Report on *On the hydrostatic approximation in rotating stratified flow* by Achim Wirth

This paper analyses the validity of the hydrostatic approximation widely used in realistic ocean simulation models, to the Naviers-Stokes equations taken as a reference. Sections 2 to 6 set out the mathematical methodology for the comparison based on projection evolution operator in Fourier space and linearised equation. Section 7 presents the error of the approximation under different forcing. Finally, in section 8, the author compares the uncertainty of the approximation with other uncertainties encountered in simulations of ocean dynamics and in the context of the Strait of Gibraltar (friction effect, Doppler). It also quantifies this approximation over a fine wave observation time. The author shows that, overall, the 2 formalisms give approximately the same results, provided that the vertical diffusion is high, which is of the order of magnitude typically used in simulations of ocean dynamics actually. Nevertheless, the difference between the 2 formalisms becomes apparent as the resolution increases and the diffusion models become finer (viscosity will have decreased). Section 8.3 is for me the most important because it implies.

I highly recommend this article, because as the author points out, with the improvement of space-time resolutions in the near future, the differences between two formalism should become apparent and this study will be even more valuable. However, to be honest, I found the text difficult to read, either because of the lack of information that you have to guess at, or because of the links between the arguments that you have to reconstruct. The author should have made it easier to read by presenting his arguments in a more orderly fashion.

I suggest that the result of **this studies would be appropriate for publication** if the authors revised the manuscript taking into account the comments below.

A) **Major comments:**

1. The Navier Stokes equations with the Boussinesq hypothesis have been presented in (1) and (2). However, it is not clear which equation in physical space was used to deduce equation (16) in Fourier space, /which relates to the acceleration of buoyancy $\hat{b}$. From (16) we deduce that:

$$\partial_t b = -(\mathbf{u}.\nabla)b - N^2.u_z + \kappa.\Delta b \tag{R1}$$

   This equation can be presented after the discussion of the term linked to ρ from line (133).

2. The original set of hydrostatic equations is not presented, even though author use them in his calculations, so it is not possible to find his results. They are not clearly written in the text, nor are the hypotheses that allow them to be written on the basis

of Navier-Stokes and the Boussinesq hypothesis. From the linearised equations (29) & (32) written by the author, if I have understood correctly, i deduce the original equation as follows:

$$
\begin{aligned}
\partial_t u_x &= -((\mathbf{u}?\ or\ \mathbf{u_H}?).\nabla)u_x + f.u_y - \partial_x P + \nu.\Delta u_x && \text{(R2)}\\
\partial_t u_y &= -((\mathbf{u}.\ or\ \mathbf{u_H}?).\nabla)u_y + -f.u_x - \partial_x P + \nu.\Delta u_x && \text{(R3)}\\
0 &= -\partial_z P + b && \text{(R4)}\\
\partial_t b &= -(\mathbf{u}.\nabla)b - N^2.u_z + \kappa.\Delta b && \text{(R5)}\\
\partial_z u_z &= -\partial_x u_x + \partial_y u_y && \text{(R6)}
\end{aligned}
$$

These equations (R2) to (R6) could be written after line (168). In addition, these equations lead to the definition of the operator $\mathbb{P}^H$ in physical space. Once these equations are clearly written in physical space, it will be possible to introduce the equation (R2) to (R6) in Fourier space which is equivalent to equations (13) (14) (15) (16). Also, for clarity, since the central tool in this article is the projection operator $\mathbb{P}$, this operator should be renamed $\mathbb{P}^N$ for the Navier-Stokes equation in (10). From these definitions, it becomes clearer to interpret geometrically the 2 operators $\mathbb{P}^N$ and $\mathbb{P}^H$ which are two operators of projection perpendicular to $\mathbf{k}$ (not clearly expressed in term of geometry in this article) but, the major difference between the two operators, it is that in the case of $\mathbb{P}^H$ one uses the incompressibility to deduce the third component whereas in the case of $\mathbb{P}^H$ the 3 components are used. Following this fundamental distinction, which should be discussed in the text before line 208.

3. Then for the T2 extension, I think that the central motivation is to be able to obtain a difference operator $\mathbb{A}$ between the two formulations $\mathbb{P}^N$ and $\mathbb{P}^H$ which is defined in (22). However, the problem comes from the definition of two different vectors $\mathbf{a}^N$ and $\mathbf{a}^H$, so we would have to write the input of $\mathbb{A}$ using a single vector $\tilde{\mathbf{a}}$ which is defined in (22), i.e. we need to find a common base. Using $\tilde{\mathbf{a}}$ defined in (22) there is only one common vector and it is possible to define in (27) the difference operator between the 2 projections $\mathbb{A}(\mathbf{a}) = \mathbb{P}^H(\mathbf{a}) - \mathbb{P}^N(\mathbf{a})$ and not have to work with $\mathbf{a}^H$ and $\mathbf{a}^N$ which cannot be applied. This point should be made clearer at the start of section 4, otherwise we won't understand the goal of the T2 extension.
4. From this clarification, the T3 extension becomes natural because it consists of writing the evolution equation (linearised) in which we must necessarily distinguish $\hat{b}$ and $\dot{\hat{b}}$ because the 2 formulations treat $\hat{b}$ and $\dot{\hat{b}}$ differently. If this point is detailed at the beginning of section 5, we can then understand the use of (28).
5. line 332: "The validity of the hydrostatic approximation relies also on how strongly the modes e_1 , e_2 on one side and e_3 on the other". I don't understand this sentence can you detailed this point on text ?
6. in section 6.2: why are restricted the space to D ? is it link to my previous question 5. ?
7. For section 8, there is generally no link with the previous study. It is not clear how the author uses the previous mathematical results to deduce physical result:
   i. "I first note that the amplitude of the balanced mode is bounded by ..." : from which section and which formula can this assertion be deduced?
   ii. section 8.2 : "The difference in the horizontal wave velocity, c, between the two formalisms is ...", why $\Delta c = \gamma.c$ and $c = \omega/k_h$? there is a mathematical formula in your article ? can you detailed the explanation in text.

iii. section 8.3: why "The difference in frequency between the two formalisms is $\sigma_f = \omega\gamma/2$" ? this is not obvious, in particular because $\omega^H = (1 + O(\gamma^2))\omega$. How do you link root mean square σ_f of difference between formalism with ω ? there is a mathematical formula in you article ? if yes which line ? Can you detailed this in text ?

iv. What are the formula used to deduce (79) and (80) ? Give details in the text.

v. "Using the Heisenberg-Gabor limit means that the observation length has to exceed $\gamma^{-2}/4\pi$..." : it is not obvious, can you detailed this point in text.

B) **Minor comments**:

1. equation (12) is not only valid for a but for any function f, ($f = a_x, a_y, ux.u_y, ...$)
2. In order to understand that it is difficult to obtain results on $\mathbb{P}$ from physical space, whereas it becomes more obvious in Fourier space, as the author has well written, the author may recall that the operator $\mathbb{P}$ in physical space is written:

$$\mathbb{P}(\mathbf{a}) = \mathbf{a} - \frac{1}{4\pi}\int\int\int \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^3}\nabla.\mathbf{a}\ d\mathbf{y} \qquad \text{(R7)}$$

3. For the sake of clarity for all readers, from line 192 onwards, the author should mention a key point which is that the incompressibility condition becomes $i.\mathbf{k}.\mathbf{u} = 0$ in Fourier space that give a geometrical interpretation $\mathbf{k} \perp \mathbf{u}$, and so the operator $\mathbb{P}^N$ and $\mathbb{P}^H$ in Fourier space becomes an operator which projects a second member into a plane perpendicular to $\mathbf{k}$. Moreover, when $kz = 0$ in (20), it should be pointed out that in this case $\mathbb{P}^N = \mathbb{P}^H$. All this geometrical interpretation also help in understanding Figure 3.
4. Figures 2 and 3 should be reversed because figure 3 is used before figure 2 in the text from the projection operator $\mathbb{P}$ in section 2. In fact, contrary to what is written on lines 175 and 217, (T2) and (T3) are not visible in figure 2 and this figure 2 is useful for illustrating what is said in section 6.
5. In equation (4), the various terms $\mathbf{a_u}$, $\mathbf{a_b}$ and $\mathbf{a_f}$ are not defined. Once these terms have been introduced, the equations in (6) ... (10) can be written using these terms, to make them uniform, compact and focused on pressure handling.
6. there is no $\mathbb{P}$ in (11)
7. Figure 3: the points representing $\hat{\mathbf{u}}$ are missing. If we can see the $\mathbf{k}$ vector (which is a good thing), the $(k_z, -k_x)$ vector is not good in terms of scale.
8. line 94: approxiamation.
9. Line 247 : a link should be made with the original hydrostatic equations.
10. line 268 : what is the norm $\sqrt{[\mathbb{A}(\tilde{\mathbf{a}}]^2}$? must to define
11. what is $\tilde{\gamma}^{\square}$ in (35) ? must be defined in (35) ($\tilde{\gamma}^H$ and $\tilde{\gamma} = 1$).
12. could you justify substitution (35). There is a common factor or a linear combination between $\tilde{e}_1, \tilde{e}_2$ and $\tilde{e}^{\square}_{1,2}$?
13. line 333 : square route.
14. line 317 : justify and discuss thermal-wind balance, it is not obvious.
15. what is link between F_d, F_r in (44) and $F = (F_x, F_y, F_z, 0, B)$? detail this in text.
16. to make it easier to read, all operators $\mathbb{L}, \mathbb{P}, ...$ should have the index N for Navier-Stokes instead of nothing

17. according your notation, $\mathtt{x} = \mathtt{x}_D$ in (50) and (51) because (51) and (48) is same. Simplify or discuss the difference.
18. what is interpretation c and s in (64) ? is-it group velocity of error ? detailed in text.
19. Section 7.3.1 : there is no link between $\mathbb{G}$ and absolute error previously defined in (49). Must to invert (67) and (65)-(66).
20. to make reading easier, the ratio deifni ligne 205 $\gamma = \sqrt{k_h^2/k_z^2}$ must be highlighted by a line break, a label (eqnarray) and the hypothesis $\gamma \ll 1$ to differentiate it from the one used in the introduction
21. to make it easier to read, the result line 310 $\omega^H \simeq (1+O(\gamma^2)).\omega$ must be highlighted by a line break, a label (eqnarray) because it is used after.