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Brief communication: velocities and thinning rates for Halfar's analytical solution to the Shallow Ice Approximation

Constantijn J. Berends[1]

[1]Institute for Marine and Atmospheric research Utrecht, Utrecht University, Utrecht, the Netherlands

Correspondence to: c.j.berends@uu.nl

Abstract. Analytical solutions to approximations of the Stokes equations are invaluable tools for verifying numerical ice-sheet models. Halfar (1981) derived a time-dependent solution to the Shallow Ice Approximation (SIA). Here, I derive the associated ice velocity vector field, and the resulting thinning rates, which may serve as additional checks for numerical ice-sheet models.

1 Introduction

Numerical ice-sheet models are the most commonly used tools to study the evolution of the Greenland and Antarctic ice sheets under anthropogenic climate change (Aschwanden et al., 2021). Assessing the performance of such models consists of two steps: *verification* (i.e. checking if the underlying equations are being solved correctly), and *validation* (i.e. checking if the correct equations are being solved). Ideally, models are verified by comparing to analytical solutions of the underlying equations. However, due to the complexity of the Stokes equations, analytical solutions exist, to the best of my knowledge, only for the two most simplified approximations to the Stokes equations: the Shallow Ice Approximation (SIA; Halfar, 1981; Bueler et al., 2005, 2007), and the Shallow Shelf Approximation (Schoof, 2006). The solutions by Halfar (1981) and Bueler (2005, 2007) describe the ice thickness as a function of time and space, whereas the solution by Schoof (2006) describes only the velocity as a function of space, for a fixed geometry (a slab of ice lying on a flat, inclined plane). These analytical solutions have been invaluable in aiding modellers to identify and remedy issues with their numerical solutions to the Stokes equations.

One component of ice-sheet models that has never been checked against analytical solutions is the vertical ice velocity. Ice-sheet models typically assume ice is incompressible, implying a constant density. Conservation of mass is thereby reduced to stating that the divergence of the ice velocity field must equal zero, i.e.:

$$\nabla \boldsymbol{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (1)$$

Compared to the frightening complexity of the Stokes equations, this equation appears simple enough that most model description papers do not describe the details of its numerical implementation. However, as anyone who has ever constructed an ice-sheet model from scratch will know, such implementations are far from trivial. The boundary conditions at the ice

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base and ice surface are not immediately obvious, particularly in the case of floating ice that is being melted by warm ocean water, so that the base itself moves due to the changing buoyancy of the melting ice, but the ice itself also flows both along and through the base. Moreover, to deal with the time-evolving ice thickness, many models use a 'terrain-following' vertical coordinate transformation to ensure that their first and last grid points in the vertical dimension respectively coincide with the ice surface and the ice base (or, in some models, vice versa; e.g. Lipscomb et al., 2019; Berends et al., 2021). Applying this coordinate transformation to both Eq. 1 and its boundary conditions is a mathematical exercise that is not undertaken lightly. Lastly, defining the extensional strain rates $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}$ at the ice margin presents additional problems, as the ice thickness and velocity there are discontinuous, and the margin generally does not coincide exactly with any grid point.

To aid ice-sheet modellers in verifying their numerical models, I here derive the ice velocity field, including the vertical component, for the time-dependent solution to the SIA by Halfar (1981).

2 Derivation

Before beginning the mathematical derivation, I will define the symbols used hereafter.

Table 1: Symbols, units, and values where applicable.

Symbol	Description	Units	Value
A	Glen's flow law factor	Pa^{-n} yr^{-1}	
g	Gravitational acceleration	m s^{-2}	9.81
H	Ice thickness	m	
H_0	Ice thickness at the divide at $t = 0$	m	
n	Glen's flow law exponent		3
R_0	Ice margin radius at $t = 0$	m	
ρ	Density of ice	kg m^{-3}	910
u	Horizontal ice velocity in x-direction	m yr^{-1}	
v	Horizontal ice velocity in y-direction	m yr^{-1}	
w	Vertical ice velocity	m yr^{-1}	

Halfar's 1981 time-dependent solution to the SIA reads:

$$H = H(x, y, t) = H_0 \left(\frac{t_0+t}{t_0}\right)^{\frac{-2}{5n+3}} \left[1 - \left(\left(\frac{t_0+t}{t_0}\right)^{\frac{-1}{5n+3}} \frac{\sqrt{x^2+y^2}}{R_0}\right)^{\frac{n+1}{n}}\right]^{\frac{n}{2n+1}}, \quad (2)$$

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$$t_0 = \frac{1}{5n+3}\frac{1}{\Gamma}\left(\frac{2n+1}{n+1}\right)^n \frac{R_0^{n+1}}{H_0^{2n+1}}, \tag{3}$$

$$\Gamma = \frac{2}{5}A(\rho g)^n. \tag{4}$$

To simplify the subsequent derivations, we first abbreviate the different exponents appearing in (2):

$$p_1 = \frac{-2}{5n+3}, \tag{5a}$$

$$p_2 = \frac{-1}{5n+3}, \tag{5b}$$

$$p_3 = \frac{n+1}{n}, \tag{5c}$$

$$p_4 = \frac{n}{2n+1}. \tag{5d}$$

We then introduce a few additional substitutions:

$$r = r(x,y) = \sqrt{x^2+y^2}, \tag{6}$$

$$f_1 = f_1(t) = \left(\frac{t_0+t}{t_0}\right)^{\frac{-2}{5n+3}} = \left(\frac{t_0+t}{t_0}\right)^{p_1}, \tag{7}$$

$$f_2 = f_2(t) = \left(\frac{t_0+t}{t_0}\right)^{\frac{-1}{5n+3}} = \left(\frac{t_0+t}{t_0}\right)^{p_2}, \tag{8}$$

$$f_3 = f_3(x,y) = \frac{r}{R_0}, \tag{9}$$

$$G = G(x,y,t) = 1 - \left(\left(\frac{t_0+t}{t_0}\right)^{\frac{-1}{5n+3}} \frac{r}{R_0}\right)^{\frac{n+1}{n}} = 1 - (f_2 f_3)^{p_3}. \tag{10}$$

Substituting (7), (8), (9), and (10) into (2) yields:

$$H = H_0 f_1 G^{p_4}. \tag{11}$$

We can derive the different partial derivatives with respect to x, y, and t of f_1, f_2, f_3, and G:

$$\frac{\partial f_1}{\partial t} = \frac{p_1}{t_0}\left(\frac{t_0+t}{t_0}\right)^{p_1-1}, \tag{12}$$

$$\frac{\partial f_2}{\partial t} = \frac{p_2}{t_0}\left(\frac{t_0+t}{t_0}\right)^{p_2-1}, \tag{13}$$

$$\frac{\partial f_3}{\partial x} = \frac{1}{R_0}\frac{\partial r}{\partial x} = \frac{x}{rR_0}, \tag{14}$$

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$$\frac{\partial G}{\partial x} = -p_3 f_2^{p_3} f_3^{p_3-1} \frac{\partial f_3}{\partial x}, \tag{15}$$

$$\frac{\partial G}{\partial t} = -p_3 f_2^{p_3-1} \frac{\partial f_2}{\partial t} f_3^{p_3}. \tag{16}$$

We can apply the product rule to (11) to find the thinning rate and the surface slope:

$$\frac{\partial H}{\partial t} = H_0 \left(\frac{\partial f_1}{\partial t} G^{p_4} + f_1 p_4 G^{p_4-1} \frac{\partial G}{\partial t} \right), \tag{17}$$

$$\frac{\partial H}{\partial x} = H_0 f_1 p_4 G^{p_4-1} \frac{\partial G}{\partial x}. \tag{18}$$

Here, it is useful to expand (18) by substituting (15) and (14) into it:

$$\frac{\partial H}{\partial x} = H_0 f_1 p_4 G^{p_4-1} \left(-p_3 f_2^{p_3} f_3^{p_3-1} \frac{\partial f_3}{\partial x} \right)$$

$$= -H_0 p_3 p_4 f_1 f_2^{p_3} f_3^{p_3-1} G^{p_4-1} \frac{\partial f_3}{\partial x}$$

$$= -H_0 p_3 p_4 f_1 f_2^{p_3} f_3^{p_3-1} G^{p_4-1} \frac{x}{r R_0}$$

$$= Q x, \tag{19}$$

$$Q = Q(x, y, t) = \frac{-H_0}{r R_0} p_3 p_4 f_1 f_2^{p_3} f_3^{p_3-1} G^{p_4-1}. \tag{20}$$

This expression can be further simplified by collecting all terms that do not depend on x, y:

$$Q_0 = Q_0(t) = \frac{-H_0}{R_0} p_3 p_4 f_1 f_2^{p3}. \tag{21}$$

Substituting (21) into (20) yields:

$$Q = \frac{Q_0}{r} f_3^{p_3-1} G^{p_4-1}. \tag{22}$$

Note that (19) implies that:

$$|\nabla H| = |Q| r. \tag{23}$$

The analytical solution to the SIA for isothermal ice on a flat bed reads:

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$$u = u(x, y, z, t) = \frac{-2A}{n+1}(\rho g)^n |\nabla H|^{n-1} \frac{\partial H}{\partial x}\left(H^{n+1} - (H - z)^{n+1}\right). \tag{24}$$

We can simplify this equation by defining:

$$c = \frac{-2A}{n+1}(\rho g)^n, \tag{25}$$

$$D_n = D_n(x, y, z, t) = H^n - (H - z)^n. \tag{26}$$

Substituting (25), (23), (19), and (26) into (24) yields:

$$u = c|\nabla H|^{n-1} \frac{\partial H}{\partial x} D_{n+1} = c(|Q|r)^{n-1} Q x D_{n+1} = c|Q|^n r^{n-1} D_{n+1} x. \tag{27}$$

We can apply the product rule to (27) to find the extensional strain rate:

$$\frac{\partial u}{\partial x} = c\left[\left(n|Q|^{n-1}\frac{\partial |Q|}{\partial x} r^{n-1} D_{n+1} x\right) + \left(|Q|^n (n-1) r^{n-2} \frac{\partial r}{\partial x} D_{n+1} x\right) + \left(|Q|^n r^{n-1} \frac{\partial D_{n+1}}{\partial x} x\right) + \left(|Q|^n r^{n-1} D_{n+1}\right)\right]. \tag{28}$$

Here, we need the partial derivatives of $|Q|$ and D_n:

$$\frac{\partial |Q|}{\partial x} = Q_0\left[\left(\frac{-1}{r^2}\frac{\partial r}{\partial x} f_3^{p_3-1} G^{p_4-1}\right) + \left(\frac{p_3-1}{r} f_3^{p_3-2}\frac{\partial f_3}{\partial x} G^{p_4-1}\right) + \left(\frac{p_4-1}{r} f_3^{p_3-1} G^{p_4-2}\frac{\partial G}{\partial x}\right)\right] * \mathrm{sign}(Q), \tag{29}$$

$$\frac{\partial D_n}{\partial x} = nH^{n-1}\frac{\partial H}{\partial x} - n(H - z)^{n-1}\frac{\partial H}{\partial x} = n\frac{\partial H}{\partial x} D_{n-1}. \tag{30}$$

Note that, in (28), only D_{n+1} and $\frac{\partial D_{n+1}}{\partial x}$ have a dependency on z. It is therefore useful to rearrange (28) to read:

$$\frac{\partial u}{\partial x} = c\left[\left(n|Q|^{n-1}\frac{\partial |Q|}{\partial x} r^{n-1} x\right) + \left(|Q|^n (n-1) r^{n-2}\frac{\partial r}{\partial x} x\right) + \left(|Q|^n r^{n-1}\right)\right] D_{n+1} + c|Q|^n r^{n-1} x \frac{\partial D_{n+1}}{\partial x}. \tag{31}$$

We abbreviate this by introducing:

$$U_1 = U_1(x, y, t) = c\left[\left(n|Q|^{n-1}\frac{\partial |Q|}{\partial x} r^{n-1} x\right) + \left(|Q|^n (n-1) r^{n-2}\frac{\partial r}{\partial x} x\right) + \left(|Q|^n r^{n-1}\right)\right]. \tag{32}$$

Substituting (32) into (31) yields:

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$$\frac{\partial u}{\partial x} = U_1 D_{n+1} + c|Q|^n r^{n-1} x \frac{\partial D_{n+1}}{\partial x}. \tag{33}$$

Substituting (30) into (33) yields:

$$\frac{\partial u}{\partial x} = U_1 D_{n+1} + c|Q|^n r^{n-1} x(n+1) \frac{\partial H}{\partial x} D_n. \tag{34}$$

We abbreviate this further by introducing:

$$U_2 = U_2(x, y, t) = c|Q|^n r^{n-1} x(n+1) \frac{\partial H}{\partial x}. \tag{35}$$

Substituting (35) into (34) yields:

$$\frac{\partial u}{\partial x} = U_1 D_{n+1} + U_2 D_n. \tag{36}$$

A similar derivation for the y-dimension yields:

$$\frac{\partial v}{\partial y} = V_1 D_{n+1} + V_2 D_n. \tag{37}$$

We can now substitute (36) and (37) into (1) to find an expression for the vertical extensional strain rate $\frac{\partial w}{\partial z}$. Since the Halfar dome assumes non-sliding, non-melting ice on a non-deforming bed, we know that the boundary condition at the ice base is $w(z=0) = 0$, which implies that:

$$w = w(x, y, z) = \int_0^z \frac{\partial w}{\partial z'} dz' = -\int_0^z \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz'. \tag{38}$$

Substituting (36) and (37) into (38) yields:

$$w = -\int_0^z [(U_1 + V_1) D_{n+1} + (U_2 + V_2) D_n] dz'. \tag{39}$$

Since U_1, U_2, V_1, V_2 are functions only of x, y, and t, they can be removed from the integral:

$$w = -(U_1 + V_1) \int_0^z D_{n+1} dz' - (U_2 + V_2) \int_0^z D_n dz'. \tag{40}$$

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Integrating (26) yields:

$$\int_0^z D_n dz' = \int_0^z (H^n - (H-z')^n)dz' = \int_0^z H^n dz' - \int_0^z (H-z')^n dz' = zH^n - \left[\frac{-1}{n+1}(H-z)^{n+1} - \frac{-1}{n+1}H^{n+1}\right]$$

$$= zH^n - \frac{1}{n+1}(H^{n+1} - (H-z)^{n+1}) = zH^n - \frac{D_{n+1}}{n+1}. \quad (41)$$

Substituting (41) into (40) yields:

$$w = -(U_1 + V_1)\left(zH^{n+1} - \frac{D_{n+2}}{n+2}\right) - (U_2 + V_2)\left(zH^n - \frac{D_{n+1}}{n+1}\right). \quad (42)$$

The velocity solution is visualised in Fig. 1, for an ice sheet with an initial margin radius $R_0 = 500$ km, initial thickness at the divide $H_0 = 3000$ m, Glen's flow exponent $n = 3$, and Glen's flow law parameter $A = 10^{-16}$ Pa^{-3} yr^{-1}.

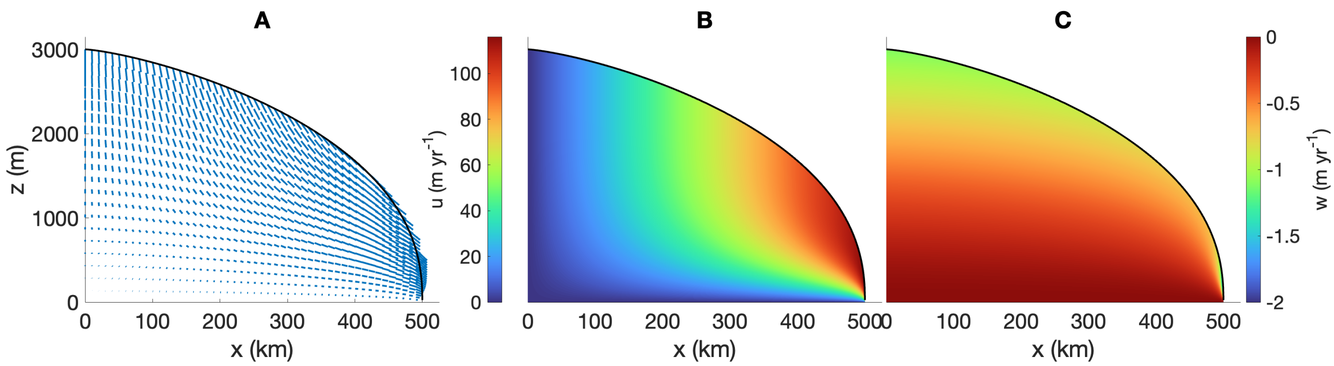

Figure 1: A) A transect of the ice sheet along the positive x-axis, showing the ice velocity field as an arrow plot, B) the horizontal velocity component *u*, C) the vertical velocity component *w*.

Note that the SIA predicts that the surface slope diverges to infinity at the ice margin. While the horizontal velocity components u, v are not affected (as the ice thickness terms in (24) tend to zero, yielding a real limit for u, v), the vertical velocity component w diverges to minus infinity. This suggests that SIA-based ice-sheet models (which are increasingly rare) will always need to include an exception for the ice margin in their calculation of w. Whether this is also the case for other approximations to the Stokes equations is not clear.

Competing interests. I declare that I have no competing interests.

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