Authors' response to reviewers report

Response to Reviewer 1

Comment # 1

I do not trust the basic result that firn results in shallower crevasse penetration that is asserted by the authors. They claim that van der Veen go the opposite result because he only considered the effect of low-density firn on the vertical stress (σ_{zz}) and not on the difference between the horizontal and vertical stress (R_{xx}). It is pretty easy to show that low-density firn has a larger effect on the magnitude of σ_{zz} than on R_{xx}. To do that I considered a simple analytic treatment of the effect of low-density firn on vertical and horizontal stresses. I assume the Weertman (1957) stress state (which the authors call the incompressible state). To make the problem super simple I assume a firn layer of thickness h_L has a density equal to half of the density of ice in the rest of the layer which has a density ρ_i. I start with the standard assumption that the vertical stress just equal to the weight of overburden so that below the firn layer the vertical stress magnitude is:

$$\sigma_{zz} = \rho_\mathrm{i} g(H - z) - \frac{\rho_\mathrm{i} g h_\mathrm{L}}{2} \tag{1}$$

where, as assumed in the paper under review, that H is the ice layer thickness, z is the distance above the base of the layer and g is the acceleration of gravity. It is easy to show that the difference between the vertical and horizontal stress through the layer is:

$$R_{xx} = \frac{\rho_\mathrm{i} g}{2}\left(H - h_\mathrm{L} + \frac{h_\mathrm{L}^2}{H}\right) - \frac{\rho_\mathrm{w} g}{2}\frac{h_\mathrm{w}^2}{H} \tag{2}$$

It makes sense to me that the magnitude of σ_{zz} is reduced by $(\rho_\mathrm{i} g h_\mathrm{L})/2$ while the magnitude of R_{xx} drops by only $(\rho_\mathrm{i} g)/2(h_\mathrm{L} - (h_\mathrm{L}^2)/H)$. For crevasses the open only within the firn the reduction in vertical stress magnitude is even greater. It is easy to calculate the depth of crevasse opening with the simple Nye (1955) assumption of no stress change on crevasse opening. Though that assumption is poor it allows for an analytic solution and in all cases I have seen the LEFM solution scales with the Nye solution. When you use the Nye assumption the crevasses are indeed deeper than for a case with uniform density. This little exercise makes me think there is something wrong in the solution derived in this paper though I have not taken the time to go through their analysis.

We thank the reviewer for their comment. Regarding the derived R_{xx} we would like to point out a minor error (or typo) in the reviewer's derivation. Starting from the same assumption as made by the reviewer, two ice layers with different densities as shown in Fig. 1, and assuming that the vertical stress (without the seawater pressure) is equal to the cryostatic stress, we can write:

$$\sigma_{zz} = \begin{cases} -\frac{\rho_\mathrm{i} g}{2}(H - z) & \text{if} \quad H - h \leq z \leq H \\ -\rho_\mathrm{i} g (H - z) + \frac{\rho_i g h}{2} & \text{if} \quad 0 \leq z \leq H - h \end{cases} \tag{3}$$

with this expression for $0 < z < H - h$ matching that of the reviewer.

The horizontal stresses are provided by:

$$\sigma_{xx} = R_{xx} + \sigma_{zz} \tag{4}$$

where, in order for the ice sheet to be in equilibrium, it is required that:

$$\int_0^H \sigma_{xx}\,\mathrm{d}z = 0 \tag{5}$$

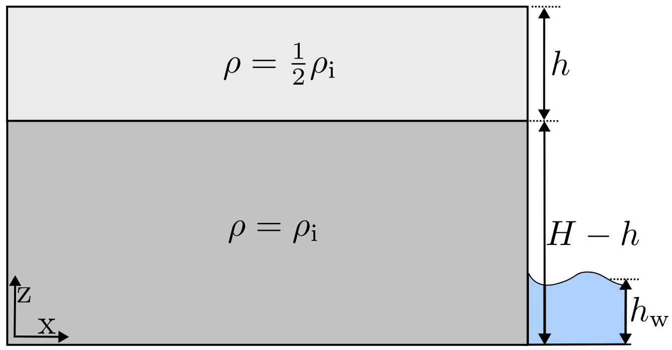


Figure 1: Composition of the ice-sheet considered in the response to comment 1.

Integrating Eq. (4) over the full thickness of the icesheet, and substituting in Eq. (5) then results in the requirement that

$$\int_0^H R_{xx}\,\mathrm{d}z + \int_0^H \sigma_{zz}\,\mathrm{d}z = 0 \tag{6}$$

This directly relates the resistive stress R_{xx} to the vertical stresses as:

$$R_{xx}H = -\int_0^H \sigma_{zz}\,\mathrm{d}z \tag{7}$$

with σ_{zz} given by Eq. (3). Calculating this integral of σ_{zz}, performed both by hand, and using the symbolic solver within MATLAB (see end of this peer review response), gives this integral as:

$$-\int_0^H \sigma_{zz}\,\mathrm{d}z = \frac{1}{2}\rho_\mathrm{i} g H(H-h) + \frac{\rho_\mathrm{i} g}{4}\frac{h^2}{H} \tag{8}$$

Leading to a resistive stress of:

$$R_{xx} = \frac{\rho_\mathrm{i} g}{2}\left(H - h + \frac{h^2}{2H}\right) \tag{9}$$

where we note the addition of a factor 1/2 which was missing from the reviewer's derivation (indicated in red). If we include the seawater pressure acting on the ice sheet, this provides an offset to R_{xx} as:

$$R_{xx} = \frac{\rho_\mathrm{i} g}{2}\left(H - h + \frac{h^2}{2H}\right) - \frac{\rho_w g}{2}\frac{h_w^2}{H} \tag{10}$$

Using this expression for R_{xx}, the horizontal stress (responsible for crevasse propagation) is obtained from Eq. (4). Fig. 2 shows the Nye zero-stress depth, related to the depth to which crevasses are likely to propagate, with the firn layer ($h = 50$ m) and without the firn layer ($h = 0$ m) for two different seawater depths ($h_\mathrm{w} = 0$ and $h_\mathrm{w} = 120$ m). We have checked these stress solutions by re-calculating R_{xx} based on the horizontal and vertical stress states, Fig. 3, which produces a R_{xx} independent of the depth (as is correct), and have compared these results to numerical simulations obtained through COMSOL (which directly takes the density distribution, and simulates the full visco-elastic rheology of the ice until a steady stress profile is reached). Comparing these stress profiles obtained with this simple two-layer model to the results from out paper where a continuous density distribution is used shows that they capture similar trends, e.g. comparing Fig. 8 in the paper (cyan/black and pink/green lines) to Fig. 2, with both showing that using a firn layer at the top reduces the stresses in the top firn layer, while (very slightly) increasing stresses below this layer.

The above exercise emphasizes an important point that addresses the reviewer's skepticism about the correctness of our solutions. From Fig. 2, it is evident that for the no seawater case ($h_\mathrm{w} = 0$, black line),

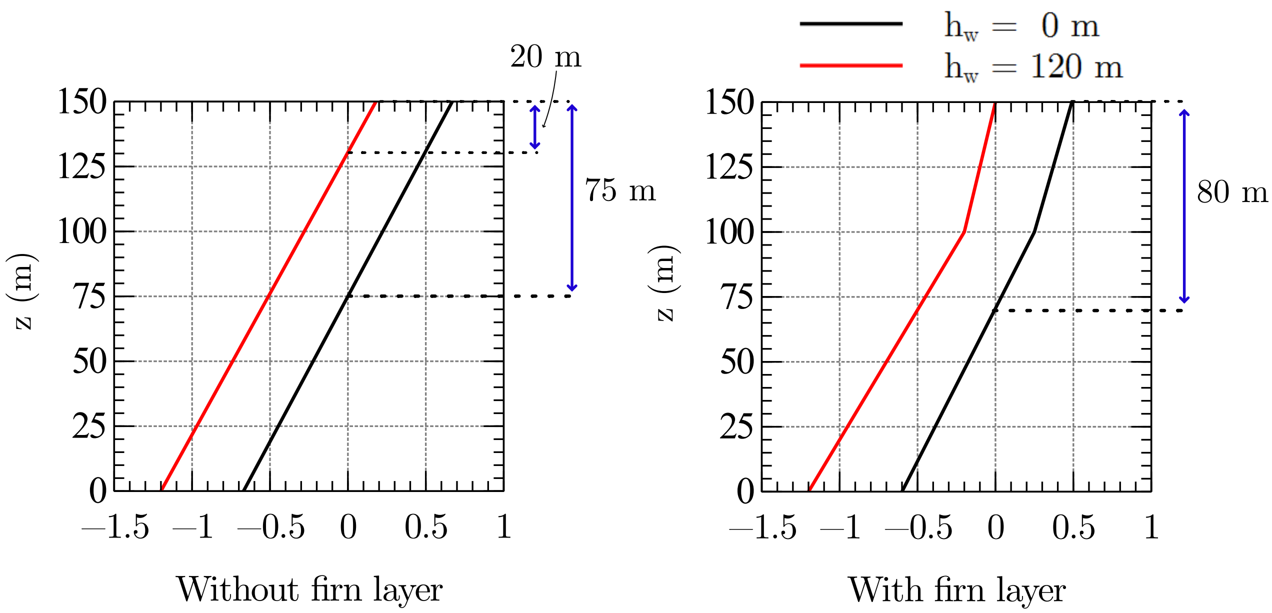


Figure 2: Horizontal stress σ_{xx} within the ice sheet as a function of the depth coordinate z, obtained from the expression from Eqs. (4) and (10) and verified using the finite element software COMSOL. The total ice thickness is $H = 150$ m and for the case with the firn layer thickness $h = 50$ m and ice layer thickness $H - h = 100$ m.

the Nye zero depth is 75 m ($H/2$) without the firn layer, whereas it is 80 m with the firn layer ($H = 150$ m and $h = 50$ m). In this case, the reviewer's argument and van der Veen's original analysis is correct that the firn layer or depth varying density leads to deeper crevasse propagation. The percentage difference in crevasse depths however is small $(80 - 75)/75 \times 100 = 6.7\%$. However, if you consider the seawater case ($h_w = 120$ m, red line) in Fig. 2, the Nye zero depth is 20 m for the 'without firn layer' case and 0 m for the 'with firn layer' case. In this case, the reviewer's and van der Veen's arguments are not correct. The percentage difference in crevasse depths in this case is $(20 - 0)/20 \times 100 = 100\%$. This is the point we were trying to make in the paper, which perhaps was not so clear from the long discussions. This can be summarized with perhaps the following simple statement, which we have added to the paper - "Considering the depth-varying properties of the firn layer in grounded glaciers leads two different regimes of crevasse growth behavior. If the resistive stress R_{xx} is large (e.g. a high oceanwater height), the firn layer promotes crevasse propagation and the crevasse penetrates to a greater depth than in the constant density case. In contrast, if the resistive stress is small, the firn layer has little influence on crevasse propagation. We note that if a Nye-zero stress criterion is used (e.g. for densily spaced cracks) instead of using LEFM to consider a single isolated crevasse, the firn layer hinders crevasse propagation and the crevasse penetrates to a lesser depth than in the constant density case for a low resistive stress. This can be seen in Fig. 8, where the zero-stress depth in the case of a non-linear viscous model is slightly deeper for the case including the firn density."

Comment # 2

In the response to reviewers the discussion of the new section 5 "Non-linear Viscous Incompressible Rheology" is wrong in several fundamental ways. First, the authors claim that their earlier analysis of the problem with a Poisson's ratio of 0.35 is based on a "common assumption if crevasse propagation occurs in a rapid brittle manner..." This completely missed to point, also made by reviewer 1, that the usual assumption is that the background stress is set by flow in the layer long before a crevasse forms. The layer is "commonly" assumed to have relaxed viscously (see Nye(1955), Weertman(1973), van der Veen

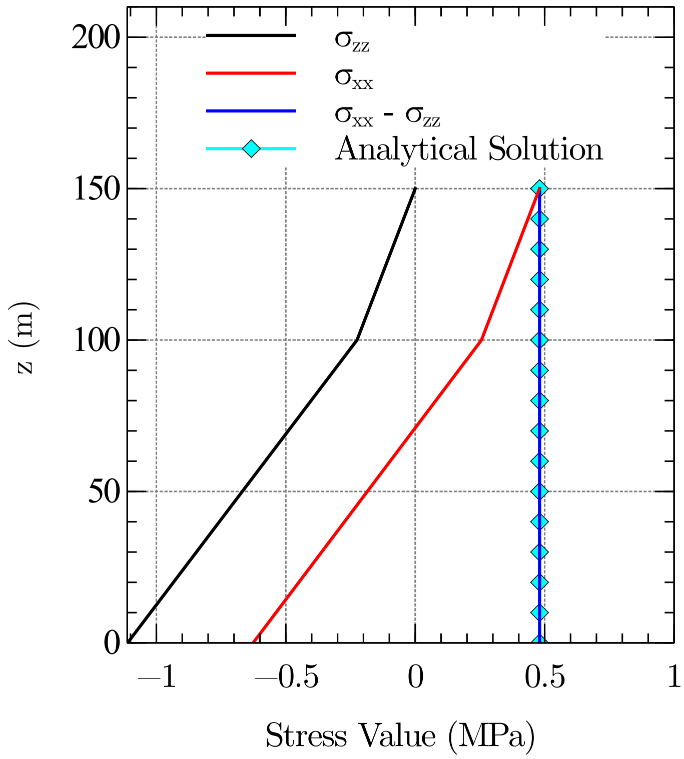


Figure 3: Verification of stress profiles by comparing $R_{xx} = \sigma_{xx} - \sigma_{zz}$ computed from finite element simulations in COMSOL with the analytical solution in Eq. (9).

(1998) and many others). Second, the deformation during crevasse opening is usually assumed to happen so fast that there is no viscous relaxation, but the opening magnitude certainly depends on the elastic constants (see Weertman (1980) for a clear discussing of this point) so the material cannot be treated as being incompressible. Finally, the paper still maintains that the purely elastic stress field for compressible cases are valid for fast crevasse propagation. That only works if an ice has no time to equilibrate through any viscous flow.

The reviewer's comment is more about semantics and mostly concerned with what is conventional and unconventional in glaciology literature. As we know, both linear elastic and nonlinear viscous constitutive models are phenomenological descriptions of material behavior and are always going to be approximations of the real system. We are aware and agree with the reviewer that the usual assumption is that the background stress is set by viscous flow in the layer. In fact, a proper handling of the glacier stress state must consider the theory of viscoelastic self gravitating bodies by Cathles (1975) and use the equilibrium equations based on the perturbation stress tensor (Lipovsky, 2020). However, what we have identified through finite element simulations is that the nearly-incompressible linear elastic model (i.e. Poisson's ratio $\nu \approx 0.5$) gives the same background stress field as the Maxwell viscoelastic model (with compressible linear elastic $\nu \approx 0.35$ and nonlinear viscous relations) and nonlinear viscous Stokes flow. Thus, the stress state is not sensitive to elasticity or viscosity, but rather to the assumption of incompressibility. We show this through a derivation of the longitudinal stress for elasticity in our prior paper (Sun et al., 2021). Therefore, our assumption is consistent with the assumptions of Nye, Weertman, and van der Veen with the background stress state.

Regarding the second comment from the reviewer regarding the opening magnitude depending on the elastic properties in a visco-elastic state, we agree that the short-term opening is indeed mainly a result of the elastic properties, but note that assuming ice to be incompressible does not prohibit an elastic crevasse opening response. However, independent of material model used, none of the methods used throughout our work, or in the works from Nye, Weertman, and van der Veen have any dependence of the crack depth on the opening width. These two phenomena occur on such a different scale, with the crevasse propagating to depths of 10m-100m, whereas opening heights are typically sub-metre for newly opened crevasses. As a result, the crevasse depth calculations do not need to consider the small changes in stress due to the crevasse width as these small changes are overshadowed by the stress changes due to the presence of the crack itself. The reverse is true, however, where the crack width does depend on the depth to which the crevasse propagates, but this width is not discussed anywhere in the submitted manuscript.

Moreover, we have verified our studies with finite element simulations conducted with the phase field model (Clayton et al., 2022) in the current paper, and also with the cohesive zone model in another recently published paper (Gao et al., 2023). The benefit of these computational models is that they do not make any explicit assumptions and consider elastic and viscous processes to occur as dictated by the Maxwell time scales, which will be spatially varying due to the dependence of viscosity on strain rate. Please see our recent work on the role of viscous deformation on turbulent hydrofracture in glaciers (Hageman et al., 2024). In response to the reviewer's concerns we have made some minor text changes in Section 5, that should clarify our methodology to mathematical glaciologists that use the approaches of Nye, Weertman, and van der Veen.

Cathles, L. M. (1975). Viscosity of the Earth's Mantle. Princeton University Press. http://www.jstor.org/stable/j.ctt13x0t47)

Lipovsky, B. P.: Ice shelf rift propagation: stability, three-dimensional effects, and the role of marginal weakening, The Cryosphere, 14, 1673–1683, https://doi.org/10.5194/tc-14-1673-2020, 2020.

Clayton, T., Duddu, R., Siegert, M. & Martínez-Pañeda, E.: A stress-based poro-damage phase field model for hydrofracturing of creeping glaciers and ice shelves. Engineering Fracture Mechanics 272, 108693 (2022).

Gao, Y., Ghosh, G., Jiménez, S. & Duddu, R.: A Finite-Element-Based Cohesive Zone Model of Water-Filled Surface Crevasse Propagation in Floating Ice Tongues. Computing in Science & Engineering 25, 8–16 (2023).

Hageman, T., Mejía, J., Duddu, R. & Martínez-Pañeda, E.: Ice viscosity governs hydraulic fracture that causes rapid drainage of supraglacial lakes. The Cryosphere 18, 3991–4009 (2024).

Response to Reviewer 2

We thank the reviewer for their recommendation to accept the paper as-is.

Derivation of assumed model from reviewer 1

Declaring symbols, add assumptions on H and h

```
syms h H rho_i z g
assume(H>0)
assume(H>h>0)
```

Defining the vertical stresses based on the hydrostatic stress state, with expressions used for the bottom and top parts of the icesheet

```
s_zz = piecewise(0<=z<=H-h, -rho_i*g*(H-z) + rho_i*g*h/2, ...
                 H-h<z<H, -rho_i*g/2*(H-z)    )
```

s_zz =

$$\begin{cases} \frac{g\,h\,\rho_i}{2} - g\,\rho_i\,(H-z) & \text{if } 0 \le z \wedge h+z \le H \\ -\frac{g\,\rho_i\,(H-z)}{2} & \text{if } z < H \wedge H < h+z \end{cases}$$

```
s_zz_nofirn = -rho_i*g*(H-z)
```

s_zz_nofirn = $-g\,\rho_i\,(H-z)$

Integrate these stresses over the full icesheet thickness

```
integral = int(s_zz, 0, H)
```

integral =

$$-\frac{g\,h^2\,\rho_i}{4} - \frac{H\,g\,\rho_i\,(H-h)}{2}$$

```
integral_nofirn = int(s_zz_nofirn, 0, H)
```

integral_nofirn =

$$-\frac{H^2\,g\,\rho_i}{2}$$

and divide by -1/H to obtain the value of R_xx

```
R_xx = -expand(1/H * integral)
```

R_xx =

$$\frac{H\,g\,\rho_i}{2} - \frac{g\,h\,\rho_i}{2} + \frac{g\,h^2\,\rho_i}{4\,H}$$

```
R_xx_nofirn = -expand(1/H * integral_nofirn)
```

R_xx_nofirn =

$$\frac{H\,g\,\rho_i}{2}$$

Substitute back to obtain s_xx

```
s_xx = expand(s_zz + R_xx)
```

s_xx =

$$\begin{cases} g\,\rho_i\,z - \frac{H\,g\,\rho_i}{2} + \frac{g\,h^2\,\rho_i}{4\,H} & \text{if } 0 \le z \wedge h + z \le H \\ \frac{g\,\rho_i\,z}{2} - \frac{g\,h\,\rho_i}{2} + \frac{g\,h^2\,\rho_i}{4\,H} & \text{if } z < H \wedge H < h + z \end{cases}$$

```
s_xx_nofirn = expand(s_zz_nofirn + R_xx_nofirn)
```

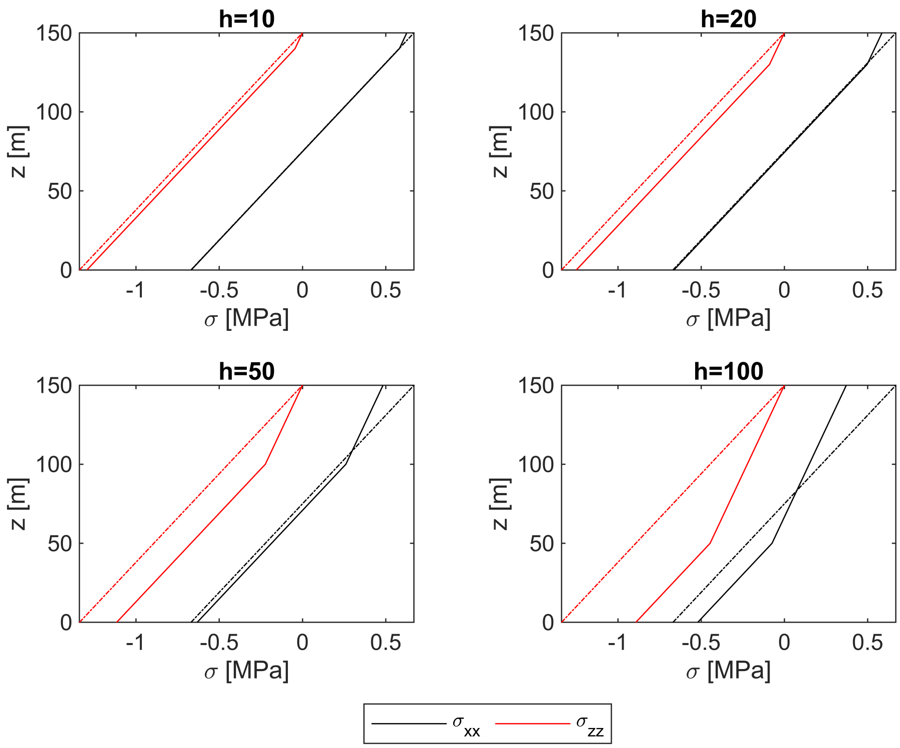
s_xx_nofirn =

$$g\,\rho_i\,z - \frac{H\,g\,\rho_i}{2}$$

Plot the obtained relations for different firn thicknesses,

```
figure
tiledlayout('flow')
for h_num = [10, 20, 50, 100]
    s_xx_num = subs(s_xx, [h, H, rho_i, g],[h_num, 150, 910, 9.81]);
    s_zz_num = subs(s_zz, [h, H, rho_i, g],[h_num, 150, 910, 9.81]);
    s_xx_nofirn = subs(s_xx_nofirn, [h, H, rho_i, g],[50, 150, 910, 9.81]);
    s_zz_nofirn = subs(s_zz_nofirn, [h, H, rho_i, g],[50, 150, 910, 9.81]);

    nexttile
    fplot(s_xx_num*1e-6, [0, 150],'k')
    hold on; view([90 -90]); xlabel('z [m]'); ylabel('\sigma [MPa]')
    fplot(s_zz_num*1e-6, [0, 150],'r')
    fplot(s_xx_nofirn*1e-6, [0, 150],'k-.')
    fplot(s_zz_nofirn*1e-6, [0, 150],'r-.')
    title("h="+string(h_num))
end
leg = legend('\sigma_{xx}','\sigma_{zz}','NumColumns',2);
leg.Layout.Tile = 'south';
```



h=10
h=20
h=50
h=100
z [m]
σ [MPa]
150
100
50
0
-1
-0.5
0
0.5
σxx
σzz