Authors' response to reviewers report

Response to Reviewer 2

The idea that it is worth considering how the properties of firn layers could affect the stresses that control surface crevasse opening is very compelling.

We thank the reviewer for their comments, and hope to address their concerns below. For the concerns raised by using a compressible linear-elastic model, we have added extra sections to the paper showing that an incompressible viscous rheology follows similar trends, which we have attached as appendix to this peer review. Please also see the responses to peer reviewer 1 regarding these additional sections.

Comment # 1

1) However, this analysis makes the radical assumption that the stresses in an ice sheet or shelf are controlled solely by the elastic deformation of compressible ice. This is fine if one simply wants to go through a mathematical exercise, but the title, abstract and body of the paper imply that the results of this analysis applies to real ice sheets and shelves. I was particularly disturbed by the fact that the abstract does not make clear that this is an exercise based on ignoring viscous flow of ice. The fact that the Maxwell time of ice is on the order of days means that an ice sheet or shelf would have to have formed in less than a day for this analysis to be applicable.

The major conclusion of the paper is that inclusion of low-density firn produces opposite effects for idealized ice sheets versus floating ice shelves. The abstract and a cursory reading of the paper makes this seem like a general conclusion. Upon closer reading it is clear that the ice shelf result is only for a particular region close to the edge of the shelf. The authors correctly note that assumption of perfect elasticity results in compression everywhere far from the shelf edge so that no surface crevasses should result for any assumed firn densities or Young's Moduli! This is confusing because the paper only discusses analytic solutions for the stress field far from a shelf edge. To get surface crevassing on a compressible ice shelf with infinite viscosity requires bending stresses close to the edge of the shelf. The authors then use a finite element model to compute those stresses at a fixed position (250 m) from the shelf edge. At that position the predicted crevasse depth is increased by a decreasing firn density and Young's Modulus. I assume that this is a robust result but it is hard to evaluate given the information in this paper. More importantly, the paper makes it seem that this is general result based on the analytical results derived in the paper, as is clear from the opening of the "Conclusions" section:

"In this paper, we derived analytical equations for the far field longitudinal stress including the effects of surface firn layers, described by depth-varying density and Young's modulus profiles based on field data. These analytic expressions were used to perform fracture propagation studies on isolated air/water-filled surface crevasses in grounded glaciers and ice shelves ..."

This certainly gave me the wrong idea when I first read the paper.

...

It is incumbent on these authors to make a case that the assumption of perfect elasticity gives insight into the opening of surface crevasses on real ice sheets and shelves.

The reviewer raises a valid point regarding the use of linear elastic compressibility instead of a non-linear viscous rheology. The assumption of ice behaving as a linear elastic compressive material was taken, due to crevasses developing in a rapid manner below the Maxwell time-scale. The model presented is also capable of capturing this incompressible stress state by setting the Poisson ratio to $\nu \approx 0.5$. This will emulate the slow development of crevasses through ice, during which the ice has sufficient time to attain an incompressible stress state.

We note that if a Poisson's ratio of $\nu = 0.5$ is considered, the far-field stress σ_{xx} in Eq. 1 (giving the longitudinal stresses for a compressible case without any depth-dependent material parameters) obtains the original expression for stress derived by Weertman (1957) assuming a non-linear viscous rheology. Thus when assuming incompressiblity, the far-field stress state is independent of rheology. To address this, a new section has been added, Section 5 in the manuscript, which shows stress profiles assuming a non-linear viscous rheology and a linear elastic incompressible rheology, as well as crevasse penetration depths versus oceanwater depth ratio for a dry crevasse, considering a compressibility and incompressibility. These results show that the analytic expressions presented in our manuscript are valid the viscous stress state.

For a more in-depth discussion regarding this point, and the changes made to the manuscript to include this discussion, see the responses to Reviewer 1, comments #1 and #2.

Comment # 2

2) The other major result of the paper is that low-density firn results in smaller crevasse depths for a grounded glacier compared to a uniform ice case. The authors note that this result contradicts the previous Linear Elastic Fracture Mechanics analysis of van der Veen (1998). I suspect that the difference with the previous study is caused by the assumption of purely elastic horizontal stresses which are less extensional at the ice sheet surface than the stresses assumed by van der Veen (1998). Thus, again I am not convinced that the results of the new analysis apply to the real world.

It is correct that the inclusion of a non-linear viscous rheology results in more extensional stresses in the upper surface of the ice sheet, leading to deeper crevasses in comparison to linear elastic compressible ice. However, the inclusion of the depth-dependent density leads to reductions in both resistive stress R_{xx} and lithostatic stress σ_{zz} for both material rheologies, a point which was neglected by van der Veen (1998) who considered R_{xx} to be independent of depth-varying density, which is a limitation of their work. From van der Veen (1998, page 36): "In the present model, the tensile resistive stress, R_{xx}, is taken constant with depth. A similar assumption is made by Rist et al. (1996) who write the full stress at depth as the sum of an arbitrary tensile surface stress and the lithostatic stress at that depth. In the notation of Eq. 4, their surface stress independent of depth corresponds to R_{xx}. However, if low-density firn is present, the tensile stress in this firn is probably less than that in the fully densified stronger ice at depth. In the limiting case of freshly fallen snow, it is highly unlikely that there is an appreciable surface stress. It would therefore be more appropriate to relate R_{xx} somehow to the firn density assumed a proxy for the firn strength, but also accounting for the generally lower surface temperatures which may increase the strength, with the tensile stress near zero at the surface. The implication of this approach would be that surface crevasses must be initiated below the firn layer, at a depth where the ice can support a tensile stress sufficiently large to initiate fracture."

For a non-linear viscous incompressible rheology ($\nu \approx 0.5$), the resistive stress R_{xx} when considering depth dependent density is given by Eq. (13), substituting $\nu = 0.5$:

$$R_{xx} = \frac{\rho_{\mathrm{i}} g H}{2} - \rho_{\mathrm{s}} g \frac{h_{\mathrm{w}}^2}{2H} - (\rho_{\mathrm{i}} - \rho_{\mathrm{f}}) g D + \frac{(\rho_{\mathrm{i}} - \rho_{\mathrm{f}}) g D^2}{H}(1 - e^{-H/D}) \tag{1}$$

which is invariant with depth z but still dependent on firn density ρ_{f}, unlike what van der Veen assumed. For example, when considering a grounded glacier of height $H = 125$m and oceanwater height of $h_{\mathrm{w}} = 0.5H$, the homogeneous density case gives a value of $R_{xx} = \frac{\rho_{\mathrm{i}} g H}{2} - \rho_{\mathrm{s}} g \frac{h_{\mathrm{w}}^2}{2H} = 404.9$ kPa, whilst the inclusion of firn density reduces the resisitive stress to $R_{xx} = 280.6$ kPa, showing that assuming this resistive stress to be independent of density depth-variations is incorrect.

For crevasses that stabilise in deeper strata, there are minimal reductions in surface crevasse depth when considering a depth-dependent density in comparison to the homogeneous case. However, as the oceanwater height increases, crevasses become shallower, and as a result, the inclusion of firn density becomes more prevalent. For example, the percentage difference in penetration depth for a dry crevasse of oceanwater height $h_{\mathrm{w}} = 0.5H$ reduces to 4%, however increasing the oceanwater height to $h_{\mathrm{w}} = 0.8H$

results in a percentage difference of 64% compared to the homogeneous case.

[1] C. J. van der Veen , 'Fracture mechanics approach to penetration of surface crevasses on glaciers', Cold Regions Science and Technology, vol. 27, pp. 31–47, Oct. 1998, https://doi.org/10.1016/S0165-232X(97)00022-0.

case, the influence of the depth-varying density is reduced when the ice shelf thickness increases, due to a larger proportion of ice being fully consolidated. However, there are still some differences in penetration depth compared to the homogeneous case for thick ice shelves ($H = 1000$ m), with a percentage difference of 14.0% for the dry crevasse, and 19.2% for $h_s/d_s = 0.75$.

Including the effects of both depth-varying density and depth-varying modulus highlights that density is the more prominent property influencing surface crevasse propagation in ice shelves. It is observed in Fig. 7 that the majority of results for depth-varying density and modulus (green lines) overlap the depth-varying density results (red lines). The exception to this is for dry crevasses in thin ice shelves, where the stabilised penetration depth is $0.194H$ compared to $0.25H$ when considering solely depth-varying density.

5 Non-linear Viscous Incompressible Rheology

The above analysis has considered ice to behave as a linear elastic compressible solid, with a Poisson ratio of $\nu = 0.35$. This is a common assumption if crevasse propagation occurs in a rapid brittle manner, such that the cracking occurs on a timescale well below the Maxwell time (typically in the order of hours-days for glacial ice). If the slow development of crevasses is to be considered, with crevasses stabilising over a span of weeks, then ice should be considered as an incompressible solid. This can be achieved by setting the Poisson ratio to $\nu \approx 0.5$ (using $\nu = 0.49$ in our studies to prevent numerical issues). In addition, we conduct a finite element simulation for a grounded glacier, including the viscous contributions of ice flow, modelled through Glen's flow law and extract numerical values of the longitudinal stress. To illustrate the influence of ice rheology, we plot the longitudinal stress profile for a land terminating ($h_w = 0$) grounded glacier, considering linear elastic compressibility ($\nu = 0.35$), linear elastic incompressibility ($\nu \approx 0.5$) and a non-linear viscous rheology in Fig. 8.

Firstly, we note that when ice is considered as linear elastic incompressible ($\nu \approx 0.5$), a stress solution is obtained which matches the steady state creep stress state derived by Weertman (1957) for a depth-independent density, and matches stress profiles obtained through simulations using a visco-elastic rheology. We observe that stresses are more extensional in the upper surface and more compressive at the base when considering incompressibility and that stress is independent of ice rheology (Glen's law creep coefficients). For the homogeneous case, the longitudinal stress varies linearly with depth and is symmetrical about the centre line $z = H/2$. Similarly to the linear elastic compressible case, the inclusion of depth-dependent density results in a reduction in both the lithostatic stress contribution σ_{zz} and the resisitive stress R_{xx}, for both material rheologies, a point which was neglected by van der Veen (1998b) who considered R_{xx} to be independent of depth-varying density.

The longitudinal stress profiles presented in Fig. 8 are used to drive crevasse propagation in the linear elastic fracture mechanics study. Values of crevasse penetration depth for an isolated dry crevasse in a grounded glacier, subject to different values of oceanwater height h_w are presented in Fig. 9. The solid line curves consider incompressible ice, whilst the dashed lines represent compressible ice of Poisson ratio $\nu = 0.35$. Considering ice as an incompressible solid leads to deeper crevasse penetration depths compared to linear elastic compressibility, but these crevasses follow a similar trend as observed for the compressible case: For surface crevasses in glaciers subject to low levels of oceanwater, the penetration depth is unaffected by firn density due to crevasses stabilising in fully consolidated strata. However, as the oceanwater height increases, crevasses

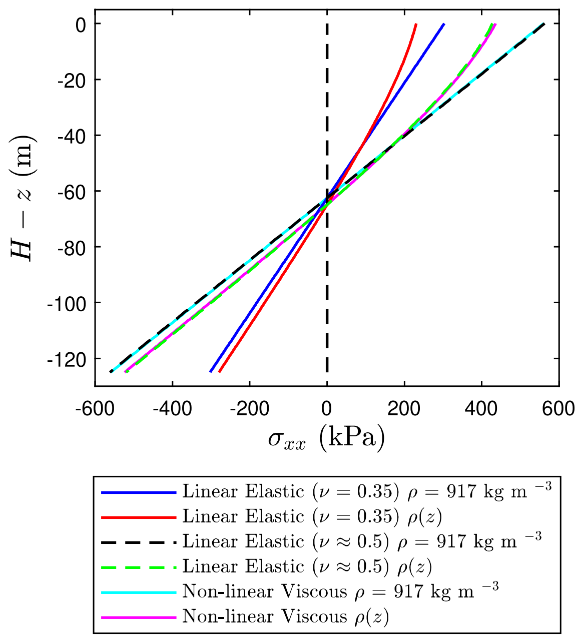

Figure 8. Far field longitudinal stress σ_{xx} throughout the depth of a land terminating glacier ($h_\mathrm{w} = 0$), showing the effects of depth-varying density $\rho(z)$; considering linear elastic compressibility ($\nu = 0.35$), linear elastic incompressibility ($\nu \approx 0.5$) and a non-linear viscous rheology.

become shallower, and as a result, the inclusion of firn density becomes more prevalent. Comparing the effects of assuming an incompressible/viscous rheology, the percentage difference in penetration depth when considering depth-dependent density, for a dry crevasse of oceanwater height $h_\mathrm{w} = 0.5H$ reduces to 4%, compared to 20% for linear elastic compressibility. The ocean-water height required to prevent any development of dry crevasses differs, with values of $h_\mathrm{w} = 0.55H$ being sufficient for compressible depth-dependent density cases whereas oceanwater levels of $h_\mathrm{w} = 0.8H$ are required for the incompressible case. Comparing this to the cases in which no density variations are considered still shows a similar trend, with higher oceanwater needed to stabilise crevasses when density variations are not considered.

Finally, we consider water-filled surface crevasses in floating ice shelves of height $H = 125$m and length $L = 5000$m, using a non-linear viscous ice rheology. Similarly to the linear elastic compressible case, we consider surface crevasses at the horizontal position $x = 4750$m (250 m from the ice shelf terminus) and extract the longitudinal stress profiles from the finite element analysis. We plot the stabilised crevasse depth versus meltwater depth ratio for the non-linear viscous (NLV) rheology in Fig. 10 along with the results for linear elastic (LE) compressibility ($\nu = 0.35$) and incompressibility ($\nu = 0.49$). When comparing the stabilised crevasse depths close to the front, we note that the penetration depth is independent of ice rheology, which is in

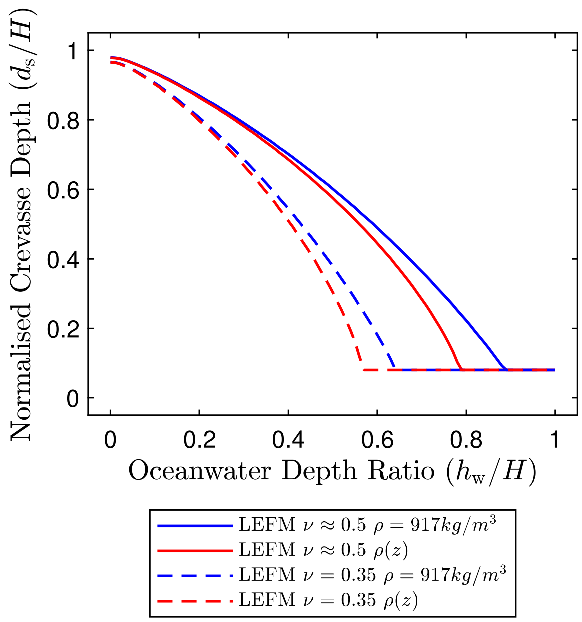

Figure 9. Normalised crevasse depth predictions versus oceanwater height ratio for a single isolated dry crevasse in a grounded glacier, considering compressible ($\nu = 0.35$) and incompressible ($\nu \approx 0.5$) ice homogeneous and depth-dependent mechanical properties

contrast to the grounded glacier case. For the homogeneous density, minimal crevasse propagation is observed for meltwater depth ratios below $h_s/d_s < 0.6$, with full thickness propagation only occurring when fractures are close to saturation. The inclusion of the depth-dependent density results in deeper crevasse penetration depths, with minimal differences in penetration depth between the linear elastic cases and non-linear viscous rheology. This likely indicates that for crevasses close to the front, fracture is driven by the flotation height and the bending stresses due to the floating condition. For depth-dependent density, the reduction in flotation height leads to an increase in tensile stress in the upper surface, due to increases in R_{xx} and increased bending stress. In addition, the lithostatic component of longitudinal stress is reduced, leading to deeper crevasse propagation when including firn density.

We also consider the propagation of an isolated surface crevasse located in the far field region ($x = 2500$ m) of a floating ice shelf, with results presented in Fig. 11. As shown previously, for the linear elastic compressible rheology the stress state is fully compressive for both the homogeneous and the depth-dependent density case, thus no crevasse propagation is observed regardless of meltwater depth ratio. By contrast, when considering the non-linear viscous rheology of ice, surface crevasses may propagate in the far field region if there is sufficient meltwater pressure present. Large increases in crevasse penetration depth are observed for meltwater depth ratios greater than $h_s/d_s = 0.50$, with full thickness propagation being observed close

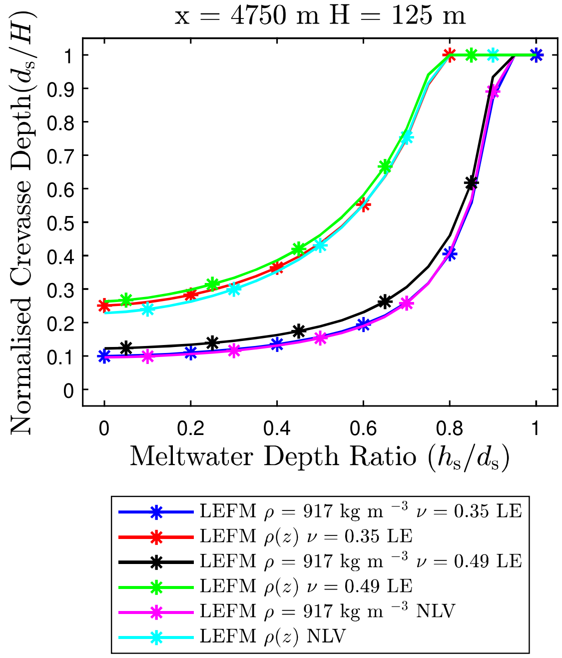

Figure 10. Normalised crevasse depth predictions versus meltwater depth ratio for a single isolated surface crevasse located close to the ice shelf front ($x = 4750$m) considering a linear elastic (LE) and non-linear viscous (NLV) rheology.

to crevasse saturation at $h_s/d_s = 0.95$. Similar to crevasses near the front, the inclusion of depth dependent density results in increased crevasse penetration depths compared to the homogeneous density scenario. Thus, similar conclusions can be drawn for both elastic and viscous rheologies.

6 Discussion

An important finding of this paper is that the inclusion of the depth-varying mechanical properties of unconsolidated ice strata results in a reduction in both the lithostatic compressive stress and the resistive tensile stress components. Contrary to the conventional understanding (van der Veen, 1998a), we find that accounting for depth-varying density and modulus can lead to an overall reduction in surface crevasse depths in grounded glaciers. This is because, in some scenarios, the reduction in resistive stress can hinder crevasse propagation more than the increase in crevasse propagation resulting from the reduction in lithostatic stress. Thus, our study suggests that firn layers can have a stabilizing effect by curtailing surface crevasse growth in grounded glaciers.

Assuming ice to be an linear elastic compressible material, we find that considering depth-varying Young's modulus has a greater influence on crevasse depths than density in thinner glaciers. For example, considering depth-varying density results

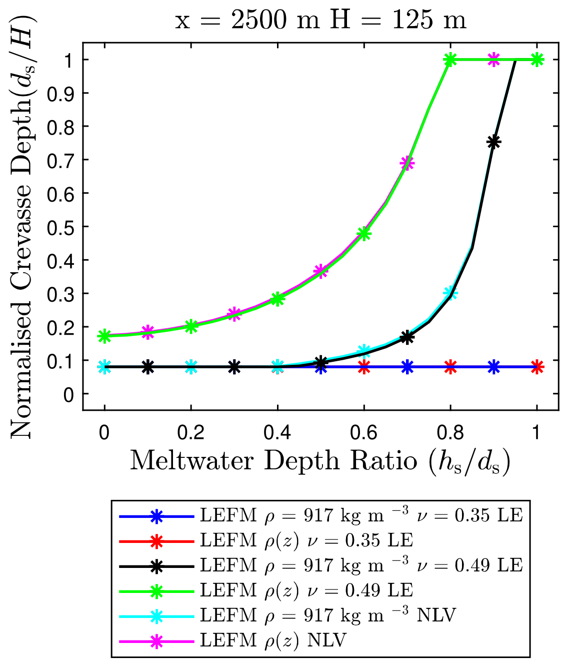

Figure 11. Normalised crevasse depth predictions versus meltwater depth ratio for a single isolated surface crevasse located in the far field region ($x = 2500$m) considering a linear elastic (LE) and non-linear viscous (NLV) rheology.

in a maximum percentage difference of 20% in the penetration depth of dry crevasses, compared to a maximum percentage difference of 45% when considering depth-varying Young's modulus. The largest reductions in crevasse depths are observed in thinner glaciers (depths of approximately 100–150 m), where the stabilizing effects of the firn layers seem to be more prominent. Larger meltwater depth ratios are required to propagate surface crevasses in thinner ice shelves; whereas in thicker glaciers, the influence of firn density is lesser (in some cases negligible), so surface crevasses propagate deeper into the fully consolidated strata. Thus, our study reveals that LEFM models assuming homogeneous ice properties are valid for crevasse depth estimation in thicker glaciers with ice thickness $H > 250$ m.

Accounting for depth-varying density in the floating ice shelf case increases the penetration depth of surface crevasses close to the ice-ocean front, with this increase caused by reductions in buoyancy height and lithostatic compressive stresses. The effect of depth-varying density is dominant in thinner ice shelves, but it can still impact surface crevasse propagation in ice shelves as thick as $H = 1000$ m, although to a lesser extend. For instance, the crevasse depth ratio increases to $d_s = 0.91H$ (188% increase compared to homogeneous case) for thin ice shelves ($H = 125$ m); whereas, a 19% increase is observed for 1 km thick ice ($d_s = 0.45H$). Considering depth-varying Young's modulus in the floating ice shelf case slightly reduces surface crevasse depth for low meltwater depths, and the effect becomes less significant in thicker ice shelves. This study suggests as the

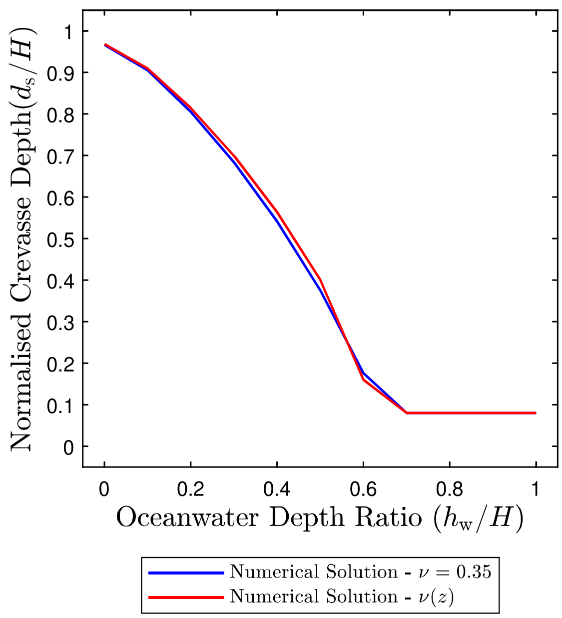

Figure E1. Normalised crevasse depth predictions versus oceanwater height ratio for a single isolated dry crevasse in a linear elastic ice sheet, considering homogeneous and depth-dependent Poisson Ratio.

Appendix E: Influence of Poisson ratio ν

E1 Depth-variable Poisson ratio

For the crevasse propagation studies previously presented, a depth invariant Poisson ratio of $\nu = 0.35$ was assumed. However, it has been suggested that Poisson ratio also exhibits a linear dependency on ice density and therefore leads to a depth-dependent profile (Smith, 1965). Furthermore, Schlegel et al. (2019) and King and Jarvis (2007) provides a depth-dependent Poisson ratio profile based on seismic velocity measurements on ice cores. To study the effect of this depth-dependent Poisson ratio, a linear elastic fracture mechanics study is performed. We assume an exponential distribution of Poisson's ratio with depth, similar to the density and Young's modulus distributions.

$$\nu(z) = \nu_i - (\nu_i - \nu_f)e^{-(H-z)/D} \tag{E1}$$

where $\nu_f = 0.07$ is the Poisson ratio of firn in the upper surface, $\nu_i = 0.35$ is the Poisson ratio of fully consolidated ice and $D = 32.5$ is the tuned constant. This profile approximates the observations from Schlegel et al. (2019), where we have scaled the length parameter D to match our density and Young's modulus profiles as this profile was obtained at a different location (with significantly different ice-sheet and firn thickness). As it is not possible to derive a fully analytic expression for the stress profiles with this depth-dependent Poisson ratio, the longitudinal stress profiles are obtained numerically through the finite element model. Once obtained, the stresses are used to drive the propagation of the surface crevasse in the linear elastic fracture mechanics study.

We consider a dry (air-filled) crevasse, with different values of oceanwater height h_w and plot the normalised crevasse penetration depth versus oceanwater height h_w in Fig. E1. This figure shows that the effect of including variations in Poisson ratio have a more limited effect compared to density and Young's modulus variations. The largest percentage difference in crevasse depth was observed for intermediate ocean-water levels, with an increase of 6% in crevasse depth with respect to the homogeneous case when considering a depth dependent Poisson ratio, for an oceanwater height of $h_\mathrm{w} = 0.5H$. This is in contrast to the inclusion of firn density and Young's modulus, which predict a reduction in stabilised crevasse depth for surface crevasses in grounded glaciers. The effect of a including a depth-dependent Poisson ratio is less influential compared to density and Young's modulus, as depth-dependent density resulted in a reduction of 20% of the crevasse depth and depth-dependent Young's modulus resulted in a reduction of 45% of the crevasse depth. We therefore conclude that the inclusion of variations in Poisson ratio does not play a significant role in crevasse propagation.

Author contributions. T. Clayton: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data Curation, Writing - Original Draft, Visualization. R. Duddu: Conceptualization, Methodology, Writing - Review & Editing. T. Hageman: Conceptualization, Writing - Review & Editing, Supervision. E. Martínez-pañeda: Conceptualization, Resources, Writing - Review & Editing, Supervision, Project administration, Funding acquisition.

Competing interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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