Supplementary Information:

When and why microbial-explicit soil organic carbon models can be unstable

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Contents

1 Derivation of kinetic parameters

We derived parameter ranges for multiplicative (m) and reverse Michaelis-Menten (r) kinetics from the parameter ranges given for forward Michaelis-Menten (f) kinetics (Hararuk et al., 2015; Tao et al., 2023).

Ranges of v_m^i ($i = p, u$) (expressed in g mgC^{-1} d^{-1}) were obtained as $\left[\frac{\min(v_f^i)}{\max(K_f^i)}, \frac{\max(v_f^i)}{\min(K_f^i)} \right]$ (where v_f^i is expressed in d^{-1} and K_f^i in mgC g^{-1}.

As concentrations of E^* are about five orders of magnitude lower than those of S^* (Fig. 5 C-D in the main text), ranges of v_r^p and K_r^p were scaled accordingly from ranges of v_f^p and K_f^p to obtain depolymerization rates of similar order of magnitude for both kinetics (Table 3 in the main text).

For density dependent mortality (Georgiou et al., 2017) a re-scaled decay rate of microbial biomass d_B' (expressed in g mgC^{-1} d^{-1}) was used as $d_B' = \frac{d_B}{0.1 \text{mgC g}^{-1}}$, where d_B is expressed in d^{-1} and 0.1 mgC g^{-1} is an estimate for the microbial biomass (Fig. 5 C-D in the main text).

Likewise, for linear uptake kinetics, ranges for v_l^u (in d^{-1}) were scaled from v_m^u as $v_l^u = v_m^u \cdot$ 0.1 mgC g^{-1}.

Parameter ranges of the organic carbon input rate I where obtained from Cotrufo & Lavallee (2022) summing up all OC input rates (excluding microbial biomass inputs) of low productive grasslands (lower bound) and highly productive forests (upper bound). Units were converted to mgC g^{-1} d^{-1} by assuming these inputs where distributed over 1 m depth, with soil bulk density ranging between 1 and 2 g cm^{-3} (upper and lower bound respectively).

2 Additional information on stability analysis

Symbol groups are defined as in the main text as

$$\alpha = d_E + l_E$$

$$\beta = d_B + m_E$$

$$\eta = (y_m - y_B)d_B + y_m m_E > 0$$

$$\omega = \alpha\beta - \alpha y_B r_B d_B - \eta d_E > 0$$

$$\pi = (1 - y_m)m_E + (1 - y_m r_B)d_B > 0$$

(see main text Table 3 for definition of symbols). All parameters and physically meaningful state variables have values ≥ 0.

Importantly, we recall that: $0 < y_B \leq y_m < 1$; $\quad 0 < x_i|_{*,1} \leq 1$; $\quad 0 \leq (f_I, f_D, r_B) \leq 1$.

2.1 SBE model

"Abiotic" equilibrium (Q_0)

With $E_0^* = 0$ for all i: $\frac{\partial P_i}{\partial S}|_{*,0} = 0$ and thus the Jacobian matrix at Q_0 is given by eq. 28 in the main text ($|_{*,0}$ indicates a term taken at Q_0).

The coefficients of the characteristic polynomial of $J_i^{SBE}|_{*,0}$ are found as

$$a_1 = \alpha + \beta + l_S - \underbrace{\left(y_m - y_B\right)}_{\geq 0} \frac{\partial P_i}{\partial E}\big|_{*,0}$$

$$a_2 = \alpha\beta - \eta \frac{\partial P_i}{\partial E}\big|_{*,0} + l_S\left(\alpha + \beta - \underbrace{\left(y_m - y_B\right)}_{\geq 0}\frac{\partial P_i}{\partial E}\big|_{*,0}\right)$$

$$a_3 = l_S\left(\alpha\beta - \eta\frac{\partial P_i}{\partial E}\big|_{*,0}\right)$$

and additionally

$$a_1 a_2 - a_3 = \left(\alpha + \beta - (y_m - y_B)\frac{\partial P_i}{\partial E}\big|_{*,0}\right)\left(\alpha\beta - \eta\frac{\partial P_i}{\partial E}\big|_{*,0} + l_S\left(\alpha + \beta - (y_m - y_B)\frac{\partial P_i}{\partial E}\big|_{*,0}\right) + l^2\right)$$

For $a_i > 0$ and $a_1 a_2 - a_3 > 0$, it is required that $\alpha + \beta > (y_m - y_B)\frac{\partial P_i}{\partial E}\big|_{*,0}$ and $\alpha\beta > \eta\frac{\partial P_i}{\partial E}\big|_{*,0}$. We can merge these two constraints on $\frac{\partial P_i}{\partial E}\big|_{*,0}$ by finding the more restrictive one – i.e. which is smaller, $\frac{\alpha+\beta}{y_m - y_B}$ or $\alpha\frac{\beta}{\eta}$? With the definitions of β and η we obtain

$$\frac{\beta}{\eta} = \frac{d_B + m_E}{d_B(y_m - y_B) + m_E\, y_m} = \frac{d_B + m_E}{(y_m - y_B)(d_B + m_E) + m_E y_B}$$

and with $m_E y_B \geq 0$ we find that $\frac{\beta}{\eta} \leq \frac{1}{y_m - y_B}$. Thus, $\alpha\frac{\beta}{\eta} \leq \frac{\alpha}{y_m - y_B}$ and since $\frac{\beta}{y_m - y_B} \geq 0$ we find that $\frac{\alpha+\beta}{y_m - y_B} \geq \alpha\frac{\beta}{\eta}$. Thus, the second condition $\alpha\beta > \eta\frac{\partial P_i}{\partial E}\big|_{*,0}$ is at least as strict as $\alpha + \beta > (y_m - y_B)\frac{\partial P_i}{\partial E}\big|_{*,0}$ and thus

$$\frac{\partial P_i}{\partial E}\big|_{*,0} < \frac{\alpha\beta}{\eta}$$

is a sufficient and necessary condition to ensure stability of Q_0.

"Biotic" equilibrium $(Q_{1,i})$
The Jacobian matrix $J_i^{SBE}\big|_{*,1}$ is given by eq. 35 in the main text. The coefficients of the characteristic polynomial are given by:

$$a_1 = \alpha\left(\underbrace{1 - x_i|_{*,1}}_{\geq 0}\right) + \beta + \frac{\partial P_i}{\partial S}\big|_{*,1} + l_S + \alpha x_i|_{*,1}\frac{m_E\, y_B}{\eta} > 0$$

$$a_2 = \alpha\beta\left(\overbrace{1 - x_i|_{*,1}}^{\geq 0}\right) + d_E\frac{\partial P_i}{\partial S}\big|_{*,1}\left(\overbrace{1 - (y_m - y_B)}^{>0}\right)$$

$$+ \left(\overbrace{1 - x_i|_{*,1} + \frac{m_E y_B}{\eta}x_i|_{*,1}}^{\geq 0}\right)\left[d_E l_S + l_E\left(l_S + \frac{\partial P_i}{\partial S}\big|_{*,1}\right)\right]$$

$$+ l_E\frac{\beta}{\eta}x_i|_{*,1}\left(\overbrace{y_m - y_B}^{\geq 0}\right)\frac{\partial P_i}{\partial S}\big|_{*,1} + d_B\frac{\partial P_i}{\partial S}\big|_{*,1}\left(\overbrace{1 - r_B y_B}^{>0}\right) + d_B l_S$$

$$+ m_E\left(l_S + \frac{\partial P_i}{\partial S}\big|_{*,1}\right) > 0$$

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$$a_3 = \frac{1}{\eta}\Bigg(\Big(\overbrace{y_m - y_B}^{\geq 0}\Big)\Big[d_B^2\, d_E\, \tfrac{\partial P_i}{\partial S}\big|_{*,1}\Big(\overbrace{1 - y_m + y_B(1 - r_B)}^{>0}\Big) + d_B^2\, \tfrac{\partial P_i}{\partial S}\big|_{*,1}\, l_E\Big(\overbrace{1 - r_B y_B}^{>0}\Big)\Big] +$$

$$\Big(\overbrace{1 - y_m}^{>0}\Big)d_E\, \tfrac{\partial P_i}{\partial S}\big|_{*,1}\, m_E^2\, y_m + \Big(\overbrace{1 - x_i|_{*,1}}^{\geq 0}\Big)\Big[2d_B^2\, d_E\, l_S\,(y_m - y_B) + \tfrac{\partial P_i}{\partial S}\big|_{*,1}\, l_E\, m_E^2\, y_m +$$

$$d_E\, l_S\, m_E^2\, y_m + l_E\, l_S\, m_E^2\, y_m + d_B\, d_E\, l_S\, m_E\, y_m + d_B\, d_E\, l_S\, m_E\,(y_m - y_B) + d_B\, l_E\, l_S\, m_E\, y_m +$$

$$d_B\, l_E\, l_S\, m_E\,(y_m - y_B)\Big] + d_B\, d_E\, \tfrac{\partial P_i}{\partial S}\big|_{*,1}\, m_E\Big[\Big(\overbrace{2y_m - y_B}^{>0}\Big)(1 - y_m) + y_B y_m\Big(\overbrace{1 - r_B}^{\geq 0}\Big)\Big] +$$

$$d_B\, \tfrac{\partial P_i}{\partial S}\big|_{*,1}\, l_E\, m_E\,[y_m(1 - r_B\, y_B) + (y_m - y_B)] + \tfrac{\partial P_i}{\partial S}\big|_{*,1}\, l_E\, m_E^2\, x_i|_{*,1}\, y_m\Bigg) > 0$$

Additionally, $a_1 a_2 - a_3$ is given by

$$a_1 a_2 - a_3$$
$$= \Big(\tfrac{\partial P_i}{\partial S}\big|_{*,1}\Big)^2\Big(\overbrace{\alpha - d_E(y_m - y_B)}^{>0} + \overbrace{\beta - d_B\, r_B\, y_B}^{>0}\Big)$$

$$+ \tfrac{\partial P_i}{\partial S}\big|_{*,1}\Bigg[\alpha\, d_E\overbrace{\Big(1 - (y_m - y_B) + x_i|_{*,1}\Big\{(y_m - y_B) - (y_m - y_B)\tfrac{m_E\, y_B}{\eta} + \tfrac{m_E\, y_B}{\eta} - 1\Big\}\Big)}^{\geq 0}$$

$$+ \alpha(l_E + 2\beta + l_S)\overbrace{(1 - x_i|_{*,1})}^{\geq 0} + \beta\, d_B\overbrace{(1 - r_{DB}\, y_B)}^{>0} + d_E\, l_S\overbrace{(1 - (y_m - y_B))}^{>0}$$

$$+ d_B\, l_S\overbrace{(2 - r_{DB}\, y_B)}^{>0} + x_i|_{*,1}\, \alpha\, d_B\, r_{DB}\, y_B\overbrace{\Big(1 - \tfrac{m_E y_B}{\eta}\Big)}^{>0} + \beta\, m_E + 2\, m_E\, l_S + d_E\, m_E\, y_B + l_E\, l_S$$

$$+ \tfrac{m_E\, y_B}{\eta}\, x_i|_{*,1}\alpha(\beta + l_S + l_E)\Bigg] + \alpha(\beta + l_S)^2(1 - x_i|_{*,1})$$

$$+ \alpha^2\beta\overbrace{\Big(1 + x_i|_{*,1}\Big[x_i|_{*,1} - 2 + m_E\tfrac{y_B}{\eta}(1 - x_i|_{*,1})\Big]\Big)}^{\geq 0}$$

$$+ \alpha^2 l_S\overbrace{\Big(1 + x_i|_{*,1}\Big[x_i|_{*,1} - 2 + m_E\tfrac{y_B}{\eta}\Big(2 - 2x_i|_{*,1} + x_i|_{*,1}\, m_E\tfrac{y_B}{\eta}\Big)\Big]\Big)}^{\geq 0} + \beta\, l_S^2 + \beta^2\, l_S$$

$$+ \tfrac{m_E y_B}{\eta}\, x_i|_{*,1}\alpha\,(l_S^2 + 2\,\beta\, l_S) > 0$$

Thus, the "biotic" equilibrium of the SBE model is without any conditions always stable for any physically meaningful equilibrium point.

Exclusive stability of "abiotic" and "biotic" equilibria in the SBE model

A physically meaningful (i.e. all state variables have positive and real valued steady-states) "biotic" equilibrium only exists if $S_i^* < S_0^*$ (compare with Table 5 in the main text). In turn, for the "abiotic" equilibrium to be stable, it is required that $\tfrac{\partial P_i}{\partial E}\big|_{*,0} < \tfrac{\alpha\beta}{\eta}$.

For multiplicative (m) kinetics:

$$\frac{\partial P_m}{\partial E}\Big|_{*,0} = v_m^p S_0^* < \frac{\alpha\beta}{\eta} \quad \rightarrow \quad S_0^* < \frac{\alpha\beta}{v_m^p \eta} = S_m^*$$

For forward Michaelis-Menten (f) kinetics

$$\frac{\partial P_f}{\partial E}\Big|_{*,0} = v_f^p \frac{S_0^*}{K_f^p + S_0^*} < \frac{\alpha\beta}{\eta} \quad \rightarrow \quad S_0^* < K_f^p \frac{\frac{\alpha\beta}{v_f^p \eta}}{1 - \frac{\alpha\beta}{v_f^p \eta}} = S_f^*$$

For reverse Michaelis-Menten (r) kinetics

$$\frac{\partial P_r}{\partial E}\Big|_{*,0} = \frac{v_r^p}{K_r^p} S_0^* < \frac{\alpha\beta}{\eta} \quad \rightarrow \quad S_0^* < K_r^p \frac{\alpha\beta}{v_r^p \eta}$$

Positivity of the "biotic" equilibrium requires $S_r^* < S_0^*$:

$$S_r^* = \frac{\alpha\beta}{v_r^p \eta} \frac{K_r^p \omega + I\eta}{\omega + l_S \frac{\alpha\beta}{v_r^p}} < \frac{I}{l_S}$$

After algebraic rearrangement this yields:

$$\frac{\alpha\beta}{v_r^p \eta} K_r^p < \frac{I}{l_S} = S_0^*$$

which contradicts with the stability criteria $S_0^* < K_r^p \frac{\alpha\beta}{v_r^p \eta}$ of the "abiotic" equilibrium.

For Equilibrium Chemistry Approximation (e) kinetics:

$$\frac{\partial P_e}{\partial E}\Big|_{*,0} = v_e^p \frac{S_0^*}{K_e^p + S_0^*} < \frac{\alpha\beta}{\eta} \quad \rightarrow \quad S_0^* < K_e^p \frac{\frac{\alpha\beta}{v_e^p \eta}}{1 - \frac{\alpha\beta}{v_e^p \eta}}$$

Positivity of the "biotic" equilibrium requires $S_e^* < S_0^*$:

$$S_e^* = \frac{\alpha\beta}{v_e^p \eta} \frac{K_e^p \omega + I\eta}{\omega - \omega\frac{\alpha\beta}{v_e^p \eta} + l_S \frac{\alpha\beta}{v_e^p}} < \frac{I}{l_S}$$

After algebraic rearrangement this yields:

$$\frac{\frac{\alpha\beta}{v_e^p \eta}}{1 - \frac{\alpha\beta}{v_e^p \eta}} K_e^p < \frac{I}{l_S} = S_0^*$$

which contradicts the requirement $S_0^* < K_e^p \frac{\frac{\alpha\beta}{v_e^p \eta}}{1 - \frac{\alpha\beta}{v_e^p \eta}}$ for stability of the "abiotic" equilibrium.

Thus, for the "abiotic" equilibrium to be stable, it must be that $S_0^* < S_i^*$ and hence the "abiotic" equilibrium is stable only when the "biotic" equilibrium has (physically non-meaningful) negative microbial biomass and extracellular enzyme concentrations.

2.2 SDB model

The Jacobian matrix around the "biotic" equilibrium for the SDB model is given by eq. 19 in the main text. From this, the coefficients of the characteristic polynomial are given by the following equations. Note that for conciseness these are formulated in a mixed form using partial differential equations of rates (P_i^{qss} and U_j) and of ODEs (i.e. $\dot{B} = \frac{dB}{dt}$).

$$a_1 = \frac{\partial P_i^{qss}}{\partial S}\big|_* + \frac{\partial U_j}{\partial D}\big|_* + l_D - \overbrace{\frac{\partial \dot{B}}{\partial B}\big|_{*,j}}^{\geq 0} > 0$$

$$a_2 = l_D \left(\frac{\partial P_i^{qss}}{\partial S}\big|_* - \frac{\partial \dot{B}}{\partial B}\big|_{*,j} \right) - \frac{\partial P_i^{qss}}{\partial S}\big|_* \frac{\partial \dot{B}}{\partial B}\big|_{*,j}$$
$$+ \frac{\partial U_j}{\partial D}\big|_* \left(\pi + d_B\, f_D\, r_B\, y_m + \frac{\partial P_i^{qss}}{\partial S}\big|_* - y_m \frac{\partial P_i^{qss}}{\partial B}\big|_* \right)$$

$$a_3 = \frac{\partial P_i^{qss}}{\partial S}\big|_* \left(-l_D \frac{\partial \dot{B}}{\partial B}\big|_{*,j} + \pi \frac{\partial U_j}{\partial D}\big|_* \right) > 0$$

and $a_1 a_2 - a_3$ is given by

$$a_1 a_2 - a_3 = X_{i \times j} + Y_{i \times j}$$

with

$$X_{i \times j} = \left(\frac{\partial U_j}{\partial D}\big|_* + \frac{\partial P_i^{qss}}{\partial S}\big|_* + l_D \right) \left(l_D \frac{\partial P_i^{qss}}{\partial S}\big|_* + \frac{\partial U_j}{\partial D}\big|_* \left[d_B\, f_D\, r_B\, y_m + \frac{\partial P_i^{qss}}{\partial S}\big|_* - y_m \frac{\partial P_i^{qss}}{\partial B}\big|_* \right] \right)$$
$$+ \pi \frac{\partial U_j}{\partial D}\big|_* \left(\frac{\partial U_j}{\partial D}\big|_* + l_D \right)$$

and

$$Y_{i \times j} = -\frac{\partial \dot{B}}{\partial B}\big|_{*,j} \left(\left[\frac{\partial P_i^{qss}}{\partial S}\big|_* + l_D \right] \left[\frac{\partial P_i^{qss}}{\partial S}\big|_* + l_D - \frac{\partial \dot{B}}{\partial B}\big|_{*,j} \right] \right.$$
$$\left. + \frac{\partial U_j}{\partial D}\big|_* \left[2 \cdot \frac{\partial P_i^{qss}}{\partial S}\big|_* + d_B\, f_D\, r_B\, y_m - y_m \frac{\partial P_i^{qss}}{\partial B}\big|_* + \pi + l_D \right] \right) .$$

For $l_D = 0$ this is equivalent to eq. 39-41 in the main text.

$\frac{\partial \dot{B}}{\partial B}\big|_{*,j}$ is given by

$$\frac{\partial \dot{B}}{\partial B}\big|_{*,j} = y_m \frac{\partial U_j}{\partial B}\big|_* - (d_B + m_E) ,$$

and as detailed in the main text $\frac{\partial \dot{B}}{\partial B}|_{*,(m,f)} = 0$, yielding $Y_{i\times(m,f)} = 0$ for $U_{(m,f)}$. Positivity of $a_1 a_2 -$ a_3 is thus only dependent on $X_{i\times j}$, which is always positive for $d_B f_D r_B y_m + \frac{\partial P_i^{qss}}{\partial S}|_* \geq y_m \frac{\partial P_i^{qss}}{\partial B}|_*$; i.e. the sufficient condition given by eq. 43 in the main text.

For any U_j independent of B (such as $U_l = v_l^u D$) $\frac{\partial \dot{B}}{\partial B}|_{*,j} = -(d_B + m_E)$ and thus, $Y_{i\times j}$ helps to ensure positivity of $a_1 a_2 - a_3$ in these cases.

With the above definitions of $\frac{\partial \dot{B}}{\partial B}|_{*,j}$: $a_1 > 0$ and $a_3 > 0$ in all of these cases and $a_2 > 0$ and $a_1 a_2 - a_3 > 0$ if the sufficient condition in eq. 43 in the main text is fulfilled.

2.3 $SDBE$ model

The Jacobian matrix around the "biotic" equilibrium for the $SDBE$ model $J_{i\times j}^{SDBE}|_*$ is given by eq. 10 in the main text. With $\frac{\partial \dot{B}}{\partial B}|_{*,(m,f)} = 0$ (i.e. the entry at position (3,3) in $J_{i\times j}^{SDBE}|_*$) the coefficients of the characteristic polynomial are given by

$$a_1 = \frac{\partial P_i}{\partial S}|_* + \frac{\partial U_j}{\partial D}|_* + d_E + l_D > 0$$

$$a_2 = \frac{\partial P_i}{\partial S}|_*(d_E + l_D) + d_E l_D + \frac{\partial U_j}{\partial D}|_*\left(d_E + d_B(1 - r_B y_m) + m_E + \frac{\partial P_i}{\partial S}|_* + d_B f_D r_B y_m\right) > 0$$

$$a_3 = \frac{\partial P_i}{\partial S}|_* d_E l_D + \frac{\partial P_i}{\partial S}|_* \frac{\partial U_j}{\partial D}|_*(m_E + d_B(1 - r_B y_m))$$
$$+ \frac{\partial U_j}{\partial D}|_* d_E \left(\pi + d_B f_D r_B y_m + \frac{\partial P_i}{\partial S}|_* - y_m \frac{m_E}{d_E} \frac{\partial P_i}{\partial E}|_*\right)$$

$$a_4 = \frac{\partial P_i}{\partial S}|_* \frac{\partial U_j}{\partial D}|_* d_E y_m \left(\frac{\partial U_j}{\partial B}|_*(1 - y_m) + d_B(1 - r_B)\right) > 0 \quad .$$

The additional condition for stability of this EP $a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0$ becomes analytically untractable. However, we observe that $\det\left(J_{i\times j}^{SDBE}|_*\right) = a_4 > 0$ and positivity of a_3 in $SDBE$ model and a_2 in the SDB model is conditional on the sign of the equivalent expressions $\Big(\pi + d_B f_D r_B y_m + \frac{\partial P_i^{qss}}{\partial S}|_* - y_m \frac{\partial P_i^{qss}}{\partial B}|_*\Big)$ respectively $\Big(\pi + d_B f_D r_B y_m + \frac{\partial P_i}{\partial S}|_* - y_m \frac{m_E}{d_E} \frac{\partial P_i}{\partial E}|_*\Big)$. This leads to the assumptions that the equivalent sufficient condition found for the SDB model might also hold for the $SDBE$ model and establish the stability of the equilibrium point. Furthermore, our numerical simulations rigorously confirm this result.

Full expression of the sufficient stability condition

Substituting the explicit formulations for $\frac{\partial P_i}{\partial S}|_*$ and $\frac{\partial P_i}{\partial E}|_*$ even for the arguably simplest kinetic formulation ($m \times m$) yields somewhat complicated terms. With the simplifications used for the analytical analysis of the $SDBE$ model ($y_B = y_m$ and $l_S = l_E = 0$), the condition given by $Z_{i\times j} \geq 0$ (eq. 46) in the main text yields for $m \times m$ kinetics

$$Z_{m\times m} = d_B f_D r_B y_m + v_m^p E_{m\times m}^* - y_m \frac{m_E}{d_E} v_m^p S_{m\times m}^* \quad ; \quad Z_{m\times m} \geq 0$$

and with $E^*_{m \times m}$ and $S^*_{m \times m}$ from Table 6 in the main text

$$Z_{m \times m} = \left(v^p_m m_E y_m (I - l_D D^*_{m \times m})^2 - d_E \pi^2 f_I I\right) \left(d_E \pi (I - l_D D^*_{m \times m})\right)^{-1} \geq 0 \quad .$$

To simplify the analysis, we define (note that for all physically meaningful biotic equilibrium points $d_E \pi (I - l_D D^*_{m \times m}) > 0$)

$$\frac{Z_{m \times m}}{d_E \pi (I - l_D D^*_{m \times m})} = Z'_{m \times m} = y_m \, m_E \, v^p_m (I - l_D D^*_{m \times m})^2 - f_I \, I \, d_E \, \pi^2 \geq 0 \quad .$$

After substituting the parameter groups, we find

$$Z'_{m \times m} = y_m \, m_E \, v^p_m \left(I - l_D \frac{d_B + m_E}{y_m v^u_m}\right)^2 - f_I \, I \, d_E \left(d_B (1 - r_B y_m) + m_E (1 - y_m)\right)^2 \geq 0 \quad .$$

Since $Z'_{m \times m}$ defines the sign of $Z_{m \times m}$, and thus whether $Z_{m \times m} \geq 0$ holds, we can gain some insights into how individual parameters affect the sufficient condition for stability ($Z_{m \times m} \geq 0$) by regarding $Z'_{m \times m}$. With all other parameters hold constant we find that:

1. an increase in y_m, v^p_m, v^u_m or r_B always increases $Z'_{m \times m}$
2. an increase in l_D, d_B, d_E or f_I always decreases $Z'_{m \times m}$
3. for $l_D = 0$ it becomes obvious that an increase in I always increases $Z'_{m \times m}$.

In fact, for $f_I = 0$ (all C input is DOC) or $d_E = 0$ (extracellular enzymes do not decay), $Z'_{m \times m}$ is always positive (for physically meaningful EPs).

For other kinetic formulations $Z_{i \times j}$, becomes analytically too cumbersome to yield mathematical insights.

We note that for $f_I = 0$ it can be shown that for all physically meaningful EPs also $Z_{f \times f} \geq 0$:

From eq. 46 in the main text $Z_{f \times f}$ is given by

$$Z_{f \times f} = \frac{\partial P_f}{\partial S}\Big|_* + y_m f_D r_B d_B - y_m \frac{m_E}{d_E} \frac{\partial P_f}{\partial E}\Big|_*$$

For $f_I = 0$ we find $\gamma_{f \times f}$ (eq. 37 in the main text) as $\gamma_{f \times f} = f_D r_B d_B y_m$ and thus from Table 6 in the main text $S^*_{1,f \times f}$ as

$$S^*_{1,f \times f} = K^p_f \left(\frac{v^p_f m_E}{d_E f_D r_B d_B} - 1\right)^{-1}$$

With this, $y_m \frac{m_E}{d_E} \frac{\partial P_f}{\partial E}\Big|_* = y_m \frac{m_E}{d_E} \left(v^p_f \frac{S^*_{1,f \times f}}{K^p_f + S^*_{1,f \times f}}\right)$ is obtained as

$$y_m \frac{m_E}{d_E} \frac{\partial P_f}{\partial E}\Big|_* = y_m f_D r_B d_B$$

and thus

$$Z_{f \times f} = \frac{\partial P_f}{\partial S}\Big|_* \quad .$$

Since for all physically meaningful EPs $\frac{\partial P_f}{\partial S}\Big|_* \geq 0$ for these also $Z_{f \times f} \geq 0$. Compare Fig. S5 for a numerical evaluation.

3 Numerical evaluation of proposed sufficient stability condition of the $SDBE$ model for other kinetics than $f \times f$
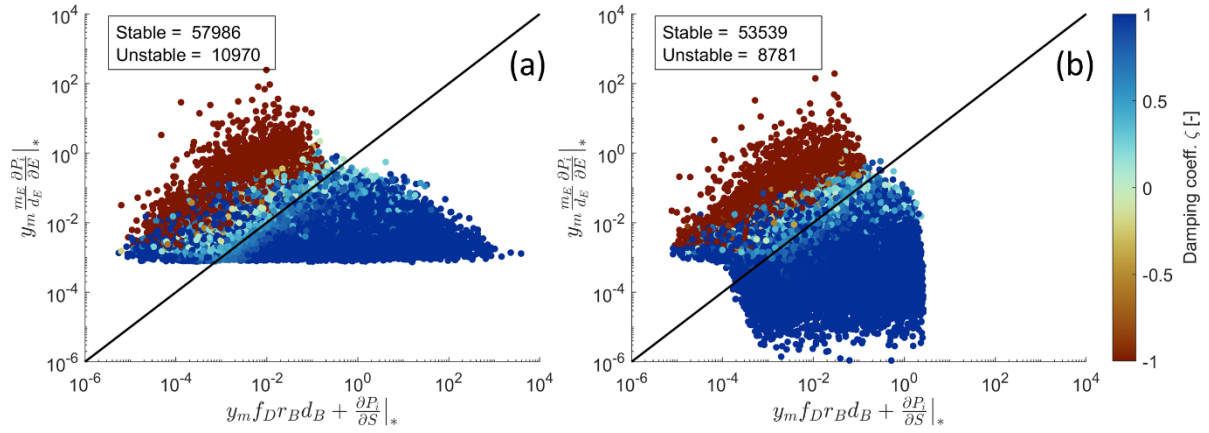

Fig. S1 Numerical evaluation of the proposed sufficient condition for stability of the $SDBE$ model with $m \times m$ (a) and $r \times f$ (b) kinetics and only constitutive ENZ production. 100 000 Monte Carlo calculations of equilibrium points were produced sampling the parameter space in Table 3. Plots illustrate the separation of all physically meaningful equilibrium points by their positive and negative terms of $Z_{i \times j}$ (the proposed sufficient condition for stability, eq. 46). Points on and below the black 1:1 line (indicates $Z_{i \times j} = 0$) fulfill the condition $Z_{i \times j} \geq 0$. The color-code indicates the value of the damping coefficient ζ.

4 Changes in damping coefficient with individual model parameters in the $SDBE$ model with $f \times f$ kinetics
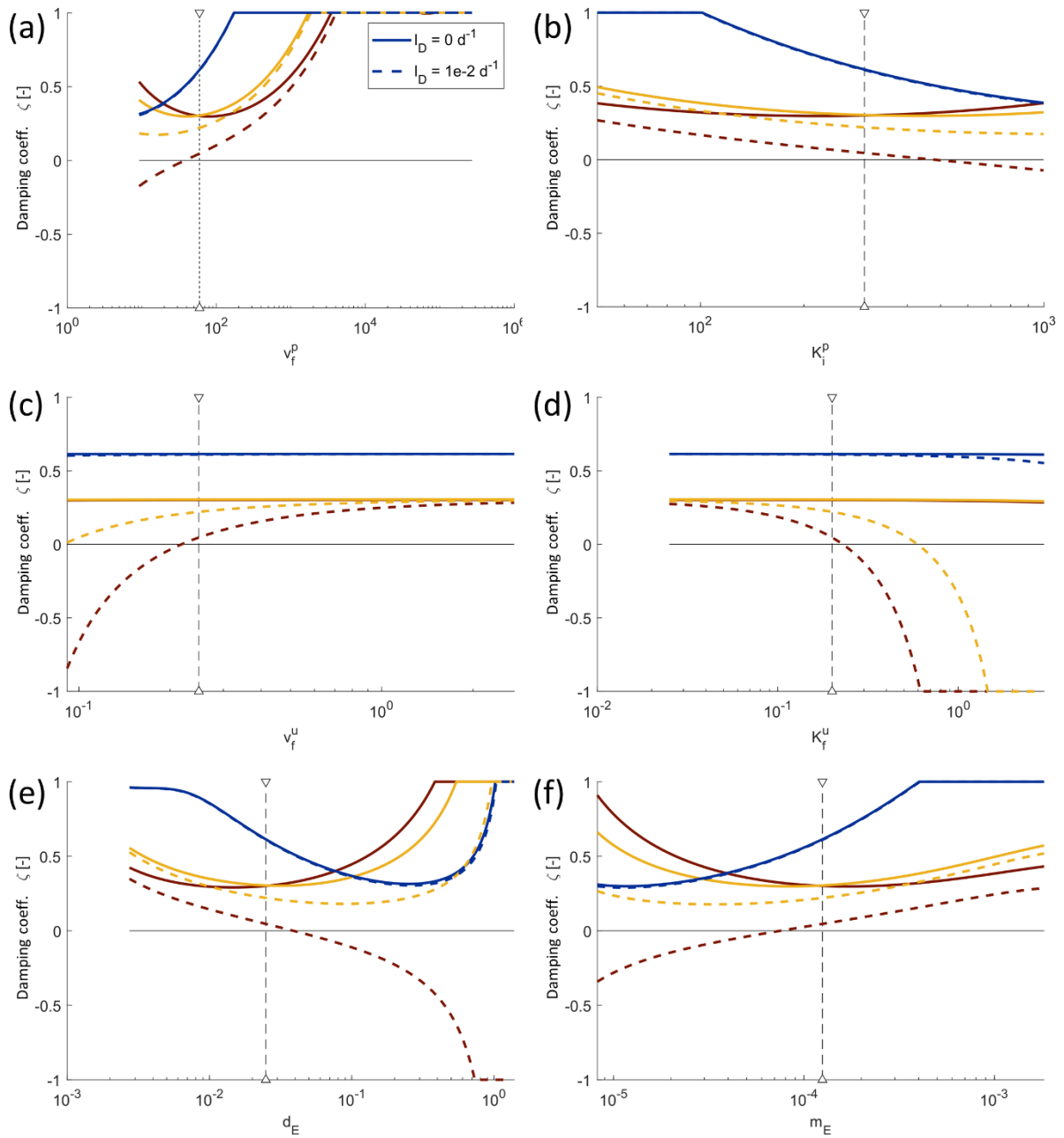

Fig. S2 Changes in the damping coefficient with changes in individual model parameters of the $SDBE$ model with $f \times f$ kinetics and constitutive ENZ production. Baseline parameter values (Table 3 in main text) are indicated by vertical dashed lines. Note that the range of m_E was extended compared to the range in Table 3 in the main text. Different line styles indicate scenarios with different DOC leaching rate coefficients (compare panel (a)); different line colors indicate scenarios with different OC input rates: 1e-2 mgC g^{-1} d^{-1} (blue), 1e-3 mgC g^{-1} d^{-1} (yellow), and 5e-4 mgC g^{-1} d^{-1} (red), respectively.
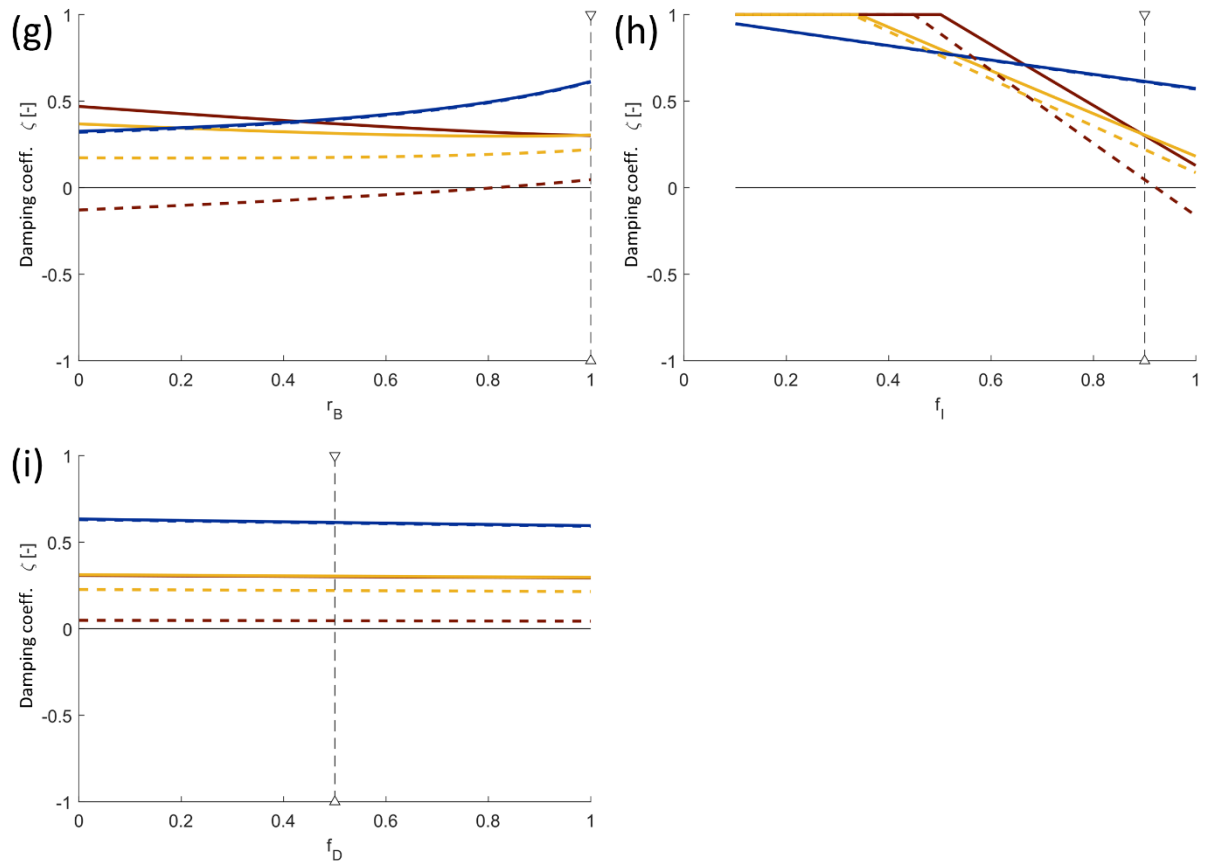
Fig. S2 continued.

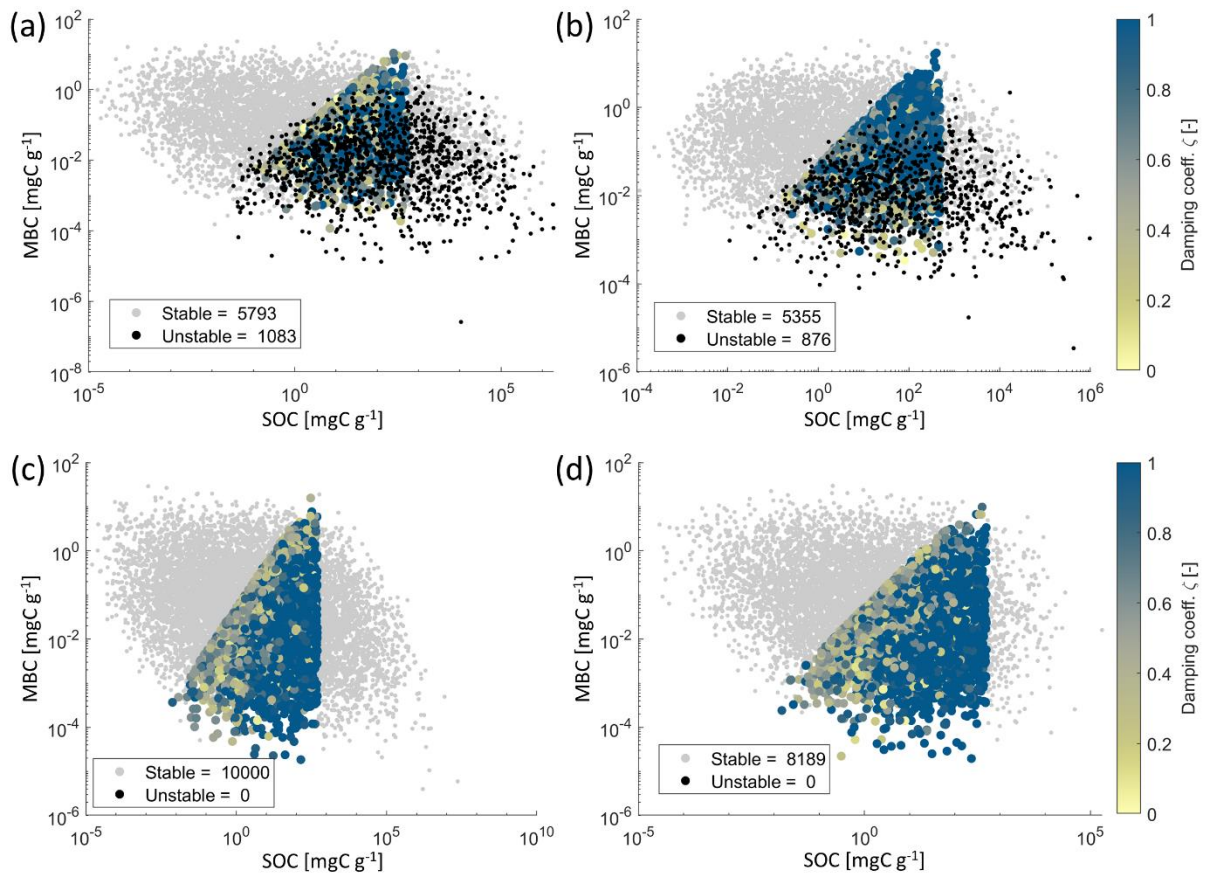# 5 Stable and unstable EPs of the $SDBE$ model for other kinetics than $f \times f$

Fig. S3 Physically meaningful (positive & real) steady states solutions from 10 000 Monte Carlo computations of equilibrium points of the $SDBE$ model with $m \times m$ (a), $r \times f$ (b), $m \times l$ (c), and $f \times l$ (d) kinetics (all considering only constitutive ENZ production with $l_D > 0$. Grey points are stable steady state solutions, color-coded points are stable steady state solutions yielding plausible results, and black points are physically meaningful but unstable steady state solutions. The color-code indicates the value of the damping coefficient. Legends indicate the numbers of physically meaningful and stable (stable + stable & plausible) or unstable EPs. Note that SOC contents > 1000 mgC g^{-1} are mathematically possible but unphysical model outcomes, as we neglect soil volume changes.

6 Density dependent mortality

Density dependent mortality was only tested in the $SDBE$ model for cases with $b = 2$ and in combination with only constitutive ENZ production ($y_B = y_m$) for $L_E = L_S = 0$. Only for $m \times m$ kinetics could DOC leaching be considered, for all other cases also $L_D = 0$.
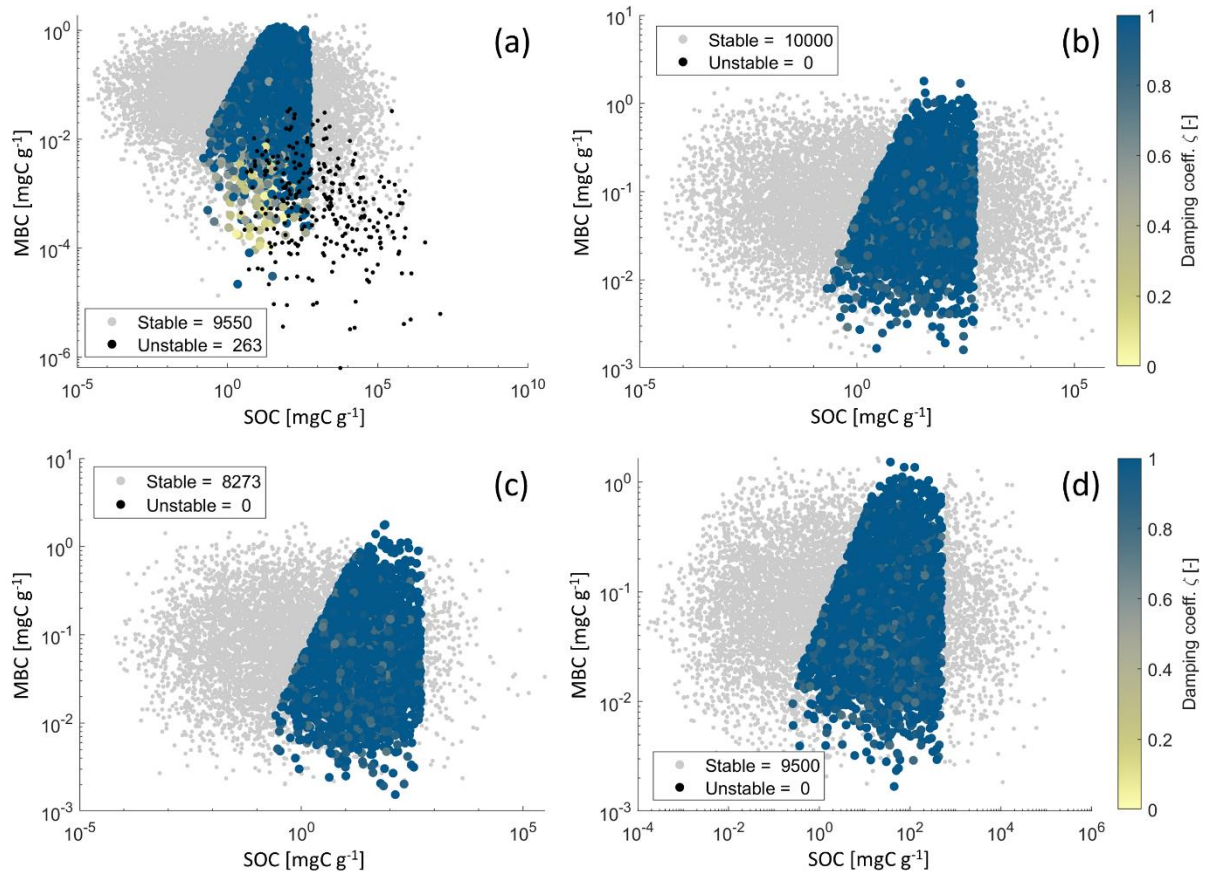

Fig. S4 Physically meaningful (positive & real) steady states solutions from 10 000 Monte Carlo simulations of the $SDBE$ model with density-dependent mortality with $m \times m$ (a & b), $f \times f$ (c), and $r \times f$ (d) kinetics (all considering only constitutive ENZ production). $L_D > 0$ is considered in (a), and $L_D = 0$ for (b-d). Grey points are stable steady state solutions, color-coded points are stable steady state solutions yielding plausible results, and black points are physically meaningful but unstable steady state solutions. The color-code indicates the value of the damping coefficient. Legends indicate the numbers of physically meaningful and stable (stable + stable & plausible) or unstable Eps. Note that SOC contents > 1000 mgC g⁻¹ are mathematically possible but unphysical model outcomes, as we neglect soil volume changes.

7 Numerical evaluation of stability of the $SDBE$ model with only DOC input ($f_I = 0$)
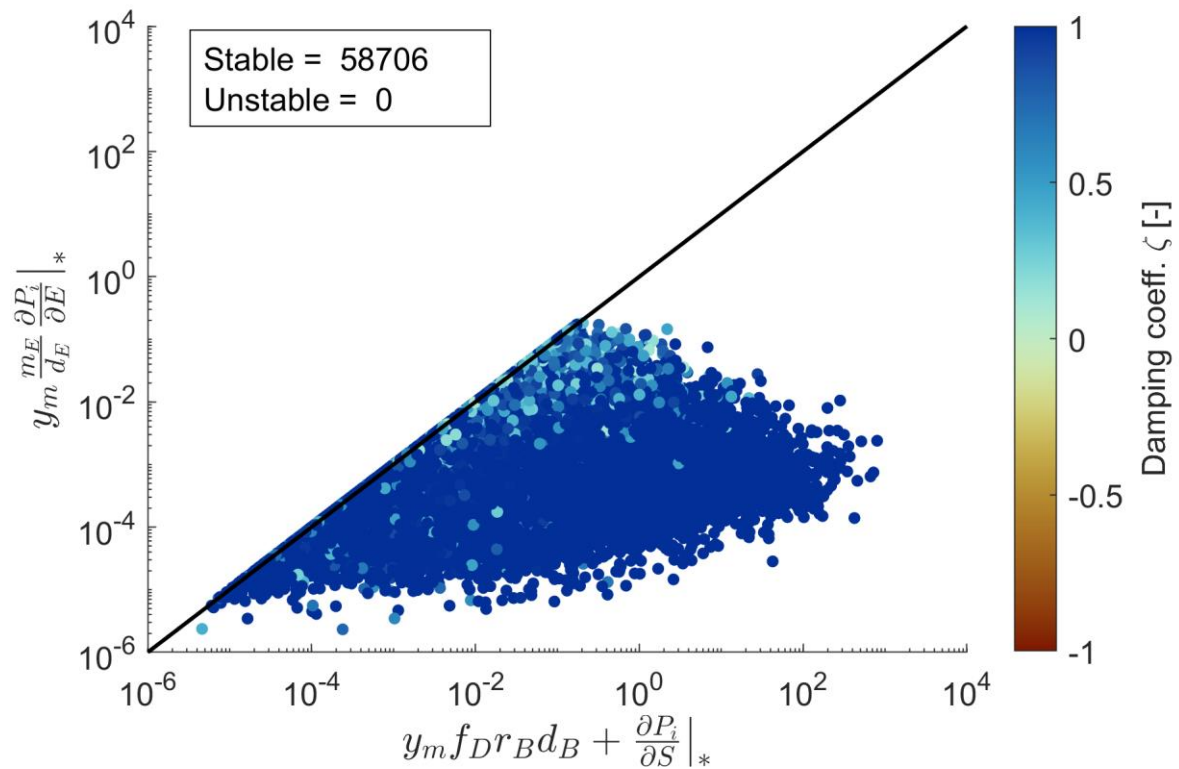

Fig. S5 Numerical evaluation of the proposed sufficient condition for stability of the $SDBE$ model with $f \times f$ kinetics and only constitutive ENZ production for $f_I = 0$. 100 000 Monte Carlo calculations of equilibrium points were produced sampling the parameter space in Table 3 (but holding $f_I = 0$). Plots illustrate the separation of all physically meaningful equilibrium points by their positive and negative terms of $Z_{f \times f}$ (the proposed sufficient condition for stability, eq. 46). Points on and below the black 1:1 line (indicates $Z_{f \times f} = 0$) fulfill the condition $Z_{f \times f} \geq 0$. The color-code indicates the value of the damping coefficient ζ.

8 References

Cotrufo, M. F., & Lavallee, J. M. (2022). Soil organic matter formation, persistence, and functioning: A synthesis of current understanding to inform its conservation and regeneration. In *Advances in Agronomy* (Vol. 172, pp. 1–66). Elsevier. https://doi.org/10.1016/bs.agron.2021.11.002

Georgiou, K., Abramoff, R. Z., Harte, J., Riley, W. J., & Torn, M. S. (2017). Microbial community-level regulation explains soil carbon responses to long-term litter manipulations. *Nature Communications*, *8*(1), 1223. https://doi.org/10.1038/s41467-017-01116-z

Hararuk, O., Smith, M. J., & Luo, Y. (2015). Microbial models with data-driven parameters predict stronger soil carbon responses to climate change. *Global Change Biology*, *21*(6), 2439–2453. https://doi.org/10.1111/gcb.12827

Tao, F., Huang, Y., Hungate, B. A., Manzoni, S., Frey, S. D., Schmidt, M. W. I., Reichstein, M., Carvalhais, N., Ciais, P., Jiang, L., Lehmann, J., Wang, Y.-P., Houlton, B. Z., Ahrens, B., Mishra, U., Hugelius, G., Hocking, T. D., Lu, X., Shi, Z., … Luo, Y. (2023). Microbial carbon use efficiency promotes global soil carbon storage. *Nature*. https://doi.org/10.1038/s41586-023-06042-3