Investigating the celerity of propagation for small perturbations and dispersive sediment aggradation under a supercritical flow

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Supplemental 1

Governing equations and eigenvalue analysis, non-negligible c_s (Morris and Williams' approach)

Morris and Williams (1996) argued that an assumption of negligible solid concentration is not appropriate for many natural streams and, therefore, determined the eigenvalues of a system of equations considering a finite c_s. The continuity and momentum equations of the mixture and the continuity equation for the sediment in Morris and Williams' approach are as follows:

$$\begin{cases} \dfrac{\partial(uh)}{\partial x}+\dfrac{\partial h}{\partial t}+\dfrac{\partial z_b}{\partial t}=0 \\ \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+g\dfrac{\partial h}{\partial x}+\dfrac{(\rho_s-\rho)gh}{2\rho_m}\dfrac{\partial c_s}{\partial x}-\dfrac{[(1-p_0)\rho_s+p_0\rho]u}{\rho_m h}\dfrac{\partial z_b}{\partial t}+g\dfrac{\partial z_b}{\partial x}=-gS_f \\ \dfrac{\partial(uhc_s)}{\partial x}+\dfrac{\partial(hc_s)}{\partial t}+(1-p_0)\dfrac{\partial z_b}{\partial t}=0 \end{cases} \tag{1}$$

where, $c_s=q_s/(q_s+q)$, and $\rho_m=c_s\rho_s+(1-c_s)\rho$ is the density of the mixture. Since the solid concentration is not negligible, the water and sediment discharge per unit width are obtained from the following equations (assuming that the solid particles move with the same velocity of water):

$$q=uh(1-c_s) \tag{2}$$

$$q_s=uhc_s \tag{3}$$

In system 1, one can substitute the terms $\partial c_s/\partial x$ and $\partial c_s/\partial t$ with following equations:

$$\frac{\partial c_s}{\partial x}=\frac{\partial c_s}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial c_s}{\partial h}\frac{\partial h}{\partial x} \tag{4}$$

$$\frac{\partial c_s}{\partial t} = \frac{\partial c_s}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial c_s}{\partial h}\frac{\partial h}{\partial t} \tag{5}$$

and write it as below:

$$\begin{cases} u\dfrac{\partial h}{\partial x} + h\dfrac{\partial u}{\partial x} + \dfrac{\partial h}{\partial t} + \dfrac{\partial z_b}{\partial t} = 0 \\ \dfrac{1}{g}\dfrac{\partial u}{\partial t} + \dfrac{u}{g}\dfrac{\partial u}{\partial x} + \dfrac{\partial h}{\partial x} + Ah\left(\dfrac{\partial c_s}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial c_s}{\partial h}\dfrac{\partial h}{\partial x}\right) - B\dfrac{u}{gh}\dfrac{\partial z_b}{\partial t} + \dfrac{\partial z_b}{\partial x} = -S_f \\ uc_s\dfrac{\partial h}{\partial x} + hc_s\dfrac{\partial u}{\partial x} + uh\left(\dfrac{\partial c_s}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial c_s}{\partial h}\dfrac{\partial h}{\partial x}\right) + c_s\dfrac{\partial h}{\partial t} + h\left(\dfrac{\partial c_s}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial c_s}{\partial h}\dfrac{\partial h}{\partial t}\right) + \\ (1-p_0)\dfrac{\partial z_b}{\partial t} = 0 \end{cases} \tag{6}$$

where, $A = (\rho_s - \rho)/(2\rho_m)$; and $B = \big((1-p_0)\rho_s + p_0\rho\big)/\rho_m$, are dimensionless parameters. In order to find the eigenvalues of this system, it is also needed to compute the differential of u, h, and z_b:

$$\frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx = du \tag{7}$$

$$\frac{\partial h}{\partial t}dt + \frac{\partial h}{\partial x}dx = dh \tag{8}$$

$$\frac{\partial z_b}{\partial t}dt + \frac{\partial z_b}{\partial x}dx = dz_b \tag{9}$$

Now, the eigenvalues of the system can be obtained by equalizing the determinant of the set of system (6) accompanied by equations (7)-(9) to zero (De Vries 1965; Cunge et al. 1980):

$$\begin{vmatrix} h\dfrac{\partial c_s}{\partial u} & h\left(c_s + u\dfrac{\partial c_s}{\partial u}\right) & c_s + h\dfrac{\partial c_s}{\partial h} & u\left(c_s + h\dfrac{\partial c_s}{\partial h}\right) & (1-p_0) & 0 \\ 0 & h & 1 & u & 1 & 0 \\ \dfrac{1}{g} & \dfrac{u}{g} + Ah\dfrac{\partial c_s}{\partial u} & 0 & 1 + Ah\dfrac{\partial c_s}{\partial h} & -\dfrac{Bh}{gh} & 1 \\ dt & dx & 0 & 0 & 0 & 0 \\ 0 & 0 & dt & dx & 0 & 0 \\ 0 & 0 & 0 & 0 & dt & dx \end{vmatrix} = 0 \tag{10}$$

This equation was expanded by Morris and Williams (1996) and the characteristic polynomial equation was obtained as follows:

$$\lambda^3\left\{Bu\frac{\partial c_s}{\partial u}-h\frac{\partial c_s}{\partial h}-[c_s-(1-p_0)]\right\}+\ \lambda^2\Big(\{Agh[c_s-(1-p_0)]-2Bu^2\}\frac{\partial c_s}{\partial u}+(2+B)uh\frac{\partial c_s}{\partial h}+ 2u[c_s-(1-p_0)]\Big)+\lambda\Big[(Bu^3-ugh\{1+A[c_s-(1-p_0)]\})\frac{\partial c_s}{\partial u}-\big((1+B)u^2h-gh^2\{1+ A[c_s-(1-p_0)]\}\big)\frac{\partial c_s}{\partial h}-(u^2-gh)[c_s-(1-p_0)]\Big]+ugh\left(u\frac{\partial c_s}{\partial u}-h\frac{\partial c_s}{\partial h}\right)=0 \tag{11}$$

and in the dimensionless form as below:

$$\acute{\lambda}^3\left\{Bu\frac{\partial c_s}{\partial u}-h\frac{\partial c_s}{\partial h}-[c_s-(1-p_0)]\right\}+\acute{\lambda}^2\Big(\{AFr^{-2}[c_s-(1-p_0)]-2B\}u\frac{\partial c_s}{\partial u}+(2+B)h\frac{\partial c_s}{\partial h}+2[c_s- (1-p_0)]\Big)+\acute{\lambda}\Big[(B-Fr^{-2}\{1+A[c_s-(1-p_0)]\})u\frac{\partial c_s}{\partial u}-(1+B-Fr^{-2}\{1+A[c_s- (1-p_0)]\})h\frac{\partial c_s}{\partial h}+(Fr^{-2}-1)[c_s-(1-p_0)]\Big]+Fr^{-2}\left(u\frac{\partial c_s}{\partial u}-h\frac{\partial c_s}{\partial h}\right)=0 \tag{12}$$

where, $\acute{\lambda}=\ \lambda/u$ = relative celerity. By solving this cubic equation, one can obtain the three celerities of the system exactly.

Reference

Morris, P. H., & Williams, D. J. (1996). Relative celerities of mobile bed flows with finite solids concentrations. *Journal of Hydraulic Engineering*, *122*(6), 311–315. https://doi.org/10.1061/(ASCE)0733-9429(1996)122:6(311)