathrm{ArcTan}[z^{\frac{1}{8}}] + \mathrm{ArcTanh}[z^{\frac{1}{8}}]) + 2\sqrt{2}\,\mathrm{ArcTanh}[\frac{\sqrt{2}z^{\frac{1}{8}}}{1+z^{\frac{1}{8}}}]\}. \end{aligned} \tag{S110}$$

Case $\alpha = \frac{2}{3}$ has the solution

$$\begin{aligned} \Upsilon(z) \quad &= \quad -\frac{1}{18z^{\frac{5}{3}}} \cdot \{(\sqrt{3} - 6i)\pi + 9z^{\frac{2}{3}} + \mathrm{ArcTan}[\frac{1+2z^{\frac{1}{3}}}{\sqrt{3}}] \\ &\quad + 3\,(2\ln[-1 + z^{\frac{1}{3}}] - \ln[1 + z^{\frac{1}{3}} + z^{\frac{2}{3}})])\}, \end{aligned} \tag{S111}$$

and case $\alpha = \frac{3}{4}$, the solution;

$$\Upsilon(z) \ = \ -\frac{7}{6z^{\frac{7}{4}}} \cdot \{2z^{\frac{3}{4}} + 3(\mathrm{ArcTan}[z^{\frac{1}{4}}] - \mathrm{ArcTanh}[z^{\frac{1}{4}}])\}. \tag{S112}$$

It is also possible to pursue an examination of the case of α = 0 by taking the limit of the α-dependent part of Equation (19) of Ivins et al [2022]. This leads to a form that contains the sum on the low and high cutoff times (τ_L and τ_H) of $Ei[-t/\tau_j]$

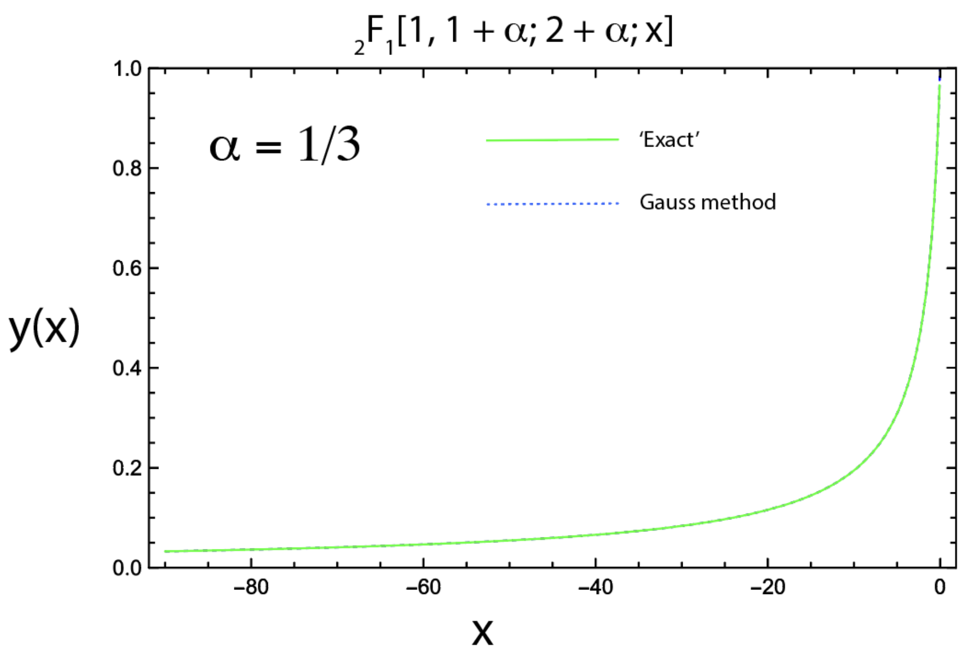

Figure S1. A comparison of the hypergeometric function computation options required of EBM/GIA.

with sum on the two cutoff times, τ_j. To complete this solution for α = 0, then requires convolving the load function with this exponential integral function, $Ei[-t/\tau_j]$, a task we can leave to the reader. It is noted that laboratory torsion experimental values of α are all larger than $\frac{1}{8}$.