# Response to reviewer #2

Feb.21, 2025

We thank the reviewer for his careful review and detailed report, especially for his suggestions on what needs to be further clarified to help the readers. We will revise the manuscript correspondingly. For communication purpose, our point-to-point response is in the following.

## Specific response to all comments

**1,** *" I think representing at least a few of the foundational equations here in index notation or providing a reference that makes such a connection would be helpful. For example the discussion surrounding the definition of Lie derivatives in eqs. (4) - (11) is very terse and abstract, and the notation used is quite different from the notation of physicists in, e.g. general relativity. "*

**Reply:** Indeed we should add the indexed version of at least some of the equations. For instance, we can add indexed version to equation (5) and (6). But maybe we will not add indexed version to equation (7) and (8) because the expression will be very lengthy. We will instead cite a textbook for the readers to learn the tensor product. We can write the indexed version for equation (9) when the manifold is an Euclidean space. We can not write the indexed version for Eq. (10) because it would be very lengthy. However we can give some specific example on how to calculate it in the appendix.

**2,** *" This section might benefit from a simple figure showing $\Phi$ effecting "flow down the integral curves" defined in (9). See for example Figures B2 and B3 in Sean Carroll's Spacetime and Geometry. You could just point the reader to such a reference. "*

**Reply:** We can not find a free online resource for this book right now. But we will keep this suggestion in mind. Thanks.

**3,** *" What are $S$, $\Omega$, and $T$ in each example? (in general comments) "*

**Reply:** $\Omega$ refers to the domain, China for example. And $T : \Omega \to \Omega$ is a smooth 1-1 and onto map whose inverse is also smooth. It does not matter whether $\Omega$ is a square or a ball. And $T$ is arbitrary too as long as $T$ is smooth, 1-1 and onto whose inverse is also smooth. $S$ is a physical field on $\Omega$. It could be wind velocity (3 dimensional or 2 dimensional vector field on $\Omega$), temperature (a function on $\Omega$), density field (a function on $\Omega$). We explicitly defined $S$ in example 2.1.1 and 2.1.2. And the governing equation of $S$ is explicitly written in examples 2.2.1 and 2.2.2. Or do you mean something else?

**4,** *" What physical constraint are you applying to constrain $T$? "*

**Reply:** Currently there is no physical constraint for $T$. The only requirement for $T$ is that it is smooth, 1-1, and onto whose inverse is also smooth. The "physically constrained" property is realized via the link between a physical field $S$ and its corresponding tensor field $\theta_S$. In fact, it does not have to be mathematical tensor fields. Basically what we really want is the correct rule of how $T$ induces the change of $S$. In this paper, it is done by first linking $S$ to a mathematical tensor field $\theta_S$. And then use the rule of $T^\#$ defined Eqs.(7) and (8) to induce a change of $\theta_S$. Finally the new $\theta_S$ induces a new $S$. How to define $\theta_S$ is determined by the physics law behind $S$. Thus $T$ can be arbitrary. The physics has no influence on $T$. The physics has no influence on $T^\#$ for $T^\#$, which acts on tensor fields, defined in Eqs.(7) and (8). But the physics do have influence on $T^\#$ for $T^\#$, which acts on $S$, defined in lines 124-127. And the physics has influence on $T^\# S$.

**5,** *" but I'm still not totally sure how $\Phi$ relates to $T^\#$. "*

**Reply:** Mathematically, $T(\cdot)$ is a map. $\Phi_u(t, \cdot)$ is a smooth family of map. For each specific $t$, $\Phi_u(t, \cdot)$ is a smooth, 1-1 and onto map whose inverse is also smooth. We can not directly solve for Eq.(1). Thus we use gradient descent method and obtain a time sequence of vector fields: $u(t, \cdot)$, for $t \in [0, t_0]$, where $t_0 > 0$ depends on when you stop the gradient descent method. $u$ is a family of smooth vector fields. It thus give us $\Phi_u(t, \cdot)$ via Eq.(9) for $t \in [0, t_0]$. Then $\Phi_u(t_0, \cdot)$ is our candidate solution to Eq.(1). This is merely a suboptimal solution to the optimization problem (1). Thank you for pointing out that we are vague here. We will add more explanation to the manuscript in the revised version.

**6,** *" Since the notation is pretty abstract, it would be helpful to make more explicit use of the simple example presented in Figure 1. What is $\Omega$ in this case? ~~Is it simply the xy plane?~~ In general would it be a more abstract state space? Is the field S in this example just the color assigned to each point? Are*

*S1 and S2 the estimate and the observation?* ~~*What is the optimal T that results in a rightward shift of the bright spot? Is it just a vector field that points everywhere rightward?*~~ *"*

**Reply:** With Fig.1, our initial goal is to convince the readers that location error should be directly and explicitly handled. $\Omega$ is the domain, China for example. It is not the space of state variables (the phase space). The value of $S$ in the domain is represented by the color. In Eq. (1), yes typically $S_1$ is the observation and $S_2$ is the background estimate.

**7,** *"Is it simply the xy plane? What is the optimal $T$ that results in a rightward shift of the bright spot? Is it just a vector field that points everywhere rightward? "*

**Reply:** It is a bit complicated here. We require $T$ to be a 1-1 and onto map. If $\Omega$ is the whole xy plane, this domain is infinite. Hence the displacement map can not be solved using the optimization problem (1) or (2). If the domain is a square with boundary, then the $T$ obtained by solving Eq.(2) can not be a uniform rightward shift, as we naturally expected. But again, this is a figure at the beginning of the paper. It is supposed to remind the readers that displacement error is obvious in human's eyes but not explicitly handled in most data assimilation algorithms. And I think we do not need to talk about it in so much detail in the introduction. But we can add some remark to address these questions somewhere after the introduction section.

**8,** *" I don't understand the difference between $T$ and $T^{\#}$. If $T^{\#}S$ mean "the map $T$ applied to the field $S$," what does $T^{\#}$ mean on its own — is this a "sharp" in the sense of the musical isomorphisms between tangent and cotangent bundles on a manifold? The paragraph from lines 43-53 seems to make use of a distinction between $T$ and $T^{\#}$, but I don't understand what that is. "*

**Reply:** $T$ is an arbitrary smooth, 1-1 and onto map whose inverse is also smooth. $T$ is a map from $\Omega$ to $\Omega$. You input a point of $\Omega$ to $T$, it outputs another point in $\Omega$. $T^{\#}$ is a map, induced by $T$, from the space of tensor fields to the space of tensor fields. We see what confuses the reviewer. $T^{\#}$ is first defined in equations (4)-(11), where $T^{\#}$ acts on the space of tensor fields. However, since lines 124-130, $T^{\#}$ directly acts on $S$. When $T^{\#}$ acts on $S$, it does not treat $S$ as a tensor field, since $S$ itself is a physical field. Thus we first link $S$ to a tensor field $\theta_S$. Then $T^{\#}$ acts on $\theta_S$ following the rule given in Eq.(4)-(11). $T^{\#}\theta_S$ is a new tensor field, it corresponds to a new $S$, which is denoted by $T^{\#}S$. Strictly speaking, we are using the same symbol $T^{\#}$ in two different ways. At first $T^{\#}$ is defined to act on tensor fields. Then we defined it again to act on physical fields. But since we always use $S$ to denote a physical field and $\theta_S$ to denote a tensor field, I think it is easy to distinguish them. However we will add more explanation in section 2.1, or maybe use different symbols for two different meanings of $T^\#$ if necessary. In lines 43-53, we are explaining that $T^\# S = S \circ T$ or $S \circ T^{-1}$ as an assumption in many papers. However, this is not good. We propose to use $T^\# S$ defined in lines 124-127. This is the main point of this paper.

**9,** *" What exactly is the operation $T^\# S$? It seems like it is usually either $S$ composed with $T$ or its inverse, but Line 280 seems to imply that this is only an assumption. When is this assumption invalid? When do you compose with $T$ and when with its inverse? "*

**Reply:** $S \circ T$ and $S \circ T^{-1}$ are two ways how a displacement map act on physical fields. But they are not the only two ways. And essentially they are the same thing. Because if you use $S \circ T^{-1}$, define another map $T_1 = T^{-1}$ then you can express $S \circ T^{-1}$ as $S \circ T_1$. Here $T_1$ is still a smooth, 1-1 and onto map whose inverse is still smooth. Indeed, the main point of this paper is that $T$ can act on $S$ in other ways. As described in lines 124-127, first we choose a specific tensor field $\theta_S$ based on the physics law of $S$. Then $T^\# \theta_S$, in the way defined in Eqs.(4)-(11), is a new tensor field. This new tensor field corresponds to the updated $S$, denoted by $S^{\text{new}}$. $T^\# S$ is defined to be $S^{\text{new}}$. $S \circ T$ and $S \circ T^{-1}$ can be invalid for example when $S$ refers to a density-like field (for instance the absolute humidity field), or a velocity field (if you want vorticity conservation). $S \circ T$ and $S \circ T^{-1}$ is valid for example when $S$ refers to the relative humidity. In general, how $T$ acts on $S$, i.e. $T^\# S$, depends on how you choose $\theta_S$. More generally, $T^\# S$ can be defined as long as you know the relation between $S$ and the fundamental state variables. In many cases, $T^\# S$ should be different from $S \circ T$ and $S \circ T^{-1}$.

**10,** *" Ex. 2.1.1 — (16) and (17) are inscrutable to me; could you include more description in the surrounding text? Does the asterisk denote the Hodge star? Some other operation? If it's a Hodge star it should be typeset in the same way as on Line 95 when it is introduced. I'm also not sure why the inverse of $T$ is used. "*

**Reply:** This notation is introduced in line 76. It is not the Hodge star operator. $T^{-1}$ is used here due to the following reason. We denote by $\Omega_A$ and $\Omega_B$ two copies of $\Omega$. $T : \Omega_A \to \Omega_B$ is a location correction map. $\theta_S$ is a differential n-form on $\Omega_A$. Our goal is to construct a differential n-form on $\Omega_B$. $T$ could induce $T^*$, the pull back operator that acts on all differential forms on $\Omega_B$. For any differential k-form $\theta$ on $\Omega_B$, $T^*\theta$ is a differential k-form on $\Omega_A$. Thus we need $T^{-1} : \Omega_B \to \Omega_A$ to pull $\theta$ to $\Omega_B$. If $X$ is a covariant k-tensor on $\Omega_A$, then $T_*$, the push forward operator which is also "defined" in line 76, constructs a covariant k-tensor on $\Omega_B$. Moreover, one always has the paring identity: $\theta(X) = ((T^{-1})^*\theta)(T_*X)$. This is called paring because $\theta$ and $X$ can both be viewed as a function of the other. You may have noticed that $^{-1}$ also appears in Eq.(6) but not in Eq.(5). The reason is that you want $T^\#$ to act on covariant tensors (vector fields for instance, see Eq.5) and contravariant tensors (differential forms, see Eq.6) in a consistent way.

**11,** *" Exs 2.2.1 and 2.2.2: I think you should be really explicit about what the $T$ (or $T^\#$?) ultimately is here. My understanding is that it is an approximation of $\Phi$ — is that correct? "*

**Reply:** Line 134 and Eq.(19) has the explicit formula for $T^\# S$. It is not an approximation of $\Phi$. The $\Phi$ in Eq.(36) refers to the displacement flow. $\Phi(t_0, \cdot)$ is supposed to be the final $T$ that we want. $T$ is a map. And $T^\#$ is the rule of how $T$ induces the change of $\theta_S$ or $S$.

**12,** *"Line 228: If $M$ is the configuration space shouldn't a state vector $S_i$ be an element of the tangent bundle of $M$ rather than of $M$ itself? "*

**Reply:** Maybe it would be more clear if we use the terminology "phase space". This will be fixed in the revision.

**13,** *" Line 315: "But the second part is merely for mathematical reasons." Should be more explicit. "*

**Reply:** Thanks for pointing it out. The second term is the smoothness constraint. It plays the same role as Tikhonov regularization. There could be other choices for this term. You can choose what you like. But you can not choose nothing. Otherwise the existence and uniqueness of solution can not be guaranteed. This will be explained in more detail in the next version.

**14,** *" It would be helpful to have more explanation of what exactly to look for in figures 3-7. For example Line 317 states that "...the difference between the morphed $h_2$ field and the target $h_1$ field is much smaller than between the morphed $\omega_2$ and the target $\omega_1$." This isn't terribly obvious from just looking at the figures — is there a statistical analysis that justifies this statement, or should it be clear just from examining the figure? Why do you suspect that this difference is attributable to the positiveness of the $h$ field? "*

**Reply:** Indeed we did not provide the numerical data that supports our statement that the difference between $\omega_1$ and $\omega_2$ is larger than the difference between $h_1$ and $h_2$. This will be added to the manuscript. I suspect that for the $h$ field it is easier to transport $h_1$ to $h_2$. I do not have rigorous justification on this statement. My motivation is that the $h$ field is a density field thus optimal transport theory guarantees that there is a map $T$ so that $T^{\#}h_1 = h_2$. Optimal transport theory only works for density fields (i.e. it must be positive and it corresponds to a differential n-form). It does not work for other types of physical fields. We have not examined if our algorithm would lead to the same solution as optimal transport theory in this case, or it would lead to another $T$ which transports $h_1$ to $h_2$ exactly. Numerically, we see that $h_2$ converges to $h_1$ much faster than the other fields. This is just our guess. We do not have rigorous justification on this. We will delete this sentence in the next version.

**15,** *" The sentence beginning on Line 328 "The space of ensemble members... to become a curved manifold." Is confusing, I would reword it. "*

**Reply:** We reword it as "Intuitively, the ensemble members are supposed to lie on a curved manifold rather than a Euclidean space." Also in line 330, we change "the Fréchet mean" is changed to "a Fréchet-type of mean".

**16,** *"Technical corrections"*

**Reply:** Thanks a lot for patiently pointing out these mistakes.