Answer to the reviewers 3 second report:

Dear Editor,

I am grateful to Reviewer 3 as his criticism is to the point and continues to increase the quality of the paper. Please find my detailed answers and corrections to the reviewer's comments (reproduced in black and blue) below, written in olive. The corrections performed to the manuscript are given in red and an updated version of the manuscript with the corrections highlighted in red is provided.

Sincerely,

Achim Wirth

Report on
On the hydrostatic approximation in rotating stratified flow
by Achim Wirth
Review 2

The new version of the text presented by the author has been much improved. Nevertheless, there are still grey areas that need to be clarified. I suggest that the result of this studies would be appropriate for publication if the author revised the manuscript taking into account the comments below. I am confident that author will carefully consider them.

A) According to author: "The equation for the evolution step is the same in both formalisms, it is Eq. 1 (the third component is never used in the hydrostatic formalism as the vertical eulerian-acceleration is calculated based on the horizontal divergence). I now added in the beginning of section 3:...".

   I cannot translate this sentence into a clear mathematical equation and there are still some elements that are mathematically unclear. For example, I think that my equations (R2) to (R6) written in my previous review correspond to the mathematical translation of the author's sentence. If so, then the author should specify equations (R2) - (R6), if not, then the author should write the equation that gives (19) (new version). This is an important point that I maintain, because it creates some confusion about the mathemat- ical formulation. However, the core of this article is the quantification of the differences between two formalisms.

   The reviewer is right, the hydrostatic equation was missing, it is now added (his eq. (R4)) as my eq. (3). His eqs. (R2) and (R3) are my eq. (1) (in vector form) the divergence condition (R6) is my equation (2). I added his eq. (R5) during the first review, as my (now) eq. (5). In section 2 I added:

When the velocity vanishes, the above equations simplify to the hydrostatic equation:

$$\partial_z P = -g\frac{\rho}{\rho_0} = \tilde{b}, \tag{1}$$

showing that the vertical pressure gradient equals the buoyancy.

And I refer to this equation in section 3: (Eq. 1)

B) According to author: "I disagree with the reviewer $\mathbb{P}^N$ is perpendicular to $\mathbf{k}$, but $\mathbb{P}^H$ is not"
In fact, to be precise, according to your Figure 2 (new version), $\mathbb{P}^N(\widehat{\mathbf{a}})$ is the orthogonal projection of â onto the plane perpendicular to $\mathbf{k}$ while $\mathbb{P}^H(\widehat{\mathbf{a}})$ (â) is the vertical (non- orthogonal) projection (non-orthogonal) projection of â in the plane perpendicular to $\mathbf{k}$. These projections for the two formalisms are always in a plane perpendicular to $\mathbf{k}$ to respect the free divergence. This geometric difference between the orthogonal projection and the vertical projection reflects the central difference between the two formalisms. This distinction must be clearly indicated in the legend of Figure 2.

The reviewer is right, my formulation in the answer was wrong. The right formulation is: $\mathbb{P}^N$ is along $\mathbf{k}$ and perpendicular onto the double red line. I now added to the legend of Figure 2:

it is orthogonal to the $\mathbf{k}$ vector.

And corrected:

(~~dashed~~ thick-blue-vector)

Other small improvements were done on the legend.

C) According to Author "The reviewer is right, there is a difference in the definition of $\mathbb{P}^N$ and its representation in the Figure explaining the projection, because $\mathbb{P}^H$ is not only the projection in the vertical "down" on the subspace of zero divergence (this only the last part), there are two more steps involved: the forgetting of the vertical dynamic acceler- ation and the addition of the buoyancy acceleration to the horizontal acceleration. The Fig. and its legend are now corrected. In the text it was ok. When I give a talk on the subject I spent 10 min on this fig. as it explains all the projection formalism. I added in the text:..."
For a reader like me, we need to know where we are going when we start a new section. Indeed, we do not know why to extend the fourth dimension to the operator, and it takes time (several hours in my case) to finally understand the interest of the mathematical tricks. This means that the author must start explaining the issues at the beginning of the section, for example: "The purpose of this section

is to express the operator $\mathbb{A}$ that measures the difference between the two formulations $\mathbb{P}^N$ and $\mathbb{P}^H$ that are defined in (20) and (22) (new version). However, the problem comes from the difference in the input vectors $\mathbf{a}^N$ and $\mathbf{a}^H$ in the operators $\mathbb{P}^N$ and $\mathbb{P}^H$. In order to express a single input of the operator $\mathbb{A}$ we use a vector $\tilde{\mathbf{a}}$ which is defined in (24) (new version) by finding a common basis between the operator $\mathbb{P}^N$ and $\mathbb{P}^H$."

I now changed the first paragraph of section 4 along the lines of the Reviewers comments:

In this section I introduce the extension of the Fourier space to 4 dimension, this technique is labeled T2 in the introduction and in Fig. **??**. It allows to use a common mathematical formalism, based on 4 dimensional linear algebra, for the Navier-Stokes and the hydrostatic projection. In the 4 dimensional space both projections apply to the same vector ($\tilde{a}$, introduced below) and it is then straightforward to obtain the difference between the two projections, four dimensional matrices ($\mathbb{A}$, given at the end of this section). Only the entries of the matrices differ between the two formalisms. Increasing the dimension from three to four is necessary as the two parts of the vertical acceleration, dynamical and buoyancy are treated differently between the formalisms.

D) According to author "The square-root of difference in the square of the horizontal wave velocity between the two formalisms... phase speed." What is the definition of c^N and c^H ? It should be specified in the text (I assume they are the phase velocities)..
I now added:

The phase velocities are given by $c^N = \omega^N/k$ and $c^H = \omega^H/k$, where the wave frequencies are defined by the dispersion relation in Eqs. **??**.