

## Response to Reviewer 2 (author's comments in blue)

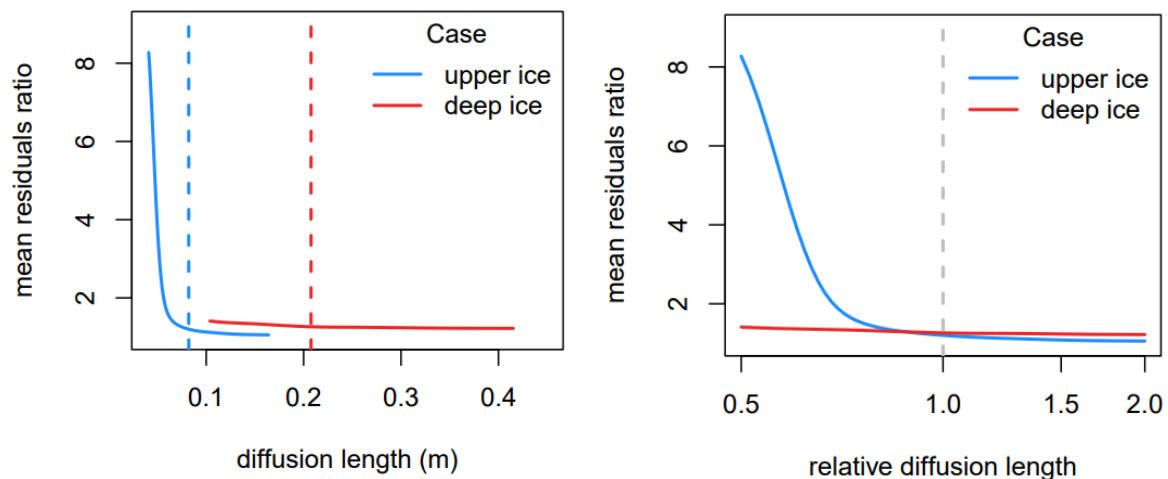
Shaw et al report an analytical method to quantify the upper limit of recoverable frequency from diffused ice core isotopic records after deconvolution, with a particular focus on the effect of measurement noise. The authors suggest that the upper part can be analytically solved where white-noise dominates. Then the authors apply the numerical solutions to the Beyond EPICA Oldest Ice Core, for which climate reconstruction is still ongoing. Overall, the study provides valuable insights into the recovery of water isotope signals in deep ice cores, especially the usefulness of the Beyond EPICA Oldest Ice Core in reconstructing climate before the Mid-Pleistocene Transition. I have a few minor comments on the manuscript.

We thank the reviewer for the positive feedback and the emphasis on the current relevance of our work.

Although the authors provide two surrogate ice core record examples, it would be helpful to generalize how reducing the measurement noise improves the deconvolved depth profile. For instance, only two different diffusion lengths are discussed in the study. How would different diffusion lengths influence the results?

For the white-noise initial PSD ( $P_0$ ) case, the frequency gain could be generalised as it only depends on the measurement noise reduction and the initial signal-to-noise ratio, and is independent of the diffusion length, as shown in Figure 4. For more complex  $P_0$ s, generalising is difficult, due to the dependence on two additional factors (diffusion length and  $P_0(f)$ ). The BE-OIC plots show the result for 9 cases, with each different coloured line representing a different Age with varying diffusion lengths and  $P_0$ s. Figures 6 and 7 offer a more qualitative picture of the potential improvement in the recovery of the isotopic data, with a specific example for both the shallow ice and deep ice cases.

To give a more general picture of the effect of measurement noise on the recovered depth series, we compute the residuals between the true and deconvolved isotopic records for both the high noise (0.1 permil) and low noise (0.01 permil) cases (as in Figures 6c and 7c). We can then find the ratio of the residuals between the two noise cases. Here, for example, a ratio of 2 means the 0.1 permil has double the mean residuals than the 0.01 permil case. Repeating this across different diffusion lengths, with 100 realisations per diffusion length, gives the following figures:



Here, the vertical dashed lines represent the diffusion lengths used in Figure 6 and 7, and the diffusion lengths range from half to double these values. The lower diffusion length values for the upper ice case show a much improved depth profile for the 0.01 permil record. This is expected, as a very small diffusion length (<5 cm) results in a less sharp decrease in power for the higher frequencies, and thus frequencies can be gained at a faster rate when reducing the measurement noise. There is minimal change in the ratio for the deep ice case.

Please check the consistency of the mathematical expressions throughout the manuscript. For instance,  $P_n$  and  $P_n(f)$  are used interchangeably.

The inconsistent notation was corrected. All power spectra terms are now initially written with frequency dependence. When assuming a constant white-noise power spectrum for a given term, the notation without frequency dependence is used for simplicity, and is now explicitly stated when used for the first time.

L86-87: it would be beneficial to provide evidence or justification for these choices?

The lowest frequencies are excluded from the fit due to the known bias in the spectral estimation method (Thomson, D. J. (1982). Spectrum estimation and harmonic analysis. Proceedings of the IEEE, 70(9), 1055–1096. <https://doi.org/10.1109/PROC.1982.12433>). We have added this justification and reference to the text.

For the upper frequency cut-off, the firn diffusion is evident from the power spectrum. It can be justified mathematically by taking the diffusion length and layer thickness in the firn and showing the loss of amplitude at frequencies above  $10 \text{ kyr}^{-1}$ .

At a depth of 30 m we can expect a diffusion length of  $\sigma \approx 0.08 \text{ m}$

For the Dome C site, the layer thickness at 30 m depth is  $42 \text{ m kyr}^{-1}$

Thus the frequency in  $\text{m}^{-1}$  corresponding to  $10 \text{ kyr}^{-1}$  at this depth is:

$$f_{depth} = 10/42 = 0.24m^{-1}$$

Therefore, the remaining power of the  $10 \text{ kyr}^{-1}$  frequency is equal to:

$$P_{remaining} = e^{-(2\pi\sigma f_{depth})^2} = e^{-(2\pi * 0.08 * 0.24)^2} = 0.986$$

Or a power loss of 1.4 %. Any frequencies at a faster timescale than this will have lost even more power to firn diffusion, and are therefore not appropriate for our analogue of the isotopic record before diffusion,  $P_0(f)$ .

We have added the above explanation to Appendix A.

Figure 3b: what is the purple curve?

The purple curve is the Wiener filter, as stated in the figure caption.

The diffusion length stated at L184 appears to differ from that in the caption for Figure 7.

Well spotted, this is a mistake, it has been corrected.

Figures 6 and 7: it is unclear how “the true, undiffused” records were obtained.

The “true, undiffused” records are stochastic depth series. They are generated by first creating white noise, scaling its Fourier transform to match the target power spectrum, and then applying the inverse Fourier transform to obtain the depth-domain signal. In the shallow ice case (Figure 6), the target power spectrum is simply white-noise with variance equal to the white-noise regime in Figure 2 ( $\epsilon$  in Equation 1). For the deep ice case (Figure 7), the power spectrum is rescaled to a power-law, defined by the low frequencies of the Figure 2 fit (given by  $\alpha f^{-\beta}$  in Equation 1). The shallow and deep ice power spectra are smoothed by diffusion lengths of 8 cm and 21 cm respectively, following Equation 5. Finally, after taking the inverse Fourier transforms, white-noise representing measurement noise is added to the resulting depth series. This is done with measurement noises with standard deviations of 0.1‰ and 0.01‰, giving the orange and blue depth series respectively (in both Figures 6 and 7).

The text has been edited to better explain this method.