Revisiting Gassmann-Type Relationships within Biot Poroelastic Theory

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Abstract. ~~Gassmann's equations , formulated several decades ago, remain a cornerstone in geophysics due to their perceived exactness~~Gassmann's equations have long served as a cornerstone of geophysics and rock physics, widely regarded as exact within their domain of applicability. However, ~~a concise and rigorous derivation rooted in thermodynamic principles and conservation laws has been missing from the literature. Additionally,~~ recent studies have ~~pointed out potential logical inconsistencies in the original formulation. This paper introduces a derivation of Gassmann~~questioned their validity, arguing that Gassmann's ~~equations, anchored in fundamental~~ s derivation involves a logical error and that an additional solid modulus is needed even for monomineralic materials. In this work, we present a general derivation of the Extended Biot poroelasticity equations, grounded in conservation laws and ~~constitutive relations, ensuring their thermodynamic consistency. Alongside this~~ classical irreversible thermodynamics. We show that the formulations of Gassmann (1951), Brown and Korringa (1975), Detournay and Cheng (1993) and Rice and Cleary (1976) emerge as special cases of this unified framework. While previous studies have analyzed the thermodynamic admissibility of standard Biot and Gassmann models, we extend ~~the discussion to include Biot's poroelastic equations, which are widely used to describe the coupled behavior of fluid-saturated porous media under mechanical deformation. By demonstrating that Gassmann's equations are a specific case within the broader framework of Biot~~this analysis to the more general theory by explicitly incorporating the off-diagonal terms arising from the second partial derivatives (Hessian) of internal energy. A key finding is that Gassmann's self-similarity condition—that porosity remains unchanged under equal changes in fluid and total pressure—is a sufficient but not necessary condition for the derivation of Gassmann-type relationship between undrained and drained bulk moduli. It holds if and only if the matrix of the second partial derivatives of internal energy is diagonal. When the off-diagonal terms in this matrix are retained, a generalized form of Gassmann's ~~theory, we further validate their relevance and applicability in geophysical contexts. Given the numerous independent rederivations and numerical verifications of these equations for diverse pore geometries, we affirm their robustness, provided the underlying assumptions are respected. To facilitate reproducibility and further exploration,~~ s equations is required, which we derive. To promote transparency and support further research, we provide symbolic Maple routines ~~are provided for the derivations presented in this study~~with thermodynamic consistency checks, ensuring full reproducibility and accessibility.

1 Introduction

~~Gassmann'~~Gassmann's equations (Gassmann, 1951), developed several decades ago, ~~stand as fundamental expressions~~ are fundamental in geophysics for analyzing the elastic properties of fluid-saturated porous media. These equations provide a means to predict ~~the~~ seismic velocities and mechanical behavior ~~of~~ in such materials. However, despite their widespread use, recent studies have ~~highlighted concerns regarding~~ questioned the logical consistency ~~in the derivation of Gassmann's equations~~of Gassmann's derivation, suggesting that it contains a logical error (Thomsen, 2023a, b, 2024, 2025). This has ~~sparked a demand for a more rigorous thermodynamically admissible framework, rooted in conservation laws and constitutive relations, to ensure their reliability and applicability~~ highlighted the need for an extended, transparent and thermodynamically consistent framework to ensure reliability in geophysical modeling and ~~exploration.~~interpretation.

This ~~article aims to address these concerns by presenting a novel derivation of Biot's poroelastic equationsand Gassmann 's equations, which strictly adheres to~~ paper presents a structured, transparent, and fully reproducible derivation of the Extended Biot poroelastic equations, with the formulations of Gassmann (1951), Detournay and Cheng (1993), Brown and Korringa (1975) and Rice and Cleary (1976) emerging as special cases. Our approach is rooted in fundamental conservation laws and ~~thermodynamic principles. In particular, we leverage the formalism of classical non-equilibrium thermodynamics as described in Lebon et al. (2008), focusing on the interrelation of fluxes and forces, entropy production, and the~~ classical irreversible thermodynamics (CIT) (Lebon et al., 2008). While earlier works have demonstrated the thermodynamic admissibility of ~~constitutive equations. This approach allows us to systematically derive the targeted equations while ensuring that the derived models are consistent with the second law of thermodynamics~~the standard Biot and Gassmann models (Coussy et al., 1998; Yarushina and Podladchiko , we extend this analysis to a broader class of models by evaluating the full Hessian matrix (i.e., matrix of second partial derivatives) of internal energy.

We respond directly to the critiques presented in Thomsen (2023a, b, 2024, 2025), adopting the CIT formalism as described in Lebon et al. (2008) and extended to poromechanics by Yarushina and Podladchikov (2015). We demonstrate the thermodynamic admissibility of the ~~derived equations and validate their integrity through~~ Extended Biot equations by incorporating entropy production constraints and the internal-variable formalism of CIT. Internal consistency is verified through both theoretical analysis and numerical ~~simulations. By incorporating the entropy production constraints and internal variables approach from classical non-equilibrium thermodynamics, we ensure that the derived models not only describe the macroscopic behavior accurately but also respect the microscopic interactions between phases in porous media. While the general methodology was outlined by Yarushina and Podladchikov (2015), this study specifically focuses on the rigorous derivation of Biot's poroelastic, Gassmann's, and effective stress law equations, along with addressing concerns related to their physical validity~~evaluation. In particular, we emphasize the interplay between thermodynamic forces and fluxes, entropy production, and the admissibility of constitutive laws.

The paper is ~~organized~~ structured as follows: ~~First, essential~~ we begin by reviewing the foundational equations of classical irreversible thermodynamics~~are presented, emphasizing the link between~~ , highlighting the roles of thermodynamic forces and fluxes. ~~Next, we introduce the resulting evolution equations applicable to poro-viscoelastoplastic media. Following this, the target Biot 's poroelastic ,~~ We then derive the evolution equations for the Extended Biot poroelastic system, followed by formulations of the Detournay–Cheng, Brown–Korringa, and Gassmann models. After we revisit Gassmann's assumptions and

delineate the specific conditions under which they remain valid. We also directly address the critiques raised in Thomsen (2023a, b, 2024, 20 regarding the validity of Gassmann's~~s , and effective stress law equations are derived within this thermodynamically consistent framework. In the discussion section, we provide a detailed analysis of the validity and applicability of Gassmann's equations, highlighting the importance of respecting thermodynamic principles in their derivation and use. To facilitate reproducibility,~~ 's equations.

To ensure full reproducibility, we provide symbolic Maple routines ~~are provided to verify the presented results. The routines archive (v1.0) is available from a permanent DOI repository (Zenodo) at~~ with detailed line-by-line commentary, enabling transparent derivation and verification. This framework also supports future extensions, including multiphase flow and viscous deformation mechanisms. All Maple scripts are available in a symbolic archive via a permanent DOI on Zenodo: https://doi.org/10.5281/zenodo.15777522 (last access: ~~October 17, 2024) (Alkhimenkov and Podladchikov, 2024)~~June 30, 2025) (Alkhimenkov and Podladchikov, 2025).

2 ~~Assumptions and~~ Scope of the ~~Study~~Manuscript

~~The following assumptions are made throughout the derivation of Biot's poroelasticand Gassmann's equations to ensure the validity of the results: The material is assumed to be linearly elastic, and the strains are small, implying small fluid pressure perturbations relative to the confining stress. The porous medium is considered homogeneous and isotropic. The interactions between the solid and fluid phases are governed by linear constitutive laws, and the fluid flow obeys Darcy's law~~

One can distinguish between two related but distinct tasks in the formulation of coupled (poroelastic) theories: (i) identifying the appropriate set of state variables that fully describe the coupled mechanical behavior, and (ii) deriving the material parameters that link these variables. Task (i) is particularly challenging and has been addressed by numerous researchers; a comprehensive review is beyond the scope of this manuscript. In this work, we build on those earlier studies and assume from the outset that the correct variables have been identified.

~~The constraint of zero dissipation (entropy production)during reversible poroelastic deformation provides an essentialconstraint on the poroelastic constitutive equation for porosity evolution.~~

~~The derivation assumes a quasi-static process, meaning inertia effects are ignored. These assumptions provide a simplified framework for the derivation and are crucial for ensuring the thermodynamic admissibility of the results. Future work may extend these derivations to include non-linear elasticity, anisotropy, and dynamic effects~~Task (ii), while relatively more straightforward, remains essential: various modifications of poroelastic theory have been proposed, often based on simplifying assumptions that affect how material parameters are defined and interpreted. The main novelty of this manuscript is the consideration of the full Hessian matrix of second derivatives of internal energy — including the off-diagonal terms (which are often neglected in classical formulations) — which enables us to derive a generalized set of Gassmann-type relations. Furthermore, we demonstrate that under appropriate mappings between poroelastic coefficients, several classical poroelastic theories can be viewed as equivalent.

3 Derivation of ~~Gassmann'~~the Extended Biot's Poroelastic Equations

3.1 General ~~Representation~~ Pattern of ~~Classical Irreversible Thermodynamics~~the Derivation

~~Porous materials can be modeled as systems consisting of two interacting phases: a solid skeleton and a saturating fluid. These phases can exchange heat, momentum, and matter, leading to complex interactions that must be captured within the framework of classical irreversible thermodynamics (Gyarmati et al., 1970; Jou et al., 1996; Lebon et al., 2008; Yarushina and Podladchikov, 2015) . Using the principles of classical non-equilibrium thermodynamics, the conservation equations governing mass, momentum, entropy, and energy for each phase are expressed in the Eulerian framework as follows:~~

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla_j \left(\rho\phi \boldsymbol{v}_j + q_\rho^j\right) = Q_p,$$

$$\frac{\partial(\rho\phi \boldsymbol{v}_i)}{\partial t} + \nabla_j \left(\rho\phi \boldsymbol{v}_i \boldsymbol{v}_j + q_{\boldsymbol{v}}^{ij}\right) = Q_{v_i},$$

$$\frac{\partial(\rho\phi \boldsymbol{s})}{\partial t} + \nabla_j \left(\rho\phi \boldsymbol{s} \boldsymbol{v}_j + q_{\boldsymbol{s}}^j\right) = Q_s,$$

$$\frac{\partial(\rho\phi \boldsymbol{e})}{\partial t} + \nabla_j \left(\rho\phi \boldsymbol{e} \boldsymbol{v}_j + q_{\boldsymbol{e}}^j\right) = Q_e,$$

~~where $\boldsymbol{v}_j$, $\boldsymbol{s}$, and $\boldsymbol{e}$ denote the velocity, specific entropy, and specific total energy per unit mass, respectively. The terms ∇_j represents the partial derivative with respect to spatial coordinates, while q_ρ^j, $q_{\boldsymbol{v}}^{ij}$, $q_{\boldsymbol{s}}^j$, and $q_{\boldsymbol{e}}^j$ correspond to the fluxes of mass, momentum, entropy, and energy, respectively. The termsQ_p, Q_{v_i}, Q_s, and Q_e represent the corresponding production rates due to irreversible processes (Yarushina and Podladchikov, 2015)~~To derive the Extended Biot poroelastic equations, one typically combines the following components:

- **Conservation laws:**
  - Conservation of linear momentum for the total stress,
  - Conservation of mass for the solid phase,
  - Conservation of mass for the fluid phase,
- **Fluid dynamics:**
  - Darcy's law for the Darcy flux $\boldsymbol{q}^D$ (assuming low-Reynolds-number flow),
- **Isothermal constitutive relations:**
  - A solid density-pressure constitutive law (equation of state),
  - A fluid density-pressure constitutive law (equation of state),
  - A porosity constitutive law (e.g., pore compressibility),

- Stress-strain relation for the deviatoric components of the stress and strain tensors.

By expressing the solid and fluid densities, as well as the medium's porosity, in terms of pressures and fluxes via these constitutive laws, one obtains the Extended Biot poroelastic equations. Under additional simplifying assumptions, the formulation reduces to the classical Biot poroelastic equations (Biot, 1962), the Brown and Korringa equations (Brown and Korringa, 1975) , Rice and Cleary (1976) equations and Gassmann's equations (Gassmann, 1951) as limiting cases. In the case of Biot poro-visco-elasticity, viscous effects are incorporated through the specific choice of the porosity evolution law (Yarushina and Podladchikov, 2015) , which can include time-dependent or rate-sensitive terms. To ensure thermodynamic consistency, these constitutive relations are derived within the framework of classical irreversible thermodynamics, which we describe in the following section.

~~**Local Entropy Production**~~

4 Thermodynamic admissibility of the extended Biot poroelasticity framework

4.1 Local Entropy Production

In the context of ~~classical non-equilibrium thermodynamics (Lebon et al., 2008), each phase within the porous medium is considered to be locally in thermodynamic equilibrium , which means that intensive variables such as temperature and chemical potential are well-defined at each point. This leads to a fundamental relationship between the~~ Classical Irreversible Thermodynamics (CIT) (Lebon et al., 2008), the hyposesis of local thermodynamic equilibrium implies that energy is well defined as a single value function at each state of the system. Moreover, for a unit mass of a solid skeleton, in agreement with the main assumption of CIT, the infinitesimal change in ~~specific internal energy U for each phase and~~ internal energy U_s follows its equilibrium relationship via the corresponding changes in ~~specific entropy S, specific volume ρ, the elastic component of porosity ϕ^e. The local entropy production is derived from the energy balance and is given by:~~ entropy S_s per unit mass, density ρ_s, and the elastic part of porosity ϕ_s^e (Yarushina and Podladchikov, 2015):

$$\frac{dU}{dt}dU_s = T\frac{dS}{dt}dS_s - p\frac{d(1/\rho)}{dt}_s d(1/\rho_s) + v\frac{dv}{dt} + \mu\frac{dC}{dt} + \frac{d\phi^e}{dt}\frac{\tau_\phi^s}{\rho_s\,\phi_s}d\phi_s^e, \tag{1}$$

~~where τ_ϕ is the~~ where: T is the absolute temperature, p_s is the solid pressure conjugated to solid density change, τ_ϕ^s is the thermodynamic variable (pressure) conjugated to porosity change (to be defined). ~~τ_ϕ~~, and $\phi_s^e = \phi_s$ is the solid volume fraction, superscript "^{e}" represents reversible (elastic) change ($\phi_f = 1 - \phi_s$, with ϕ_f being the medium's porosity). τ_ϕ^s can be viewed as analogy to pressure as conjugate variable to volume change. $\frac{d}{dt} = \frac{\partial}{\partial t} + v_i\nabla_i$ ~~denotes the Lagrangian (material) derivative with respect to a specific phase, $\frac{d\phi^e}{dt}$ is the reversible part of the porosity change.~~ The individual terms in this energy balance are interpreted as:

- ~~$T\frac{dS}{dt}$~~ TdS_s: Heat stored in internal energy ~~U~~ U_s.
- $\frac{p}{\rho^2}\frac{d\rho}{dt}$ $p_s d(1/\rho_s)$: Energy change due to ~~volumetric deformation (~~compressibility of solid grains (volumetric Hooke's Law).

- ~~$v\dfrac{dv}{dt}$: Newtonian mechanics (kinetic energy, e.g., $v\frac{dv}{dt} = \frac{1}{2}\frac{dv^2}{dt}$).~~
- ~~$\mu\dfrac{dC}{dt}$: Energy due to changes in composition (chemical potential), which is zero in the present derivation.~~
- $\dfrac{\tau_\phi}{\rho\phi}\dfrac{d\phi^e}{dt}$ ~~:~~ $\dfrac{\tau^s_\phi}{\rho_s\phi_s}d\phi^e_s$: Poroelastic effects: reversible part of the energy change due to the changes in porosity.

Note, that ~~τ_ϕ is not defined yet.~~ *τ^s_ϕ is not defined yet.*

~~Entropy Production (TQ_s)~~

4.2 Entropy Production for Poroelastic Loading

~~Solving the local entropy productionequation for Q_s and multiplying both sides by T, we have (for details see Appendix B):~~ In the context of poroelasticity, the most important outcome from Appendix B is an expression for entropy production, Q_s^{poro}, associated with elastic (reversible) porosity change:

$$TQ_s^{poro} = \left[(p_s - \tau^s_\phi) - p_f\right]\frac{d\phi^e_s}{dt}, \tag{2}$$

where p_s is the solid pressure and p_f is the fluid pressure. Entropy production must be zero for reversible poroelastic deformation; therefore $(p_s - \tau^s_\phi) - p_f = 0$ (!). This implies that (Yarushina and Podladchikov, 2015):

$$TQ_s\tau^s_\phi = \eta\phi^2 + p\upsilon + \mu Q_\rho C_s - \upsilon Q_v - Q_\rho G_{\text{Gibbs}} + Q_u + p - \tau_{\phi f} \tag{3}$$

~~This expression represents the entropy production, which must be non-negative according to the second law of thermodynamics. This formulation, which assumes local thermodynamic equilibrium for only the solid and fluid phases, is less strict than Biot's classical assumption of a single internal energy potential for the entire~~ We also notice that $\tau^s_\phi = p_e/(1-\phi_f)$, where $p_e = \bar{p} - p_f$ represents the effective pressure (total pressure is defined as $\bar{p} = (1-\phi_f)p_s + \phi_f p_f$). For an explanation of the Maple script used in the derivation and analysis of entropy production in a single-phase medium, see Appendix A. Appendix B provides a similar explanation for the entropy production derivation in a two-phase ~~system in the linear poroelastic case (Yarushina and Podladchikov, 2015).~~ porous medium.

4.3 ~~Extended Thermodynamic Admissibility~~

4.3 Internal energy of the solid frame

~~Building upon the concepts from Lebon et al. (2008) and the nonlinear viscoelastoplastic framework developed by Yarushina and Podladchi , the derivation of Gassmann's and Biot's equations must satisfy the constraints of thermodynamic admissibility. Specifically, the entropy production Q_s must be non-negative.~~ We begin with the internal energy of representative infinitesimal solid skeleton (frame) linked to material points (grains) of the solid skeleton in a Lagrangian fashion, $U_s(V_s, \phi_s)$, per unit mass. Here, V_s is

the (Lagrangian) solid volume and $\phi_s = V_s/V_t$ is the solid volume fraction, V_t is the (Lagrangian) total volume. A first-order Taylor expansion about an equilibrium state (V_s^0, ϕ_s^0) yields:

$$U_s(V_s,\phi_s) = U_s(V_s^0,\phi_s^0) + \frac{\partial U_s}{\partial V_s}(V_s^0,\phi_s^0)\Delta V_s + \frac{\partial U_s}{\partial \phi_s}(V_s^0,\phi_s^0)\Delta\phi_s + o(\epsilon), \tag{4}$$

where $\Delta V_s = V_s - V_s^0$ and ~~the constitutive relations must be derived in a way that ensures compliance with the second law of thermodynamics.~~ $\Delta\phi_s = \phi_s - \phi_s^0$. The energy increment ΔU_s is:

$$\Delta U_s = U_s(V_s,\phi_s) - U_s(V_s^0,\phi_s^0) = \frac{\partial U_s}{\partial V_s}(V_s^0,\phi_s^0)\Delta V_s + \frac{\partial U_s}{\partial \phi_s}(V_s^0,\phi_s^0)\Delta\phi_s. \tag{5}$$

4.3.1 ~~Thermodynamic Constraints on Fluxes and Productions~~

The internal energy U_s is a scalar potential defined on a smooth, convex state space, where the Hessian matrix is symmetric:

$$\begin{bmatrix} \frac{\partial U_s}{\partial V_s}(V_s,\phi_s) \\ \frac{\partial U_s}{\partial \phi_s}(V_s,\phi_s) \end{bmatrix} = \begin{bmatrix} \frac{\partial U_s}{\partial V_s}(V_s^0,\phi_s^0) \\ \frac{\partial U_s}{\partial \phi_s}(V_s^0,\phi_s^0) \end{bmatrix} + \mathbf{H} \begin{bmatrix} \Delta V_s \\ \Delta\phi_s \end{bmatrix}, \tag{6}$$

where $\mathbf{H}$ is the Hessian matrix of second derivatives of the internal energy with respect to V_s and ϕ_s:

$$\mathbf{H} = \begin{bmatrix} \frac{\partial^2 U_s}{\partial V_s^2}(V_s^0,\phi_s^0) & \frac{\partial^2 U_s}{\partial V_s \partial \phi_s}(V_s^0,\phi_s^0) \\ \frac{\partial^2 U_s}{\partial \phi_s \partial V_s}(V_s^0,\phi_s^0) & \frac{\partial^2 U_s}{\partial \phi_s^2}(V_s^0,\phi_s^0) \end{bmatrix}. \tag{7}$$

The increment of the first derivatives of ΔU_s are:

$$\begin{bmatrix} \Delta\frac{\partial U_s}{\partial V_s}(V_s^0,\phi_s^0) \\ \Delta\frac{\partial U_s}{\partial \phi_s}(V_s^0,\phi_s^0) \end{bmatrix} = \mathbf{H} \begin{bmatrix} \Delta V_s \\ \Delta\phi_s \end{bmatrix}. \tag{8}$$

For isothermal processes and in agreement with CIT (equation (1)), ΔU_s can be also expressed via mechanical variables only:

$$\Delta U_s(V_s,\phi_s) = -p_s\Delta V_s + \tau_\phi^s \frac{V_s}{\phi_s}\Delta\phi_s \equiv -p_s\Delta V_s + (p_s - p_f)\frac{V_s}{\phi_s}\Delta\phi_s. \tag{9}$$

By comparing equations (8) and (9), we identify:

$$\Delta\frac{\partial U_s}{\partial V_s}(V_s^0,\phi_s^0) = -\Delta p_s, \qquad \Delta\frac{\partial U_s}{\partial \phi_s}(V_s^0,\phi_s^0) = -\Delta\left(\frac{V_s}{\phi_s}(p_f - p_s)\right) \approx \frac{V_s}{\phi_s}\Delta(p_f - p_s). \tag{10}$$

Therefore, the following linear system holds:

$$\begin{bmatrix} -\Delta p_s \\ -\frac{V_s}{\phi_s}\Delta(p_f - p_s) \end{bmatrix} = \mathbf{H} \begin{bmatrix} \Delta V_s \\ \Delta\phi_s \end{bmatrix}. \tag{11}$$

We then use the following equation of state for the fluid for isothermal processes:

$$\frac{\Delta V_f}{V_f} = -\beta_f \Delta p_f, \tag{12}$$

where β_f is the fluid compressibility. Equations (11) and (12) are used by Yarushina and Podladchikov (2015) (assuming simplified diagonal Hessian matrix $\mathbf{H}$) as a constitutive closure relationships (their equations 6-8).

5 Derivation of the original Gassmann and Biot equations

We here provide a derivation which is similar to the one proposed by Yarushina and Podladchikov (2015) in terms of underlying constitutive closer relationships. Unlike Yarushina and Podladchikov (2015), we start from the Hessian matrix $\mathbf{H}$ and provide a detailed derivation, without skipping any intermediate steps

5.1 Derivation of the original Biot-Gassmann equations

We consider a simplified diagonal version of the full compliance matrix $\mathbf{H}$ (equation (11)):

$$\begin{bmatrix} -\Delta p_s \\ -\frac{V_s}{\phi_s}\Delta(p_f - p_s) \end{bmatrix} = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} \begin{bmatrix} \Delta V_s \\ \Delta \phi_s \end{bmatrix}. \tag{13}$$

We further use the following relation between density increments and solid volume change:

$$\frac{\Delta \rho_s}{\rho_s} = -\frac{\Delta V_s}{V_s}, \tag{14}$$

In addition, we use the following identity:

$$\Delta \phi_s = -\Delta \phi_f. \tag{15}$$

Equation (13) can be now re-written as:

$$\begin{bmatrix} -\Delta p_s \\ -\frac{V_s}{(1-\phi_f)}\Delta(p_f - p_s) \end{bmatrix} = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} \begin{bmatrix} V_s \frac{\Delta \rho_s}{\rho_s} \\ -\Delta \phi_f \end{bmatrix}. \tag{16}$$

We solve (16) with respect to $\Delta\phi_f/\phi_f$ and $\Delta\rho_s/\rho_s$. The resulting expressions are cumbersome and can be directly accessed via the provided Maple scripts:

$$\frac{\Delta \phi_f}{\phi_f} = f(H_{11}, \Delta(p_f - p_s), \phi_f, \Delta\phi_f, V_s), \tag{17}$$

$$\frac{\Delta \rho_s}{\rho_s} = f(H_{22}, \Delta(p_f - p_s), \phi_f, \Delta\phi_f, V_s). \tag{18}$$

5.2 The incremental formulation

The ~~second law of thermodynamics requires that the total entropy production of the system remains non-negative. This condition applies both to the intra-phase and inter-phase entropy production within a porous medium. Mathematically, this is expressed as:~~ next step is to substitute the resulting equations for $\frac{\Delta\phi_f}{\phi_f}$ and $\frac{\Delta\rho_s}{\rho_s}$ into the mass conservation equations, which is explored below. Now, we transition from differentials into the incremental formulation and use the following identity:

$$\sum_{\text{phases}} Q_s \underset{\sim}{\Delta} = \sum_{\text{phases}} Q_s^{\text{intra}} + Q_s^{\text{inter}} \geq 0. \frac{d^s \cdot}{dt}, \tag{19}$$

~~Here, Q_s^{intra} represents the intra-phase entropy production within each phase, while Q_s^{inter} accounts for the inter-phase contributions due to interactions between the solid skeleton and the fluid phase. To satisfy the second law, both components must be non-negative.~~ where we adopt material (Lagrangian) time derivatives. We use the following notation: $\frac{d^s}{dt} = \frac{\partial}{\partial t} + v_i^s \nabla_i$ denotes the Lagrangian (material) derivative with respect to solid and $\frac{d^f}{dt} = \frac{\partial}{\partial t} + v_i^f \nabla_i$ denotes the Lagrangian (material) derivative with respect to fluid, where v_i^f and v_i^s are the fluid and solid velocities, respectively. The Einstein summation convention is used: summation is applied over repeated indices.

~~Entropy Production and Compaction Mechanisms~~

We re-write equation (12) in a rate form:

$$\frac{d^f p_f}{dt} = \frac{d^s p_f}{dt} + \left(v_i^f - v_i^s\right) \nabla_i p_f. \tag{20}$$

We adopt the following approximate relations, which are strictly valid under small strains:

$$\frac{d^f p_f}{dt} \approx \frac{d^s p_f}{dt}, \tag{21}$$

$$\frac{d^f \phi_f}{dt} \approx \frac{d^s \phi_f}{dt}. \tag{22}$$

~~In the context of poroelasticity, the most important outcome from expression (B5) is in the two terms, which describe porosity change:~~

$$TQ_s^{poro} = p\frac{d\phi}{dt} - \tau_\phi \frac{d\phi^e}{dt} = \sum_{\text{phases}} \left(p\frac{d\phi}{dt} - \tau_\phi \frac{d\phi^e}{dt}\right).$$

Approximations (21)-(22) are implicitly assumed in Yarushina and Podladchikov (2015). For equation (21), this approximation is valid when the relative velocity between fluid and solid phases is small, or when the fluid pressure gradient is negligible.

5.3 Conservation of mass in a rate form

Conservation of mass for fluid phase in rate form is

$$\frac{\partial(\phi_f \rho_f)}{\partial t} + \nabla_j \left(\phi_f \rho_f v_j^f\right) = 0, \tag{23}$$

~~We assume that the porosity evolution can be decomposed into elastic and dissipative components, which together with the negativity of entropy production requires that inelastic porosity equation takes the form (Yarushina and Podladchikov, 2015):~~ and conservation of mass for the solid phase in rate form is:

$$-\frac{\partial((1-\phi_f)\rho_s)}{\partial t} + \nabla_j \left((1-\phi_f)\rho_s v_j^s\right) = -0. \tag{24}$$

~~where ϕ^e denotes the elastic portion of porosity, $p_e = \bar{p} - p_f$ represents the effective pressure (~~Equations (23)-(24) can be reformulated for divergences $\nabla_j v_j^s$ and $\nabla_j q_j^D$:

$$\nabla_j v_j^s = -\frac{1}{\rho_s}\frac{d^s \rho^s}{dt} + \frac{1}{1-\phi_f}\frac{d^s \phi_f}{dt}, \tag{25}$$

$$\nabla_j q_j^D = -\frac{\phi_f}{\rho_f}\frac{d^f \rho^f}{dt} - \frac{d^f \phi_f}{dt} - \phi_f \nabla_j v_j^s, \tag{26}$$

where $q_i^{\mathrm{D}} = \phi_f(v_i^f - v_i^s)$ is the Darcy flux.

5.4 Relations between total, solid and fluid pressures

Note that the material derivatives of the total pressure, ~~$\bar{p} = (1-\phi)p_s + \phi p_f$, minus fluid pressure, p_f, respectively), and η_ϕ stands for the effective bulk viscosity. Using the definition (B8), we can rewrite expression (??)~~ $\bar{p}$, and the solid pressure, p_s, are related via:

$$\frac{d^s \bar{p}}{dt} = (1-\phi)\frac{d^s p_s}{dt} + \phi\frac{d^s p_f}{dt} + \frac{d^s \phi}{dt}(p_f - p_s), \quad \Rightarrow \quad \frac{d^s p_s}{dt} = \frac{1}{1-\phi_f}\left(\frac{d^s \bar{p}}{dt} - \phi_f \frac{d^s p_f}{dt} - \frac{d^s \phi_f}{dt}(p_f - p_s)\right) \tag{27}$$

Equation 27 for solid pressure p_s can be simplified by neglecting the porosity derivative term:

$$\frac{d^s p_s}{dt} \approx \frac{1}{1-\phi_f}\left(\frac{d^s \bar{p}}{dt} - \phi_f \frac{d^s p_f}{dt}\right). \tag{28}$$

5.5 Resulting equations of Biot-Gassmann theory

We then adopt the relation (28) and replace p_s in favor of $\bar{p}$. By simplifying equations (25)-(26), we can write the following relation

$$\begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \frac{d^s \bar{p}}{dt} \\ \frac{d^s p_f}{dt} \end{pmatrix}. \tag{29}$$

We note that $a_{12} = a_{21}$, which is explicitly derived rather than imposed (this fact is explored in more details for the case of the full matrix $\mathbf{H}$ and is provided below). Let us define the following compressibilities:

$$\beta_d = -a_{11}, \tag{30}$$

which gives:

$$H_{1,1} = -H_{2,2}\frac{(\phi_f^2 - 2\phi_f + 1)}{V_s\left(\phi_f^3 H_{2,2}\beta_d - 3\phi_f^2 H_{2,2}\beta_d + 3\phi_f H_{2,2}\beta_d - H_{2,2}\beta_d + V_s\right)}. \tag{31}$$

Then we introduce α as

$$\alpha = \frac{a_{12}}{\beta_d}, \tag{32}$$

which gives

$$H_{2,2} = \frac{V_s}{\left(\alpha\phi_f^2 + \phi_f^3 - 2\alpha\phi_f - 2\phi_f^2 + \alpha + \phi_f\right)\beta_d}. \tag{33}$$

Finally, we introduce B as

$$B = -\frac{\alpha\beta_d}{a_{22}} \equiv \frac{\beta_d - \beta_s}{\phi_f(\beta_f - \beta_s) + \beta_d - \beta_s}. \tag{34}$$

By using the definitions (30)-(34), we can rewrite (29) in the following form:

$$TQ_s^{poro}\begin{pmatrix}\nabla_k v_k^s \\ \nabla_k q_k^D\end{pmatrix} = \sum_{\text{phases}}(p_s - \tau_\phi^s) - (p_f - \tau_\phi^f)\frac{d\phi^e}{dt}\beta_d \begin{pmatrix}1 & -\alpha \\ -\alpha & \frac{\alpha}{B}\end{pmatrix}\begin{pmatrix}\frac{d^s\bar{p}}{dt} \\ \frac{d^s p_f}{dt}\end{pmatrix}, \tag{35}$$

~~In equilibrium conditions, the entropy production tends to zero, which implies that the term $\left[(p_s - \tau_\phi^s) - (p_f - \tau_\phi^f)\right] = 0$ (!). The fluid phase does not contain the porosity term, meaning that $\tau_\phi^f = 0$. It implies that $\left[(p_s - \tau_\phi^s) - (p_f - \tau_\phi^f)\right] = 0$ corresponds to $\tau_\phi^s = p_s - p_f$ (Yarushina and Podladchikov, 2015). We also notice that $\tau_\phi^s = p_e/(1-\phi)$. By definition the poroelastic constant K_ϕ~~ which is the original Biot poroelastic equation (Biot, 1962), extended to an incremental large-strain formulation (Yarushina and Podladc . Equation (35) reduces exactly to original Biot formulation (Biot, 1962) if we assume small strains. We also note that the expression (32) for α can be written as

$$\alpha = 1 - \frac{\beta_s}{\beta_d}. \tag{36}$$

5.6 Key observations

To derive the original Biot–Gassmann poroelasticity relations, one should use the proposed rheological relationship (13) and the two equalities (21) and (22). The relationship (13) implies the following identity:

$$\frac{d^s\phi_f^e}{dt} \overset{1}{=} -\beta_\phi(1-\phi_f)\frac{d^s\tau_\phi^s}{dt} \overset{2}{=} -\beta_\phi\frac{d^s p_e}{dt}, \tag{37}$$

where the poroelastic constant (compressibility) β_ϕ is defined that as linear rheological relationship during reversible poroelastic part of deformation.

Equality (1) in equation (37) is the primary assumption made by Biot (1962) and by Gassmann (1951) (also used by Yarushina and Podladchikov (2015)). It postulates that equal changes in total and fluid pressure leave porosity unchanged. This assumption is often referred to as the *self-similarity hypothesis* and is equivalent to assuming that the matrix of second-order derivatives of internal energy, $\mathbf{H}$, is diagonal (see equation (13)). **Equality (2)** in equation (37) results from the thermodynamic admissibility condition of Yarushina and Podladchikov (2015), which leads to the relation $\tau_\phi^s = p_s - p_f = p_e/(1-\phi_f)$, derived in section 4.2.

We can infer the expression for β_ϕ introduced in equation (37), which directly follows from equation (13) once we substitute expressions for $H_{1,1}$ and $H_{2,2}$:

$$\frac{d\phi^e}{dt} = K_\phi(1-\phi)\frac{d\tau_\phi^s}{dt} = K_\phi\frac{dp_e}{dt},$$

$$\beta_\phi = \beta_d(1-\phi_f) - \beta_s. \tag{38}$$

~~The statement (37) means that changes in porosity are proportional to changes via τ_ϕ^s, which is the pressure difference $p_e/(1-\phi)$. Due to the requirement of zero entropy production, this statement provides us with the definition that equal changes in pressures leave porosity unchanged.~~

~~One of the key assumptions made during the original derivation of Gassmann's equations (Gassmann, 1951) is~~ **The proposed rheological relationship** (13) **and the equalities** (17) **and** (18) **inserted into the mass conservation equations** (25) **and** (26) **fully define the original Biot–Gassmann poroelasticity framework (Gassmann, 1951; Biot, 1962).** As a consequence, the theory contains three exact constitutive laws: (i) the *effective stress law* (explored below), (ii) *Gassmann relation* for the undrained bulk modulus $K_u = 1/\beta_u$ (β_u is the undrained compressibility), and (iii) the *relation between the effective compressibility* β_ϕ, the solid grains' compressibility β_s, and the drained (or dry) frame compressibility β_d.

5.7 Effective stress law

Nur and Byerlee (1971) provided an exact expression for the effective stress law, which is widely regarded as a fundamental result in poroelasticity. It is defined by the following relation:

$$dp_{\text{eff}} = d\bar{p} - \alpha\, dp_f \equiv d\bar{p} - \left(1 - \frac{\beta_s}{\beta_d}\right) dp_f, \tag{39}$$

where the drained compressibility, β_d, can be measured experimentally as:

$$\frac{1}{\beta_d} = -\frac{1}{\nabla_k v_k^s}\frac{dp_{\text{eff}}}{dt}\bigg|_{\text{undrained}}. \tag{40}$$

The exact effective stress law given by equation (39) follows directly from the derived poroelastic expressions.

5.8 Resulting equations of Biot-Gassmann theory for bulk moduli

To derive the original Biot poroelastic equations (Biot, 1962) in stiffness form, we invert the coefficient matrix in equation (35):

$$\left[\frac{1}{K_d}\begin{pmatrix} 1 & -\alpha \\ -\alpha & \dfrac{\alpha}{B}\end{pmatrix}\right]^{-1} = \frac{K_d}{1-\alpha B}\begin{pmatrix} 1 & B \\ B & \dfrac{B}{\alpha}\end{pmatrix}, \tag{41}$$

where $K_d = 1/\beta_d$ is the drained bulk modulus (i.e., β_d is the drained compressibility). The resulting expression for stiffness is:

$$\begin{pmatrix} \dfrac{d\bar{p}}{dt} \\ \dfrac{dp_f}{dt}\end{pmatrix} = -K_u\begin{pmatrix} 1 & B \\ B & \dfrac{B}{\alpha}\end{pmatrix}\begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D\end{pmatrix}, \tag{42}$$

where $K_u = K_d(1-\alpha B)^{-1}$. The poroelastic constants used in equation (42) are:

$$\alpha = 1 - \frac{K_d}{K_s}, \tag{43}$$

$$B = \frac{1/K_d - 1/K_s}{1/K_d - 1/K_s + \phi(1/K_f - 1/K_s)}, \tag{44}$$

where the bulk moduli are defined as the reciprocals of the corresponding compliance parameters: $\beta_s = 1/K_s$, and $\beta_f = 1/K_f$.

5.8.1 Original Gassmann's equations

The relation between the undrained bulk modulus K_u (see equation (42) under the constraint $\nabla_k q_k^D = 0$) and the drained bulk modulus K_d is known as Gassmann's equation (Gassmann, 1951):

$$K_u = K_d\,(1-\alpha B)^{-1}. \tag{45}$$

According to Gassmann's theory, the shear modulus of a fluid-saturated rock G_u, is equal to the shear modulus of the dry (drained) rock G_d:

$$G_u = G_d. \tag{46}$$

The expression (45) is obtained by inverting the coefficient matrix in equation (35), leading to the stiffness form given in equation (42). An English translation of the original German-language article by Gassmann (1951) is provided in Pelissier et al. (2007) . Gassmann's relation (45) can also be rewritten in terms of bulk modulus as:

$$K_u = K_d + \frac{(1 - K_d/K_s)^2}{\phi_f K_f^{-1} + (1-\phi_f)K_s^{-1} - K_d/K_s^2}. \tag{47}$$

5.8.2 Assumptions behind the derivation of original Gassmann's equations

The following assumptions are made throughout the derivation of Biot's poroelastic and Gassmann's equations to ensure the validity of the results:

- The material is assumed to be linearly elastic, and the strains are small.
- The porous medium is considered homogeneous and isotropic and a fully interconnected pore network.
- The interactions between the solid and fluid phases are governed by linear constitutive laws, and the fluid flow obeys Darcy's law (or equivalently, the fluid is governed by the quasi-static Navier–Stokes equations for a compressible fluid).
- The self-similarity hypothesis: that equal changes in pore (fluid) pressure and confining (total) pressure ~~leave the porosity unchanged. This assumption holds when considering a homogeneous elastic frame material (Korringa, 1981; Alkhimenkov, 2024) .Any discrepancy in~~ result in no change in porosity ϕ_f. This is equivalent to assuming a diagonal compliance matrix $\mathbf{H}$ (see equation (6)).
- The derivation assumes a quasi-static process, such that inertial effects can be neglected.

These assumptions provide a simplified framework for the derivation and are thermodynamically admissible. One of the key assumptions in the original derivation of Gassmann's equations (Gassmann, 1951) is the *self-similarity hypothesis* — equal changes in total and fluid pressure ~~changes will lead to porosity changes as follows from equation (37). As highlighted by Korringa (1981), applying confining (external) pressure to a homogeneous elastic frame material causes it to behave as a linear mapping. **Note that in the present thermodynamically admissible model, this is not assumed, but derived as a condition necessary to ensure zero entropy production during reversible poroelastic processes.**~~ leave porosity unchanged — explicitly stated in the original manuscript.

~~After simplifying and collecting terms (see Appendix B), the total entropy production becomes:~~

6 Derivation of the Extended Biot's poroelasticity formulation: General case

6.1 Goal

Recall the structure of the original Biot–Gassmann formulation (35):

$$\begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix} = -\beta_d \begin{pmatrix} 1 & -\alpha \\ -\alpha & \dfrac{\alpha}{B} \end{pmatrix} \begin{pmatrix} \dfrac{d^s \bar{p}}{dt} \\ \dfrac{d^s p_f}{dt} \end{pmatrix}, \tag{48}$$

This relationship was originally derived under the assumption that the Hessian matrix $\mathbf{H}$ is diagonal. Here, we aim to extend this result by retaining the full matrix $\mathbf{H}$, including its off-diagonal terms, and derive an analogous relationship that preserves the original structure and introduces generalized parameters. To this end, we follow the same steps as outlined in Section 5, with the goal of obtaining Gassmann-type relationships for the Extended Biot poroelastic theory.

6.2 Derivation

We now consider the full compliance matrix $\mathbf{H}$ (equation (6)):

$$TQ_{s,\text{total}} \begin{bmatrix} -\Delta p_s \\ -\dfrac{V_s}{\phi_s}\Delta(p_f - p_s) \end{bmatrix} = \frac{p_e}{(1-\phi)}^2 + \eta_t v^{s\,2} + \frac{(q^D)^2 \eta_{\text{dV}}}{\phi} + \frac{\lambda_t}{T}\frac{\partial T}{\partial x}^2 - \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \Delta V_s \\ -\Delta\phi_f \end{bmatrix}. \tag{49}$$

- ~~$\dfrac{1}{\eta_\phi}\left(\dfrac{p_e}{(1-\phi)}\right)^2$: Entropy production due to poroelastic deformation (poroelastic coefficient η_ϕ and pressure difference p_e).~~
- ~~$\eta_t\,(\text{div}\, v^s)^2$: Entropy production due to viscous dissipation in the solid phase.~~
- ~~$\dfrac{(q^D)^2\eta_{\text{dV}}}{\phi}$: Entropy production due to viscous dissipation in fluid flow (Darcy flow) .~~
- ~~$\dfrac{\lambda_t}{T}\left(\dfrac{\partial T}{\partial x}\right)^2$: Entropy production due to heat conduction (Fourier's law).~~

~~**The non-negative nature of each term ensures the overall positivity of entropy production, thereby confirming the thermodynamic validity of the system**.~~

~~For detailed derivations and applications of these principles to specific pore geometries and boundary conditions, readers are encouraged to refer to Appendix A, Appendix B, and~~ Note that $H_{12} = H_{21}$ due to the structure of the matrix $\mathbf{H}$: the ~~discussions provided by Yarushina and Podladchikov (2015). Additionally, symbolic Maple routines used to reproduce and validate the theoretical results presented in this article are available in a permanent DOI repository (Zenodo) *will be provide after review, now see suppl. material*. For a detailed explanation of the Maple script used in the derivation and analysis of entropy production in a single-phase medium, see Appendix A. Appendix B provides a similar explanation for the entropy production derivation in a two-phase porous medium.~~ off-diagonal component H_{12} corresponds to the second mixed partial derivative of internal energy, first with respect to V_s and then ϕ_f, and must be equal to H_{21}, which is the derivative taken in the opposite order. This symmetry holds because the internal energy is assumed to be a *smooth* (twice continuously differentiable)

scalar function of its state variables. (This is also known as the symmetry of second derivatives). Then, we follow a the same steps as in section 5 by using identities (14)-(15) and arrive to the following equations:

$$\frac{\Delta\phi_f}{\phi_f} = f(H_{11}, H_{12}, \Delta p_s, \Delta(p_f - p_s), \phi_f, \Delta\phi_f, V_s), \tag{50}$$

$$\frac{\Delta\rho_s}{\rho_s} = f(H_{22}, H_{12}, \Delta p_s, \Delta(p_f - p_s), \phi_f, \Delta\phi_f, V_s), \tag{51}$$

which are cumbersome and can be found in the Maple script. We then use identities (21)-(22). Following the steps provided in section 5, we substitute the resulting equations for $\frac{\Delta\phi_f}{\phi_f}$ (equation (50)) and $\frac{\Delta\rho_s}{\rho_s}$ (equation (51)), re-written in a rate form, into the mass conservation equations (25)-(26).

6.3 Resulting equations of the Extended Biot poroelastic theory

We again adopt the relation (28) and express p_s in terms of $\bar{p}$. Substituting equations (50)–(51) into the mass conservation equations (25)–(26) yields

$$\begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix} = \begin{pmatrix} a_{11}^{\mathrm{EB}} & a_{12}^{\mathrm{EB}} \\ a_{21}^{\mathrm{EB}} & a_{22}^{\mathrm{EB}} \end{pmatrix} \begin{pmatrix} \frac{d^s\bar{p}}{dt} \\ \frac{d^s p_f}{dt} \end{pmatrix}. \tag{52}$$

We note that $a_{12}^{\mathrm{EB}} = a_{21}^{\mathrm{EB}}$, which is not imposed by symmetry but emerges naturally from the substitution of equation (49) into the mass conservation equations (25)–(26). This symmetry is a direct consequence of the algebra.

6.4 ~~Two-phase media: fluid-saturated porous material~~

Following the approach of Section 5, we now define the compressibilities. First, we define

$$\beta_d^{\mathrm{EB}} = -a_{11}^{\mathrm{EB}}, \tag{53}$$

which gives:

$$\beta_d^{\mathrm{EB}} = -\frac{(-1+\phi_f)^2 H_{2,2} + V_s\left(V_s H_{1,1} - 2H_{1,2}(-1+\phi_f)\right)}{(-1+\phi_f)^3\left(H_{1,1}H_{2,2} - H_{1,2}^2\right)V_s}. \tag{54}$$

Then we introduce α^{EB} as

$$\alpha^{\mathrm{EB}} = \frac{a_{12}^{\mathrm{EB}}}{\beta_d^{\mathrm{EB}}} \equiv \frac{-V_s\phi_f^2 H_{1,2} + \phi_f^3 H_{2,2} + V_s^2 H_{1,1} - 2\phi_f^2 H_{2,2} + V_s H_{1,2} + \phi_f H_{2,2}}{(-1+\phi_f)^2 H_{2,2} + \left(V_s H_{1,1} - 2H_{1,2}(-1+\phi_f)\right)V_s}, \tag{55}$$

which gives

$$H_{2,2} = \frac{V_s\left(\alpha^{\mathrm{EB}}\phi_f H_{1,2}\beta_d^{\mathrm{EB}} - \alpha^{\mathrm{EB}}H_{1,2}\beta_d^{\mathrm{EB}} - \phi_f H_{1,2}\beta_d^{\mathrm{EB}} + H_{1,2}\beta_d^{\mathrm{EB}} + 1\right)}{\beta_d^{\mathrm{EB}}\left(\alpha^{\mathrm{EB}}\phi_f^2 - \phi_f^3 - 2\alpha^{\mathrm{EB}}\phi_f + 2\phi_f^2 + \alpha^{\mathrm{EB}} - \phi_f\right)} \tag{56}$$

Finally, we introduce B^{EB} as

$$B^{\mathrm{EB}} = -\frac{\alpha^{\mathrm{EB}}\beta_d^{\mathrm{EB}}}{a_{22}^{\mathrm{EB}}} \equiv \frac{(\beta_d^{\mathrm{EB}} - \beta_s'^{\mathrm{EB}})\left(1 + H_{1,2}(1-\phi_f)^2\beta_d^{\mathrm{EB}}\right)}{(1-\phi_f)^2\left((\beta_d^{\mathrm{EB}})^2 + (\phi_f\beta_f - 2\beta_s'^{\mathrm{EB}})\beta_d^{\mathrm{EB}} + (\beta_s'^{\mathrm{EB}})^2\right)H_{1,2} + \beta_d^{\mathrm{EB}} + (\beta_f - \beta_s'^{\mathrm{EB}})\phi_f - \beta_s'^{\mathrm{EB}}}, \tag{57}$$

where $\beta_s'^{\mathrm{EB}}$ is defined by the following relation: $\alpha^{\mathrm{EB}} = 1 - \frac{\beta_s'^{\mathrm{EB}}}{\beta_d^{\mathrm{EB}}}$. By using the definitions (53)-(57), we can rewrite (52) in the following form:

$$\begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix} = -\beta_d^{\mathrm{EB}} \begin{pmatrix} 1 & -\alpha^{\mathrm{EB}} \\ -\alpha^{\mathrm{EB}} & \frac{\alpha^{\mathrm{EB}}}{B^{\mathrm{EB}}} \end{pmatrix} \begin{pmatrix} \frac{d^s\bar{p}}{dt} \\ \frac{d^s p_f}{dt} \end{pmatrix}, \tag{58}$$

which is the incremental form of the large strain Extended Biot poroelastic formulation. Note that we did not define a particular expression for $H_{1,2}$ which can be set arbitrarily via introduction of a new parameter $\beta_s''^{\mathrm{EB}}$.

~~The equations governing fluid flow in poro-viscoelastoplastic media can be formulated based on the conservation laws and constitutive equations for both fluid and solid phases~~To derive the Extended Biot poroelasticity relations, we used only the proposed rheological relationship (49) and the two equalities (21) and (22). The relationship (49) denotes the following identity:

$$\frac{d^s\phi_f}{dt} = \frac{(1-\phi_f)^2\,\beta_s'^{\mathrm{EB}}\,H_{1,2}\left((1-\phi_f)\beta_d^{\mathrm{EB}} + \beta_s'^{\mathrm{EB}}\right)}{1 + H_{1,2}(1-\phi_f)^2\beta_d^{\mathrm{EB}}}\frac{d^s p_f}{dt} - \left((1-\phi_f)\beta_d^{\mathrm{EB}} - \beta_s'^{\mathrm{EB}}\right)\frac{d^s(\bar{p} - p_f)}{dt} \tag{59}$$

where the poroelastic constant (compressibility) β_ϕ^{EB} can be defined as a coeficient in front of effective pressure $d^s p_e = d^s(\bar{p} - p_f)$:

$$\beta_\phi^{\mathrm{EB}} = \beta_d^{\mathrm{EB}}(1-\phi_f) - \beta_s'^{\mathrm{EB}}. \tag{60}$$

Therefore, equation (59) can be written now as:

$$\frac{d^s\phi_f}{dt} = \frac{(1-\phi_f)^2\,\beta_s'^{\mathrm{EB}}\,H_{1,2}\left((1-\phi_f)\beta_d^{\mathrm{EB}} + \beta_s'^{\mathrm{EB}}\right)}{1 + H_{1,2}(1-\phi_f)^2\beta_d^{\mathrm{EB}}}\frac{d^s p_f}{dt} - \beta_\phi^{\mathrm{EB}}\frac{d^s p_e}{dt} \tag{61}$$

To further simplify the notation, we can introduce $\beta_s''^{\mathrm{EB}}$ and solve for $H_{1,2}$ the following equation:

$$\frac{(1-\phi_f)^2\,\beta_s'^{\mathrm{EB}}\,H_{1,2}\left((1-\phi_f)\beta_d^{\mathrm{EB}} + \beta_s'^{\mathrm{EB}}\right)}{1 + H_{1,2}(1-\phi_f)^2\beta_d^{\mathrm{EB}}} = \beta_s'^{\mathrm{EB}} - \beta_s''^{\mathrm{EB}}, \tag{62}$$

which gives

$$H_{1,2} = \frac{\beta_s'^{\mathrm{EB}} - \beta_s''^{\mathrm{EB}}}{(1-\phi_f)^2\left((\beta_s'^{\mathrm{EB}})^2 + \beta_d^{\mathrm{EB}}(\phi_f - 2)\beta_s'^{\mathrm{EB}} + \beta_d^{\mathrm{EB}}\beta_s''^{\mathrm{EB}}\right)}. \tag{63}$$

Substituting equation (63) in the expression for B (equation (57)) gives simplified relation:

$$B^{\mathrm{EB}} = \frac{\beta_d^{\mathrm{EB}} - \beta_s'^{\mathrm{EB}}}{(\beta_f - \beta_s'^{\mathrm{EB}})\phi_f + \beta_d^{\mathrm{EB}} - \beta_s''^{\mathrm{EB}}}. \tag{64}$$

We also note that the expression (55) for α^{EB} can be written as

$$\alpha^{\mathrm{EB}} = 1 - \frac{\beta_s'^{\mathrm{EB}}}{\beta_d^{\mathrm{EB}}}. \tag{65}$$

Furthermore, the equation (62) can now be re-written as

$$\frac{d^s \phi_f}{dt} = (\beta_s'^{\mathrm{EB}} - \beta_s''^{\mathrm{EB}})\frac{d^s p_f}{dt} - \beta_\phi^{\mathrm{EB}} \frac{d^s p_e}{dt}. \tag{66}$$

6.4 Relations between poroelastic parameters and H

We can write the relations between poroelastic parameters and $\mathbf{H}$ as follows:

$$\beta_s'^{\mathrm{EB}} = \frac{(1-\phi_f)H_{2,2} - V_s H_{1,2}}{V_s\left(H_{1,1}H_{2,2} - H_{1,2}^2\right)(1-\phi_f)} \tag{67}$$

and

$$\beta_s''^{\mathrm{EB}} = \frac{V_s(\phi_f - 2)H_{1,2} + (1-\phi_f)H_{2,2}}{V_s(H_{1,1}H_{2,2} - H_{1,2}^2)(1-\phi_f)}. \tag{68}$$

The relations between poroelastic parameters β_d^{EB} (equation (54)), $\beta_s'^{\mathrm{EB}}$ (equation (67)), $\beta_s''^{\mathrm{EB}}$ (equation (68)), α^{EB} (equation (55)), and B^{EB} (equation (57), in which β_d^{EB} and $\beta_s'^{\mathrm{EB}}$ are substituted) are fully expressed in terms of the components of the Hessian matrix $\mathbf{H}$.

6.4.1 ~~Conservation of linear momentum and Darcy's law~~

6.5 Gassmann-type relation

The ~~conservation of linear momentum is~~equations for the undrained compressibility in the framework of the Extended Biot poroelastic formulation is:

$$\nabla_j(-\delta_{ij} + {}_{ij}) - g_i \beta_u^{\mathrm{EB}} = 0\beta_d^{\mathrm{EB}} \left(1 - \alpha^{\mathrm{EB}} B^{\mathrm{EB}}\right), \tag{69}$$

~~where $\bar{p} = (1-\phi)p_s + \phi p_f$~~ which has a structure similar to the original Gassmann equation (45).

7 Comparison against previous poroelasticity models

In this section, we assume small strains to enable a direct comparison with other classical poroelasticity models, which are typically formulated within the infinitesimal deformation framework.

7.1 Comparison against poroelasticity model of Detournay and Cheng (1993)

7.1.1 Rheology

Detournay and Cheng (1993) postulate linear rheological relationships that connect the volumetric response of the porous medium to increments in fluid and effective pressures:

$$\begin{pmatrix} \frac{\Delta V_t}{V_t} \\ \frac{\Delta V_p}{V_p} \end{pmatrix} = - \begin{pmatrix} \beta_s'^{\mathrm{DC}} & \beta_d^{\mathrm{DC}} \\ \beta_s''^{\mathrm{DC}} & \beta_p'^{\mathrm{DC}} \end{pmatrix} \begin{pmatrix} dp_f \\ dp_e \end{pmatrix} \tag{70}$$

These expressions describe how the total volume V_t and pore volume V_p deform in response to changes in fluid pressure p_f and effective pressure $p_e = \bar{p} - p_f$, where $\bar{p}$ is the total pressure. The mechanical interpretation of the four compressibilities β_d^{DC}, β_p', ~~$\bar{\tau}_{ij}$ is the deviatoric stress tensor, δ_{ij} is the Kronecker delta, $i, j = \overline{1..3}$ and Einstein summation convention isused (summation is applied over repeated indexes). Viscous fluid flow through porous media is governed by Darcy's law:~~

$$q_i^{\mathrm{D}} = -\frac{k}{\eta_f}(\nabla_i p^f + g_i \rho^f),$$

$\beta_s'^{\mathrm{DC}}$, and $\beta_s''^{\mathrm{DC}}$ has been defined in Detournay and Cheng (1993). Note that by invoking the Betti-Maxwell reciprocal theorem, Detournay and Cheng (1993) suggest that $K_p' = \frac{\phi_f}{\alpha^{\mathrm{DC}} \beta_d^{\mathrm{DC}}}$, and $\beta_p'^{\mathrm{DC}} = 1/K_p'^{\mathrm{DC}}$.

7.1.2 Geometry and kinematics

Detournay and Cheng (1993) use exact relations that connect the total, solid, and pore volumetric responses with porosity changes. Assuming control volumes and using finite changes, the following identities hold:

$$\frac{\Delta V_t}{V_t} = \frac{\Delta V_s}{V_s} + \frac{\phi_f}{1-\phi_f}\frac{\Delta \phi_f}{\phi_f}, \tag{71}$$

$$\frac{\Delta V_p}{V_p} = \frac{\Delta V_s}{V_s} + \frac{1}{1-\phi_f}\frac{\Delta \phi_f}{\phi_f}. \tag{72}$$

~~where $q_i^D = \phi(v_i^f - v_i^s)$ denotes Darcy's flux, v_i^f denotes the fluid velocity, v_i^s denotes the solid velocity, k is permeability, η_f is fluid shear viscosity.~~

7.1.3 ~~Conservation of mass~~

7.1.3 Porosity evolution and solid-volume change

~~Conservation of mass for fluid phase is~~

$$\frac{\partial(\phi \rho_f)}{\partial t} + \nabla_j \left(\phi \rho_f v_j^f \right) = 0,$$

Combining the rheological relations (70) with the geometric identities (71)–(72) yields compact expressions for the porosity variation and the solid-volume strain (Detournay and Cheng, 1993):

$$\frac{\Delta\phi_f}{\phi_f} = -\frac{(\beta_\phi^{\mathrm{DC}})}{\phi_f}\, dp_e + (\beta_s'^{\mathrm{DC}} - \beta_s''^{\mathrm{DC}})\, dp_f, \tag{73}$$

$$\phi_s \frac{\Delta V_s}{V_s} = -\beta_s'^{\mathrm{DC}}\, dp_e - \left(\beta_s'^{\mathrm{DC}} - \phi_f\, \beta_s''^{\mathrm{DC}}\right) dp_f, \tag{74}$$

where ~~ρ_f denotes fluid density and conservation of mass for solid phase is~~ $(\beta_\phi^{\mathrm{DC}}) = \beta_d^{\mathrm{DC}}(1-\phi_f) - \beta_s'^{\mathrm{DC}}$.

The resulting representation of Detournay and Cheng (1993) is:

$$\begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix} = -\beta_d^{\mathrm{DC}} \begin{pmatrix} 1 & -\alpha^{\mathrm{DC}} \\ -\alpha^{\mathrm{DC}} & \dfrac{\alpha^{\mathrm{DC}}}{B^{\mathrm{DC}}} \end{pmatrix} \begin{pmatrix} \dfrac{d\bar{p}}{dt} \\ \dfrac{dp_f}{dt} \end{pmatrix} \tag{75}$$

The inverse form, expressing the time evolution of pressure fields in terms of mechanical and hydraulic divergence rates, reads:

$$\begin{pmatrix} \dfrac{d\bar{p}}{dt} \\ \dfrac{dp_f}{dt} \end{pmatrix} = -K_u^{\mathrm{DC}} \begin{pmatrix} 1 & B^{\mathrm{DC}} \\ B^{\mathrm{DC}} & \dfrac{B^{\mathrm{DC}}}{\alpha^{\mathrm{DC}}} \end{pmatrix} \begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix}, \tag{76}$$

The poroelastic constants used in equations (75)–(76) are ($K_d' = 1/\beta_d^{\mathrm{DC}}$, $K_s'^{\mathrm{DC}} = 1/\beta_s'^{\mathrm{DC}}$, $K_s''^{\mathrm{DC}} = 1/\beta_s''^{\mathrm{DC}}$):

$$\alpha^{\mathrm{DC}} = 1 - \frac{\beta_s'^{\mathrm{DC}}}{\beta_d^{\mathrm{DC}}}, \tag{77}$$

$$B^{\mathrm{DC}} = \frac{\beta_d^{\mathrm{DC}} - \beta_s'^{\mathrm{DC}}}{\beta_d^{\mathrm{DC}} - \beta_s'^{\mathrm{DC}} + (\beta_f - \beta_s''^{\mathrm{DC}})\,\phi_f}, \tag{78}$$

$$\beta_u^{\mathrm{DC}} = \beta_d^{\mathrm{DC}} \left(1 - \alpha^{\mathrm{DC}}\, B^{\mathrm{DC}}\right), \tag{79}$$

$$K_u^{\mathrm{DC}} = K_d' \left(1 - \alpha^{\mathrm{DC}}\, B^{\mathrm{DC}}\right)^{-1} \equiv K_d + \frac{\left(1 - K_d'/K_s'^{\mathrm{DC}}\right)^2}{\phi\left(K_f^{-1} - (K_s''^{\mathrm{DC}})^{-1}\right) + (K_s'^{\mathrm{DC}})^{-1} - K_d'\,(K_s'^{\mathrm{DC}})^{-2}}. \tag{80}$$

This expression has a similar structure to the original Gassmann equation (45). We emphasize that these expressions arise naturally as a special case of the present Extended Biot poroelastic formulation, which is shown below. In particular, the Detournay–Cheng model assumes small strains and constant poroelastic parameters, whereas in our framework — large strain incremental formulation is adopted, thus, porosity evolution is present and the coupling coefficient $B^{\mathrm{EB}}(\phi_f)$ vary with porosity.

7.2 Comparison against the poroelasticity model of Brown and Korringa (1975) and Rice and Cleary (1976)

The poroelasticity formulation of Brown and Korringa (1975) can be rewritten using the notation introduced by Thomsen (2025) , in terms of the drained bulk modulus $K_d^{\mathrm{BK}} = 1/\beta_d^{\mathrm{BK}}$, the "mean" grain modulus $K_M^{\mathrm{BK}} = 1/\beta_M^{\mathrm{BK}}$ and the overall modulus of the heterogeneous solid constituent of the rock $K_S^{\mathrm{BK}} = 1/\beta_S^{\mathrm{BK}}$.

$$\begin{pmatrix} \dfrac{\Delta V_t}{V_t} \\ \dfrac{\Delta V_p}{V_p} \end{pmatrix} = - \begin{pmatrix} \beta_M^{\mathrm{BK}} & \beta_d^{\mathrm{BK}} \\ \beta_\phi^{\mathrm{BK}} & \beta'^{\mathrm{BK}} \end{pmatrix} \begin{pmatrix} dp_f \\ dp_e \end{pmatrix} \tag{81}$$

The drained compressibility is defined as (Brown and Korringa, 1975; Thomsen, 2025):

$$+\nabla_j \beta_d^{\mathrm{BK}} = -\frac{1}{V_t} \left((1-\phi)\rho_s v_j^s \frac{\partial V_t}{\partial p_e} \right) = 0_{p_f}, \tag{82}$$

where ~~ρ_s denotes solid density. Equations (23)-(24) can be reformulated for divergences $\nabla_j v_j^s$ and $\nabla_j q_j^D$:~~ p_e is the effective (or differential) pressure, $p_e = \bar{p} - p_f$. The compressibility with respect to pore pressure at constant total stress is (Brown and Korringa, 1975; T :

$$\beta_M^{\mathrm{BK}} = -\frac{1}{V_t} \left(\frac{\partial V_t}{\partial p_f} \right)_{p_e}. \tag{83}$$

The undrained compressibility is (Brown and Korringa, 1975; Thomsen, 2025):

$$\beta_u^{\mathrm{BK}} = -\frac{1}{V_t} \left(\frac{\partial V_t}{\partial \bar{p}} \right). \tag{84}$$

Brown and Korringa (1975); Thomsen (2025) introduce the following compressibilities for the pore volume:

$$\beta'^{\mathrm{BK}} = -\frac{1}{V_t} \left(\frac{\partial V_p}{\partial p_e} \right)_{p_f}, \tag{85}$$

$$\beta_\phi^{\mathrm{BK}} = -\frac{1}{V_p} \left(\frac{\partial V_p}{\partial p_e} \right)_{p_f}, \tag{86}$$

$$\beta_f = -\frac{1}{V_p} \left(\frac{\partial V_p}{\partial p_f} \right)_{p_e}. \tag{87}$$

Thus, the variation of pore volume can be written as (Brown and Korringa, 1975; Thomsen, 2025):

$$\phi_f \beta_f \delta p_f = \beta'^{\mathrm{BK}} \Delta p_e + \beta_\phi^{\mathrm{BK}} \Delta p_f. \tag{88}$$

Finally, the undrained compressibility can be written as:

$$\beta_u^{\mathrm{BK}} = \beta_d^{\mathrm{BK}} - \frac{(\beta_d^{\mathrm{BK}} - \beta_M^{\mathrm{BK}})^2}{\phi_f(\beta_f - \beta_\phi^{\mathrm{BK}}) + (\beta_d^{\mathrm{BK}} - \beta_M^{\mathrm{BK}})}. \tag{89}$$

Thomsen (2025) used the following identity:

$$\beta_M^{\mathrm{BK}} = \phi_f \beta_\phi^{\mathrm{BK}} + (1-\phi_f)\,\beta_S^{\mathrm{BK}}. \tag{90}$$

Brown and Korringa (1975) also showed that $\beta'^{\mathrm{BK}} = \beta_d^{\mathrm{BK}} - \beta_M^{\mathrm{BK}}$. Finally, the resulting expression of Brown and Korringa (1975) for the undrained compressibility β_u^{BK} in the notation provided by Thomsen (2025):

$$\nabla_j v_j^s \beta_u^{\mathrm{BK}} = \beta_d^{\mathrm{BK}} - \pm \frac{(\beta_d^{\mathrm{BK}} - \beta_M^{\mathrm{BK}})^2}{\phi_f(\beta_f - \beta_S^{\mathrm{BK}}) + (\beta_d^{\mathrm{BK}} - \beta_S^{\mathrm{BK}} - 2\beta_M^{\mathrm{BK}})}, \tag{91}$$

~~and~~ or, in terms of bulk moduli, which can be explicitly written as ($K_u^{\mathrm{BK}} = 1/\beta_u^{\mathrm{BK}}$, $K_d^{\mathrm{BK}} = 1/\beta_d^{\mathrm{BK}}$, $K_S^{\mathrm{BK}} = 1/\beta_S^{\mathrm{BK}}$, $K_M^{\mathrm{BK}} = 1/\beta_M^{\mathrm{BK}}$, $K_f = 1/\beta_f$) (Thomsen, 2025):

$$\nabla_j q_j^D K_u^{\mathrm{BK}} = K_d' \left(1 - \underline{-}\underline{-}\phi \nabla_j v_j^s \alpha^{\mathrm{BK}} B^{\mathrm{BK}}\right)^{-1} \equiv K_d^{\mathrm{BK}} + \frac{(1 - K_d^{\mathrm{BK}}/K_M^{\mathrm{BK}})^2}{\phi_f\left(K_f^{-1} - (K_S^{\mathrm{BK}})^{-1}\right) + (K_S^{\mathrm{BK}})^{-1} - K_d^{\mathrm{BK}}/(K_M^{\mathrm{BK}})^2}, \tag{92}$$

~~where $\frac{d^s}{dt} = \frac{\partial}{\partial t} + v_i^s \nabla_i$ denotes the Lagrangian (material) derivative with respect to solid and $\frac{d^f}{dt} = \frac{\partial}{\partial t} + v_i^f \nabla_i$ denotes the Lagrangian (material)derivative with respect to fluid~~where

$$\alpha^{\mathrm{BK}} = 1 - \frac{\beta_M^{\mathrm{BK}}}{\beta_d^{\mathrm{BK}}} \tag{93}$$

and B^{BK} can be calculated from the equality (92).

7.3 Equivalence of the Brown–Korringa (BK) model and Detournay–Cheng (DC) model

The Detournay–Cheng (DC) model is fully equivalent to the Brown–Korringa model if a proper mapping between the poroelastic parameters is established (i.e., $K_s'^{\mathrm{DC}}$ and $K_s''^{\mathrm{DC}}$ to K_M^{BK} and K_S^{BK}). Using the assignments:

$$K_M^{\mathrm{BK}} = K_s'^{\mathrm{DC}}, \qquad K_S^{\mathrm{BK}} = \frac{\phi_s\, K_s'^{\mathrm{DC}}\, K_s''^{\mathrm{DC}}}{K_s'^{\mathrm{DC}} - \phi_f\, K_s'^{\mathrm{DC}}}, \tag{94}$$

we find that the two models — the DC model and the Brown–Korringa model — are algebraically identical. When $K_s'^{\mathrm{DC}} = K_s''^{\mathrm{DC}}$, it immediately follows that $K_M^{\mathrm{BK}} = K_S^{\mathrm{BK}}$, and the two models reduce to the classical Biot–Gassmann formulation.

The algebraic equivalence between these formulations can be also established by the following exact relation:

$$\frac{1}{K_s'^{\mathrm{DC}}} - \phi_f \frac{1}{K_s''^{\mathrm{DC}}} = \frac{\phi_s}{K_S^{\mathrm{BK}}}. \tag{95}$$

7.3.1 ~~Constitutive relations~~

This analysis shows that the Brown–Korringa model is distinct from the Detournay–Cheng formulation in terms of the parameter definitions and the physical interpretation and experimental measurability of the poroelastic coefficients.

~~Elastic compressibility for fluid and solid densities is formulated as (Yarushina and Podladchikov, 2015):~~

7.4 Equivalence of the present Extended Biot formulation and Detournay–Cheng (DC) model

Here we show that the present Extended Biot formulation contains the Detournay–Cheng (DC) model as a special case. Indeed, if we set $\beta_s'^{\mathrm{EB}} = \beta_s'^{\mathrm{DC}}$, and choose

$$H_{1,2} = \frac{\phi_f \left(\beta_s'^{\mathrm{EB}} - \beta_s''^{\mathrm{EB}}\right)}{\phi_f^3 \beta_d^{\mathrm{EB}} \beta_s''^{\mathrm{EB}} - \phi_f^2 \beta_d^{\mathrm{EB}} \beta_s'^{\mathrm{EB}} - 2\phi_f^2 \beta_d^{\mathrm{EB}} \beta_s''^{\mathrm{EB}} + \phi_f^2 (\beta_s'^{\mathrm{EB}})^2 + 2\phi_f \beta_d^{\mathrm{EB}} \beta_s'^{\mathrm{EB}} + \phi_f \beta_d^{\mathrm{EB}} \beta_s''^{\mathrm{EB}} - 2\phi_f (\beta_s'^{\mathrm{EB}})^2 - \beta_d^{\mathrm{EB}} \beta_s'^{\mathrm{EB}} + (\beta_s'^{\mathrm{EB}})^2}, \tag{96}$$

$$\frac{K_s}{\rho_s} \frac{d^s \rho_s}{dt} = \frac{1}{1-\phi} \left(\frac{d^s \bar{p}}{dt} - \phi \frac{d^f p_f}{dt} \right),$$

the present Extended Biot formulation will be exactly equivalent to the Detournay–Cheng (DC) model in the small strain regime. We refer to the provided Maple script for more details.

7.5 Equivalence of the present Extended Biot formulation and Brown–Korringa (BK) model

Here we show that the present Extended Biot formulation contains the Brown–Korringa (BK) model as a special case. Indeed, if we set $\beta_s'^{\mathrm{EB}} = \beta_M^{\mathrm{BK}}$, use identity (94), and choose

$$H_{1,2} = \frac{\beta_M^{\mathrm{BK}} - \underline{\beta}_S^{\mathrm{BK}}}{\phi_f^2 \beta_d^{\mathrm{EB}} \underline{\beta}_S^{\mathrm{BK}} - 2\phi_f \beta_d^{\mathrm{EB}} \underline{\beta}_S^{\mathrm{BK}} + \phi_f \left(\beta_M^{\mathrm{BK}}\right)^2 + \beta_d^{\mathrm{EB}} \underline{\beta}_S^{\mathrm{BK}} - \left(\beta_M^{\mathrm{BK}}\right)^2}, \tag{97}$$

~~where K_f denotes the fluid bulk modulus and K_s denotes the solid bulk modulus. A closing relation is the equation governing porosity evolution (Maxwell viscoelastic volumetric response) :~~ the present Extended Biot formulation will be exactly equivalent to the Brown–Korringa (BK) model in the small strain regime. We refer to the provided Maple script for more details.

8 A closed system of equations of the extended Biot poroelastic framework

The conservation of linear momentum is given by:

$$= -+ \nabla_j (p_f - \bar{p}\delta_{ij} + \bar{\tau}_{ij}) - g_i \bar{\rho} = 0, \tag{98}$$

where ~~K_ϕ is the poroelastic constant defined by equation (37).~~ $\bar{\tau}_{ij}$ is the deviatoric stress tensor, δ_{ij} is the Kronecker delta, and $i, j = 1, 2, 3$. The total density is given by $\bar{\rho} = \phi_s \rho^s + \phi_f \rho^f$, where ρ^s and ρ^f are the solid and fluid densities, respectively. The vector g_i denotes the components of gravitational acceleration.

8.0.1 ~~Resulting evolution equations for poro-viscoelastoplastic media~~

Viscous fluid flow through the porous medium is governed by Darcy's law:

$$q_i^{\mathrm{D}} = -\frac{k}{\eta_f}(\nabla_i p_f + g_i \rho^f), \tag{99}$$

where k is the permeability of the medium, and η_f is the fluid shear viscosity

~~By eliminating the time derivatives of densities and porosity in equations (25)-(??) using expressions (??)-(??), the following system of equations for compressibilities is obtained (Yarushina and Podladchikov, 2015):~~

The matrix of coefficients in equation (58) can be inverted, yielding:

$$\begin{pmatrix} \dfrac{d^s \bar{p}}{dt} \\ \dfrac{d^f p_f}{dt} \end{pmatrix} = -\frac{1}{\beta_u^{\mathrm{EB}}(\phi_f)} \begin{pmatrix} 1 & B^{\mathrm{EB}}(\phi_f) \\ B^{\mathrm{EB}}(\phi_f) & \dfrac{B^{\mathrm{EB}}(\phi_f)}{\alpha^{\mathrm{EB}}} \end{pmatrix} \begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix} -, \tag{100}$$

where the abbreviated definition $\beta_u^{\mathrm{EB}} = \beta_u^{\mathrm{EB}}(\phi_f) = \beta_d^{\mathrm{EB}}\left(1 - \alpha^{\mathrm{EB}} B^{\mathrm{EB}}\right)$ is used, and the parameters are functions of porosity ϕ_f, meaning that $B^{\mathrm{EB}} = B^{\mathrm{EB}}(\phi_f)$.

Deviatoric stresses are related to solid velocity gradients through the ~~Maxwell viscoelastic relationship(Beuchert and Podladchikov, 2010~~ following relationship:

$$\frac{1}{G_u}\frac{d^{\nabla}\bar{\tau}_{ij}}{dt} + = \frac{1}{2}(\nabla_j v_i^s + \nabla_i v_j^s) - \frac{1}{3}(\nabla_k v_k^s)\delta_{ij}, \tag{101}$$

where ~~G_{sat} is the~~ G_u is the undrained shear modulus of the ~~fluid-saturated porous material,~~ ~~$\frac{d^{\nabla}\bar{\tau}_{ij}}{dt} = \frac{d^s\bar{\tau}_{ij}}{dt} - \bar{\tau}_{ik}\omega_{kj} - \bar{\tau}_{jk}\omega_{ki}$~~ ~~correspond to~~ saturated porous medium (it is assumed that the dry or drained shear modulus is equivalent to G_u, i.e., $G_d = G_u$), and

$$\frac{d^{\nabla}\bar{\tau}_{ij}}{dt} = \frac{d^s\bar{\tau}_{ij}}{dt} - \bar{\tau}_{ik}\omega_{kj} - \bar{\tau}_{jk}\omega_{ki} \tag{102}$$

is the Jaumann objective stress rate ~~and $\omega_{ki} = \frac{1}{2}(\nabla_k v_i^s - \nabla_i v_k^s)$~~. The tensor $\omega_{ki} = \frac{1}{2}(\nabla_k v_i^s - \nabla_i v_k^s)$ denotes the antisymmetric part of the solid velocity gradient. ~~The~~

The poroelastic constants in expression (100) can be defined in terms of compliance parameters as:

$$\alpha^{\mathrm{EB}} \equiv \alpha^{\mathrm{EB}} = 1 - \frac{\beta_s'^{\mathrm{EB}}}{\beta_d^{\mathrm{EB}}}, \tag{103}$$

$$B^{\mathrm{EB}} \equiv B^{\mathrm{EB}}(\phi_f) = \frac{\beta_d^{\mathrm{EB}} - \beta_s'^{\mathrm{EB}}}{(\beta_f - \beta_s'^{\mathrm{EB}})\phi_f + \beta_d^{\mathrm{EB}} - \beta_s''^{\mathrm{EB}}}\cdot, \tag{104}$$

$$\beta_u^{\mathrm{EB}} \equiv \beta_u^{\mathrm{EB}}(\phi_f) = \beta_d\left(1 - \alpha^{\mathrm{EB}} B^{\mathrm{EB}}(\phi_f)\right), \tag{105}$$

where β_d^{EB} corresponds to the drained (or dry) compressibility and β_u^{EB} denotes the undrained compressibility. Note that the porosity ϕ_f evolves according to the evolution equation (66), which in turn affects the poroelastic parameter $B^{\mathrm{EB}} = B^{\mathrm{EB}}(\phi_f)$ at each loading increment. Finally, we can use the Carman–Kozeny relationship ~~for permiability~~ to model permeability evolution as a function of porosity ~~is~~

$$k = k_0 \left(\frac{\phi}{\phi_0} \right)^{n_k},$$

~~where $n_k = 3.$~~ (where ϕ_0 is the reference porosity of the medium and k_0 is the reference permeability), given by:

$$k = k_0 \left(\frac{\phi_f}{\phi_0} \right)^{n_k}, \quad \text{where, e.g., } n_k = 3. \tag{106}$$

8.1 ~~Linear elastic limit ($\eta_\phi \to +\infty$): Biot's poroelastic equations~~

Equations (98)–(106) fully represent the quasi-static Extended Biot poroelasticity formulation.

~~Under the small strain approximation and infinite η_ϕ, a linear elastic limit of expression (35) can be derived which is know as Biot's poroelastic equations (Biot, 1962):~~

$$\begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix} = -\frac{1}{K_d} \begin{pmatrix} 1 & -\alpha \\ -\alpha & \dfrac{\alpha}{B} \end{pmatrix} \begin{pmatrix} \dfrac{d\bar{p}}{dt} \\ \dfrac{dp_f}{dt} \end{pmatrix}.$$

~~The system of equations (??) can be rewritten for stiffness. For that let us invert the matrix of coefficients:~~

$$\left[\frac{1}{K_d} \begin{pmatrix} 1 & -\alpha \\ -\alpha & \dfrac{\alpha}{B} \end{pmatrix} \right]^{-1} = \frac{K_d}{\alpha/B - \alpha^2} \begin{pmatrix} \dfrac{\alpha}{B} & \alpha \\ \alpha & 1 \end{pmatrix} \equiv \frac{K_d}{1 - \alpha B} \begin{pmatrix} 1 & B \\ B & \dfrac{B}{\alpha} \end{pmatrix}.$$

~~The resulting expression for stiffness is:~~

$$\begin{pmatrix} \dfrac{d\bar{p}}{dt} \\ \dfrac{dp_f}{dt} \end{pmatrix} = -K_u \begin{pmatrix} 1 & B \\ B & \dfrac{B}{\alpha} \end{pmatrix} \begin{pmatrix} \nabla_k v_k^s \\ \nabla_k q_k^D \end{pmatrix},$$

~~where $K_u = K_d(1 - \alpha B)^{-1}$. Poroelastic constants in the expressions (35)-(42) are the following:~~

$$\alpha = 1 - \frac{K_d}{K_s}$$

~~and~~

$$B = \frac{1/K_d - 1/K_s}{1/K_d - 1/K_s + \phi(1/K_f - 1/K_s)}.$$

9 Numerical studies supporting Gassmann's equations for monomineralic frame

~~The relation between K_d, K_s and K_ϕ (defined by equation (37)) is~~

$$\frac{1}{K_\phi} = \frac{1-\phi}{K_d} - \frac{1}{K_s}.$$

~~Various poroelastic constants can be calculated numerically (Alkhimenkov, 2023) or measured using physical experimentation in a laboratory (Makhnenko and Podladchikov, 2018)~~Alkhimenkov (2023) performed a numerical validation of Gassmann's equations considering a 3D numerical setup and relatively complex pore geometry that included narrow regions (cracks) and large pore space (Figure 1a-b). The numerical model consisted of a solid phase representing the grain matrix and a pore space. The model was cubic, with dimensions of $0.44 \times 0.44 \times 0.44\,\mathrm{m}$. The pore space comprised cracks, modeled as flat cylinders, connected to an internal cubic cavity, as illustrated in Figure 1a-b. The material properties used in the simulations are listed in Table 1, while the geometrical characteristics of the pore space are provided in Table 2.

9.1 ~~Gassmann's equations~~

Table 1. Material properties used in all simulations.

Material parameter	Solid grains	Fluid
Bulk modulus K	36 GPa	4.3 GPa
Shear modulus μ	44 GPa	– GPa
Shear viscosity η	– Pa·s	1.414 Pa·s

~~The relation between undrained response, K_u (see expression (42) under $\nabla_k q_k^D = 0$) , and drained response, K_d, is known as~~

Table 2. Geometrical properties of the model.

Geometrical parameter	Value
Flat cylinder (crack) radius, b (m)	0.2
Flat cylinder (crack) thickness, h (m)	0.016
Crack aspect ratio, $\alpha = h/(2b)$	0.04
Side of internal cubic pore (m)	0.25
Volume of the pore space (m^3)	0.01854
Total porosity	≈ 0.2176465

Alkhimenkov (2023) applied a 3D finite-element method to resolve the conservation of linear momentum coupled with the stress-strain relations for the solid phase and the quasi-static linearized compressible Navier-Stokes momentum equation for the fluid phase. The resulting system of equations was solved using a direct PARDISO solver (Schenk and Gärtner, 2004). Alkhimenkov (2023) conducted a convergence study showing that, for finer resolution, the result of the numerical solution

converges towards the result obtained from the original Gassmann's equation~~(Gassmann, 1951):~~

$$K_u = K_d\,(1-\alpha B)^{-1}.$$

~~According to~~ . Such a convergence analysis validates the accuracy of Gassmann's ~~equations, shear modulus of a fluid-saturated rock, G_{sat}, is equivalent to the shear modulus of a dry rock, G_d (equivalent to a drained response) :~~

$$G_{sat} = G_d.$$

~~The expression (??) is derived from the equation (??) via inversion of matrix of coefficients leading to the expression (42). Note that English translation of the the original paper by Gassmann(Gassmann, 1951) is presented by Pelissier et al. (2007)~~equation for a particular (but arbitrary) pore geometry. Furthermore, the pore geometry that was used did not contain any special features (among all possible geometries) that were tailored to make it consistent with Gassmann's equations (Alkhimenkov, 2024). There are also other 3D numerical studies that consider different geometries of the pore space and that are consistent with Gassmann's equations (Alkhimenkov et al., 2020a, b; Alkhimenkov and Quintal, 2022a, b).

9.1 ~~Effective stress law~~

~~Nur and Byerlee (1971) provided the exact expressions for the effective stress law, which can be treated as an exact result in poroelasticity. It is defined by the following expression (Yarushina and Podladchikov, 2015):~~

$$dp_{\text{eff}} = d\bar{p} - \alpha\, dp_f \equiv d\bar{p} - \left(1 - \frac{K_d}{K_s}\right) dp_f,$$

~~where K_d can be measured as~~

$$K_d = -\left.\frac{1}{\nabla_k v_k^s}\frac{dp_{\text{eff}}}{dt}\right|_{\text{undrained}}.$$

~~The exact effective stress law given by the formula (39) strictly follows from the derived expression (??)~~We here extend the results of Alkhimenkov (2023) for a denser finite element mesh (achieving $2,025,916$ elements)) and report the convergence study showing that, for finer resolution, the result of the numerical solution converges towards the result obtained from the original Gassmann's equation (Figure 1c-d).

10 Discussion

10.1 Physical ~~Interpretation~~ interpretation of the ~~Derived Equations~~~~The derived~~ present Extended Biot's poroelastic framework

The derived Extended Biot's poroelastic equations describe the coupled mechanical and fluid flow behavior of a fluid-saturated porous medium under general conditions. Specifically, they account for the interaction between ~~the~~ solid matrix deformation

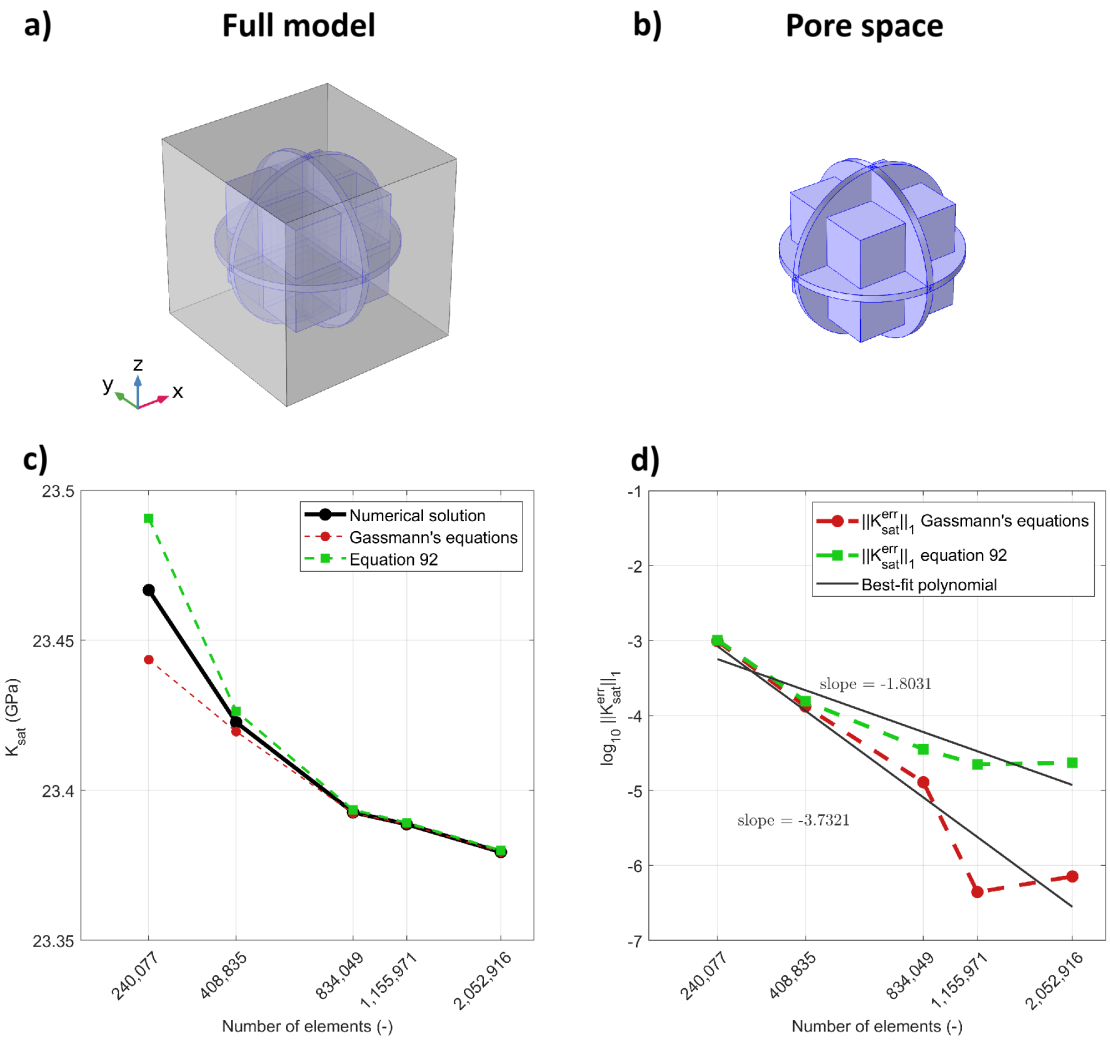

Figure 1. Panels (a)-(b) show sketch illustrating the model geometry. Panel (c) shows the numerical solution of K_u, the analytical solution via Gassmann's equations (47), and the analytical solution via equation (92) as a function of the numerical resolution. Panel (d) shows the error magnitudes between (i) the numerically evaluated bulk modulus K_u and the analytically evaluated bulk modulus via Gassmann's equations (47) and (ii) the numerically evaluated bulk modulus and the analytically evaluated bulk modulus via equation (92).

and the changes in pore fluid pressurechanges. The effective stress law, which modifies the classical elastic stress by incorporating fluid pressure, plays a key role in understanding how external loads and fluid injection or extraction influence the stability and deformation of the porous medium.

. Classical Biot's equations (Biot, 1962) and Gassmann's equations (Gassmann, 1951) are special cases of the presented theory. Gassmann's equations provide a relation between the bulk moduli of the dry and drained (or dry) and undrained fluid-saturated rock, offering insights insight into how fluid properties and porosity affect the seismic influence the mechanical response of the material. The results show that under the assumption of quasi-static conditions and small perturbations, the derived equations capture the essential physics of wave propagation and attenuation in fluid-saturated media.

10.2 Derivation Other derivations of Gassmann's equationsand relation to poroelasticity

Gassmann's equations are directly related to the quasi-static (Biot, 1941) and dynamic poroelasticity (Biot, 1956, 1962)formulation of poroelasticity developed by Biot (1941), and later extended to dynamic settings by Biot (1956, 1962). While the roots of the elastodynamic poroelasticity(e.g., conceptual foundation of elastodynamic poroelasticity—such as the presence of the fast P-wave, slow P-wave and shear wave in fluid-saturated porous media) were provided —was introduced by Frenkel (1944) (see also Pride and Garambois (2005)), a rigorous derivation of poroelastic equations and parameters were presented a few years later rigorous derivations of the poroelastic parameters were provided subsequently by Biot (1941); Biot and Willis (1957); Biot (1962). Many researchers have fully

Numerous researchers have rederived Gassmann's equations relying on different methods (or explored using various approaches or examined specific aspects of Gassmann's equations in the framework of poroelasticity) (Brown and Korringa, 1975; Korringa, 1981; Bur . Of course, a these equations within the poroelasticity framework (Brown and Korringa, 1975; Rice and Cleary, 1976; Korringa, 1981; Bur . Some modifications of small-strain poroelasticity to include non-reciprocal effects are given by Sahay (2013); Müller and Sahay (2019) . While the full list of scientist who contributed to poroelasticity is large, and while contributors to the field is extensive and beyond the scope of this paper, we acknowledge their extensive contributions, our intention in this short article is not to provide an exhaustive list. An interested reader is referred foundational work.

We refer the reader to Sevostianov (2020), which provides an extensive review of Gassmann'presents a comprehensive overview of Gassmann's equations. There are several books that also might be useful , e.g., In addition, several books may be useful for readers interested in poroelasticity and its applications, including: Bourbié et al. (1987), Zimmerman (1990), Wang (2000), Ulm and Coussy (2003), Coussy (2004, 2011), Guéguen and Boutéca (2004), Dormieux et al. (2006), Cheng (2016), and Mavko et al. (2020).

10.2.1 Thermodynamically admissible conditions for the diagonal structure of matrix H

The main assumptions behind the applicability of Gassmann's equations (??) (46) (45)–(47) are: (i) Linear elasticity; linear elasticity, (ii) Small strains; small strains, (iii) Isotropic an isotropic, homogeneous frame material ; and isotropic, homogeneous solid grains, (iv) Isotropic an isotropic dry response (note that although Gassmann's original publication contains includes an

extension to anisotropy)~~;~~, and (v) ~~Assumption~~ self-similarity hypothesis: the assumption that equal changes in pore (fluid) pressure and confining (total) pressure leave the porosity unchanged (Korringa, 1981; Alkhimenkov, 2024).

Assumption (v) ~~holds~~ may hold for isotropic homogeneous frame ~~material (Korringa, 1981)~~materials (Korringa, 1981), but it must be derived rigorously. In the framework of the present study, this condition is satisfied ~~and~~ when the compliance matrix $\mathbf{H}$ is diagonal, and it is required for ~~thermodynamic admissibility (see expressions (??)-(37) and the explanation therein): "The constraint of zero dissipation (entropy production)during reversible poroelastic deformation provides an essential constraint on the poroelastic constitutive equation for porosity evolution." In other words, in the present thermodynamically admissible model , (v) is not an assumption but a strict requirement for zero entropy production during reversible poroelastic processes.~~ the thermodynamic admissibility of the model (see Appendix B). As stated there: *"The constraint of zero dissipation (entropy production) during reversible poroelastic deformation provides an essential constraint on the poroelastic constitutive equation for porosity evolution."*

10.3 ~~Numerical validation of Gassmann's equations~~

10.2.1 When the solid compressibilities coincide ($\beta_s'^{\mathrm{EB}} = \beta_s''^{\mathrm{EB}} = \beta_s = \beta_s'^{\mathrm{DC}} = \beta_s''^{\mathrm{DC}} = \beta_M^{\mathrm{BK}} = \beta_S^{\mathrm{BK}}$)

~~Alkhimenkov (2023) performed a numerical validation of Gassmann's equations considering a~~ Strictly speaking, the most general model should always use the full matrix $\mathbf{H}$ (equation (6)). However, in certain special cases—such as isotropic and homogeneous rock frames—additional constraints may hold. Several researchers have pointed out that for monomineralic, isotropic materials, the self-similarity hypothesis is valid, and therefore Gassmann's equations apply and are exact (Brown and Korringa, 19

.

In general, various poroelastic constants can be computed numerically (Alkhimenkov, 2023), derived analytically using effective medium theory (Yarushina and Podladchikov, 2015), or measured experimentally in laboratory settings (Makhnenko and Podladch

.

The distinction between the solid compressibilities lies in the structure of the matrix $\mathbf{H}$, which depends on the particular choice of rheological relationships. The definitions (Detournay and Cheng, 1993):

$$\beta_s = \frac{1}{K_s}, \qquad \beta_s'^{\mathrm{DC}} = \frac{1}{K_s'^{\mathrm{DC}}}, \qquad \beta_s''^{\mathrm{DC}} = \frac{1}{K_s''^{\mathrm{DC}}} \tag{107}$$

are only necessary when the rock microstructure allows the bulk frame, solid grains, and pore space to deform differently under unjacketed loading (Makhnenko and Podladchikov, 2018) (K_s is the bulk modulus of solid grains). Note that the rheological assumptions in the Brown–Korringa (BK) model differ from those in the Detournay–Cheng (DC) and the presented Extended Biot formulations. As a result, the interpretation and estimation of the parameters in (107) differ between models.

- The poroelastic parameters (107) can be computed numerically with arbitrary precision. Numerical studies conducted in 3D ~~numerical setup and relatively complex pore geometry that includes narrow regions (cracks)and large pore space. Numerical calculations were performed using a finite element method and the resulting system of equations was solved~~

~~using a robust direct PARDISO solver (Schenk and Gärtner, 2004). Alkhimenkov (2023) conducted a convergence study showing that, for finer resolution, the result of the numerical solution converges towards the result obtained from the original Gassmann's equation. Such a converges analysis validates the accuracy of Gassmann's equation for a particular (but arbitrary)pore geometry. Furthermore, the pore geometry that was used did not contain any special features (among all possible geometries) that were tailored to make it consistent with Gassmann's equations (Alkhimenkov, 2024). There are also other~~ confirm that for isotropic (or cubic), monomineralic rock frames with isotropic grains and a fully interconnected pore space, the three parameters in equation (107) are equal (Alkhimenkov, 2023, 2024).

- These parameters can also be measured experimentally in laboratory settings, enabling practical application. In many practical situations, the differences between these parameters (107) are small, and one can safely adopt a single solid modulus K_s. The condition $\beta_s = \beta_s^{\prime \mathrm{DC}} = \beta_s^{\prime\prime \mathrm{DC}}$ typically holds when the rock has a monomineralic, isotropic, and uniform skeleton; a fully interconnected pore network; and is subjected to pressures below the onset of micro-fracturing or mineral phase transitions. Under such assumptions, the unjacketed compression test measures the intrinsic mineral bulk modulus, and both the whole-specimen ($K_s^{\prime \mathrm{DC}}$) and pore-volume ($K_s^{\prime\prime \mathrm{DC}}$) moduli may collapse (as suggested by several studies) to $K_s = 1/\beta_s$, reducing the DC model to the original Biot–Gassmann formulation. That is, under unjacketed conditions, the entire solid surface is subjected to a uniform pressure increment Δp, and if the rock is microscopically isotropic and homogeneous, both the solid grains and bulk framework undergo uniform volumetric strain, resulting in no change in porosity (Tarokh and Makhnenko, 2019). Typical examples include dense quartz sands, clean limestones below micro-crack initiation stress, and synthetic rock samples.

- Even for multi-mineral skeleton, the differences between these parameters (107) are small, which is shown in the 3D ~~numerical studies that consider different geometries of the pore space and are consistent with Gassmann's equations (Alkhimenkov et al., 2020a, b; Alkhimenkov and Quintal, 2022a, b)~~numerical study by Alkhimenkov (2025) and in laboratory settings (Makhnenko and Podladchikov, 2018).

- Finally, these parameters can also be derived using effective medium theory. This is the most rigorous way to establish under which conditions the three poroelastic parameters are equivalent. The application of effective medium theory is outside the scope of the present study but remains an important direction for future work.

We note that when a rock frame consists of two or more minerals with different elastic properties (e.g., shales, poorly consolidated sandstones, or cracked carbonates), the distinction $\beta_M^{\mathrm{BK}} \neq \beta_S^{\mathrm{BK}}$ in the BK framework is present. In such cases, the assumptions underlying the self-similarity hypothesis break down, and Gassmann's equations serve only as an (very good) approximation within the framework of the Extended Biot formulation (Alkhimenkov, 2025).

To further assess the magnitude of the off-diagonal components of the matrix $\mathbf{H}$, we perform a Taylor expansion of $\beta_\delta^{\mathrm{EB}} = \beta_s^{\prime \mathrm{EB}} - \beta_s^{\prime\prime \mathrm{EB}}$ (without imposing any assumption on mono- or multi-mineral composition of the frame):

$$H_{1,2} = \frac{1}{(1-\phi_f)^2\left((\beta_s^{\prime \mathrm{EB}})^2 + \beta_d^{\mathrm{EB}}(\phi_f - 2)\beta_s^{\prime \mathrm{EB}} + \beta_d^{\mathrm{EB}}\beta_s^{\prime \mathrm{EB}}\right)}\beta_\delta^{\mathrm{EB}} + O\left(\beta_\delta^2\right) \tag{108}$$

which demonstrates that the off-diagonal terms of $\mathbf{H}$ are small.

10.3 ~~Applicability~~ Comparison of Gassmann's equations and Thomsen's alternative formulation

~~Gassmann's equation (Gassmann, 1951) represented by expression (??) can be rewritten in the following form:~~

$$K_u = K_d + \frac{\left(1 - K_d K_s^{-1}\right)^2}{\phi K_f^{-1} + (1-\phi) K_s^{-1} - K_d / K_s^2}.$$

Thomsen (2023b) argued that the original derivation of Gassmann's equations contains a logical error ~~and~~ — namely, an incorrect application of Love's theorem to hydraulically open and closed systems. In the present derivation, we rely on classical irreversible thermodynamics and not rely on any assumptions regarding whether the porous material system is open or closed.

Thomsen (2023b) provided an updated version of these relations (see also Brown and Korringa (1975)):

$$K_u = K_d + \frac{\left(1 - K_d K_M^{-1}\right)^2}{\phi K_f^{-1} + (1-\phi) K_s^{-1} - K_d / K_M^2} \frac{\left(1 - K_d (K_M^{\mathrm{BK}})^{-1}\right)^2}{\phi_f K_f^{-1} + (1-\phi_f) K_s^{-1} - K_d^{\mathrm{BK}} / (K_M^{\mathrm{BK}})^2}, \tag{109}$$

where ~~K_M~~ K_M^{BK} is a new parameter ~~, so-called "mean~~referred to as the "~~incompressibility (or "~~mean" bulk modulus ~~)~~ (Thomsen, 2023b). Note the similarity between expressions ~~(??)~~ (47) and (109). ~~Relation (109) contains one more parameter, K_M, compared to~~ Thomsen's relation introduces one additional parameter, K_M^{BK}, beyond the original Gassmann ~~'s equation (??)~~equation (47). Thomsen (2023b) also provided ways to evaluate ~~K_M by using the following expressions:~~ K_M^{BK}, including:

$$K_M{}^{\mathrm{BK}} = \left[1/K_d \frac{1}{K_d^{\mathrm{BK}}} - \frac{1}{B^{\mathrm{BK}}} \left(\frac{1}{K_d^{\mathrm{BK}}} - \frac{1}{K_u^{\mathrm{BK}}} \right) \right]^{-1}, \tag{110}$$

where ~~B~~ B^{BK} (Skempton coefficient) is directly observable in ~~a~~ quasi-static ~~experiment~~experiments. Alternatively, expression (110) ~~for K_M can be exactly reformulated as:~~ can be rewritten as:

$$K_M{}_M^{\mathrm{BK}} = \left[\frac{B\left(\phi K_f^{-1} + (1-\phi) K_s^{-1}\right) - (1-B) K_d^{-1}}{2B - 1} \frac{B^{\mathrm{BK}} \left(\phi_f K_f^{-1} + (1-\phi_f) (K_s^{\mathrm{BK}})^{-1} \right) - (1 - B^{\mathrm{BK}}) (K_d^{\mathrm{BK}})^{-1}}{2B^{\mathrm{BK}} - 1} \right]^{-1}. \tag{111}$$

Importantly, Thomsen's formulation reduces to Gassmann's when $K_M^{\mathrm{BK}} = K_S^{\mathrm{BK}}$.

Thomsen (2023b) argued that this additional parameter K_M^{BK} must be independently measured, even for mono-mineralic rocks, and that equation (109) should be used instead of the original Gassmann relation (47). As follows from equation (111), evaluating K_M^{BK} requires an independent measurement of the Skempton coefficient B^{BK}. Thomsen (2023b) further noted that the porosity ϕ_f is not constant under equal changes in fluid pressure p_f and total pressure $\bar{p}$, and argued that for mono-mineralic rocks, K_M^{BK} generally differs from K_S^{BK}. This implies a sensitivity of porosity variation—either increasing or decreasing — depending on the sign of $K_M^{\mathrm{BK}} - K_S^{\mathrm{BK}}$.

We note the following:

- Gassmann explicitly stated the self-similarity hypothesis in his original manuscript (Gassmann, 1951). Therefore, claims of a logical error (Thomsen, 2023b) in the derivation are unfounded.

- The claims made by Thomsen (2023b) are not supported by rigorous theoretical developments (e.g., exact solutions in effective medium theory) that explicitly demonstrate that $K_M^{\mathrm{BK}} \neq K_S^{\mathrm{BK}}$ for mono-mineralic rocks.

- Several 3D numerical studies confirm that the self-similarity hypothesis holds for homogeneous, isotropic (or cubic) dry responses and isotropic solid grain materials. This has been verified numerically for both cubic and transversely isotropic symmetries (Alkhimenkov et al., 2020a, b; Alkhimenkov and Quintal, 2022a, b; Alkhimenkov, 2023, 2024).

- A recent 3D numerical study of a heterogeneous frame material composed of two solids with different bulk and shear moduli showed that the difference $K_M^{\mathrm{BK}} - K_S^{\mathrm{BK}}$ is below 0.11 GPa — practically insignificant.

- Laboratory experimentations show that even for multi-mineral skeleton, the differences between K_M^{BK} and K_S^{BK} is small (Makhnenko and Podladchikov, 2018).

- This all suggests that, in relatively homogeneous rock samples, the distinction between different solid grain moduli has negligible practical impact.

- The mechanics of rocks includes additional important aspects such as nonlinearity in their mechanical response; differences in mechanical properties under extension versus compression (which can differ by several percent); intrinsic anisotropy of the solid grains; effective anisotropy of the rock sample; and irreversible damage under applied loads. All of these factors contribute to a much more complex mechanical behavior of rocks. These additional constraints may have a significantly greater impact on rock response than potential deviations from the self-similarity hypothesis.

Alkhimenkov (2023) conducted a numerical convergence study ~~showing that K_M is converging to K_s~~ demonstrating that $K_M^{\mathrm{BK}} \to K_s$ (where K_s is the solid bulk modulus) for monomineralic rock as the resolution increases~~(in the numerical experiment K_M was calculated independently using expression (110), so B was calculatedin addition to other parameters).~~ In this study, K_M^{BK} was computed independently using equation (111), with the Skempton coefficient B^{BK} also calculated. Consequently, the result of ~~the expression (109) is converging~~ expression (109) converges to the original Gassmann ~~'s formulation (??) as the resolution increases. As a result, there is no difference between the two formulations (equations (??) and (109)) since $K_M \equiv K_s$, that validates~~ relation (47) in the mono-mineralic, isotropic (or cubic symmetry) case where $K_M^{\mathrm{BK}} \equiv K_s$ (within numerical precision), thereby validating the original Gassmann ~~'s formulation~~ formulation for a particular pore-space and solid material geometry.

~~We fully agree with the proposal by Thomsen (2023b) that an additional measurement (or an additional parameter) can significantly improve the characterization of fluid-saturated rocks. Indeed, rocks are usually composed by several anisotropic minerals ; rocks have~~

10.4 Limitations

Often, natural rocks are composed of multiple minerals that are anisotropic, and typically exhibit some degree of ~~anisotropy; rocks contain~~ intrinsic anisotropy. They may also contain a combination of compliant cracks (~~or grain-to-grain~~ e.g., grain contacts) and stiff pores~~that behave differently under loading; rocks may have some degree of heterogeneity that cannot be represented via~~ , which respond differently under mechanical loading. Additionally, a rock's heterogeneity can violate the assumptions of a representative volume element. ~~Furthermore, the elastic moduli might be different~~ It is also well established that elastic moduli can vary by several percent under compression ~~or extension. All these divergences of ideal small-strain elasticity suggest more degrees of freedom and, as a consequence,~~ versus extension. These deviations from ideal small-strain elasticity suggest the need for additional effective parameters, and thus more experimental (or numerical) measurements~~are needed to fully characterize the fully saturated realistic rocks~~, to accurately characterize fully saturated and realistic rock samples.

11 Conclusions

This study has presented a ~~novel~~ structured, transparent, and thermodynamically admissible derivation of ~~both Gassmann's and Biot's poroelastic equations, which are crucial~~ the quasi-static Extended Biot's poroelasticity framework. The well-known classical Gassmann equations and Biot poroelastic formulation — fundamental tools for characterizing the ~~elastic and coupled~~ poroelastic mechanical behavior of fluid-saturated porous media~~in geophysics. By~~ —are derived here as special cases of the general theory. While the thermodynamic admissibility of the original Biot equations has been previously demonstrated, the present work extends this admissibility to a more general model using the framework of classical irreversible thermodynamics. We emphasize clarity, accessibility, and full reproducibility throughout the derivation. The main novelty of this study is the development of the Extended Biot's poroelasticity framework, which incorporates off-diagonal components of the Hessian matrix. The relations between the new set of poroelastic parameters are fully expressed in terms of the components of the Hessian matrix $\mathbf{H}$.

By strictly adhering to conservation laws and ~~constitutive relations~~thermodynamic principles, we have ~~addressed concerns about logical inconsistencies in the original derivation~~ also addressed recent claims by Leon Thomsen regarding the validity of Gassmann's ~~equations and extended the theoretical framework to include Biot's equations, which describe the interaction between solid deformation and pore fluid pressure. These results provide a robust foundation for future research and applications. The inclusion of Symbolic Maple routinesfacilitates the reproducibility of our findings, enhancing accessibility and verification within the scientific community~~formulation. In particular, we have shown that the key self-similarity assumption — that porosity remains unchanged under equal changes in fluid and total pressure — is a sufficient but not necessary condition for the derivation of Gassmann-type relationship between undrained and drained bulk moduli. Indeed the Extended Gassmann poroelastic equation (69) is derived in this contribution without relying on the Gassmann's assumption of self-similarity.

To promote transparency and support future developments, we provide symbolic Maple routines. These materials ensure full reproducibility of the derivations and offer a practical foundation for extending the framework to more complex scenarios, such as multiphase fluid systems and related phenomena.

Code availability

The software developed and used in this study is licensed under the MIT License. The latest version of the symbolic Maple routines is available from a permanent DOI repository (Zenodo) at: https://doi.org/10.5281/zenodo.15777522 (last accessed: 30 June 2025) (Alkhimenkov and Podladchikov, 2025). The repository contains code examples and can be readily used to reproduce the results presented in the manuscript. The codes are written in the Maple programming language.

Appendix A: Explanation of the Maple Script for a single phase media

The following Maple script provides a step-by-step derivation of the entropy production for a one-dimensional system using the principles of ~~classical non-equilibrium thermodynamics~~Classical Irreversible Thermodynamics. It uses the volume-specific formulation for mass conservation and the principles of local thermodynamic equilibrium (LTE) to establish the relationship between different thermodynamic fluxes and forces. The script calculates the entropy production, $Q[s]$, and demonstrates the impact of various choices for flux definitions. Below is a detailed explanation of each step in the script.

```
restart;
V := 1/rho:
dVdt := -diff(q[V](x), x)/rho(x): # mass balance (using volume and not density)
dUdt := -diff(q[e](x), x)/rho(x): # conservation of energy
dsdt := -diff(q[s](x), x)/rho(x) + Q[s]/rho(x): # balance of entropy
LTE := dUdt = T(x)*dsdt + P(x)*dVdt: # local thermodynamic equilibrium
Q[s] := solve(LTE, Q[s]); # solving for entropy production

q[e](x) := T(x)*q[s](x); # choice for energy flux
q[V](x) := v: # Galileo's principle for volume flux
q[s](x) := -lambda*diff(T(x), x): # Fourier's law for entropy flux
Q[s] := simplify(eval(Q[s])); # final expression for entropy production
```

Listing 1. Maple Script for Entropy Production

Below, we provide a detailed explanation of each line in the script.

Initialization and Mass Conservation

```
restart;
V := 1/rho:
```

Here, `V` is defined as the specific volume, which is the inverse of density, ρ.

```
dVdt := -diff(q[V](x), x)/rho(x):
```

This line represents the mass conservation equation using the volume-specific formulation. It calculates the time derivative of the specific volume as the negative divergence of the volume flux `q[V](x)` divided by the local density.

Conservation of Energy

```
dUdt := -diff(q[e](x), x)/rho(x):
```

This represents the conservation of energy, where `dUdt` is the time derivative of the specific internal energy, `q[e](x)` is the energy flux, and the equation states that the change in internal energy is equal to the negative divergence of the energy flux divided by the density.

Entropy Balance

```
dsdt := -diff(q[s](x), x)/rho(x) + Q[s]/rho(x):
```

~~The equation~~ This line represents the entropy balance. Here, `dsdt` is the time derivative of specific entropy, `q[s](x)` is the entropy flux, and `Q[s]` is the entropy production rate per unit volume. This equation states that the change in entropy is equal to the negative divergence of the entropy flux plus the entropy production term.

Local Thermodynamic Equilibrium (LTE)

```
LTE := dUdt = T(x)*dsdt + P(x)*dVdt:
```

This equation expresses the principle of local thermodynamic equilibrium (LTE). It relates the internal energy change `dUdt` to the product of temperature `T(x)` and entropy change `dsdt`, plus the product of pressure `P(x)` and the volume change `dVdt`.

Solving for Entropy Production

```
Q[s] := solve(LTE, Q[s]);
```

The script solves the LTE equation for the entropy production term `Q[s]`.

Choice for Energy Flux

```
q[e](x) := T(x)*q[s](x);
```

The energy flux `q[e](x)` is chosen as the product of temperature `T(x)` and the entropy flux `q[s](x)`. This is a common assumption based on the linear coupling between the energy and entropy fluxes.

Flux Definitions

```
q[V](x) := v: # Galileo's principle for volume flux
q[s](x) := -lambda*diff(T(x), x): # Fourier's law for entropy flux
```

The volume flux `q[V](x)` is represented by velocity `v` following Galileo's principle. The entropy flux `q[s](x)` is defined according to Fourier's law, where it is proportional to the temperature gradient `diff(T(x), x)` with thermal conductivity `lambda`.

Final Expression for Entropy Production

```
Q[s] := simplify(eval(Q[s]));
```

The final expression for entropy production `Q[s]` is simplified to:

$$Q[s] = \frac{\lambda}{T(x)} \left(\frac{dT(x)}{dx} \right)^2, \tag{A1}$$

This result shows that the entropy production is non-negative and is proportional to the square of the temperature gradient, divided by temperature, which is a classical result in non-equilibrium thermodynamics.

Appendix B: Explanation of the Maple Script for Two-Phase Fluid-Saturated Media

This appendix provides a detailed explanation of the Maple script used to derive the governing equations and analyze the behavior of a two-phase fluid-saturated medium. The script covers the conservation laws, flux definitions, and the derivation of entropy production for the coupled fluid and solid phases, using principles from classical non-equilibrium thermodynamics .

B1 General Representation of Classical Irreversible Thermodynamics

General Conservation Equations

First, we define the Porous materials can be modeled as two-phase systems composed of a solid skeleton and a saturating fluid. These phases exchange mass, momentum, and energy, leading to complex coupled processes that are naturally described using the framework of classical irreversible thermodynamics (CIT) (Gyarmati et al., 1970; Jou et al., 1996; Lebon et al., 2008; Yarushina and Po . In this formulation, conservation equations for a general quantity $A(t,x)$ and massconservation for density $\rho(t,x)$:

```
restart; #some useful relations
eqA := diff(rho(t, x)*A(t, x), t) + diff(rho(t, x)*A(t, x)*Vx(t, x) + qx(t, x), x) - QA;
eqM := diff(rho(t, x), t) + diff(rho(t, x) * Vx(t, x), x) - Qrho;
```

- eqA represents the conservation of a general quantity $A(t,x)$, incorporating the advective term $\rho(t,x)A(t,x)v_x(t,x)$ and an additional flux $q_x(t,x)$. - eqM is the mass conservation equation for density $\rho(t,x)$ with velocity $v_x(t,x)$ and a source term Q_ρ. The difference between these equations is simplified to derive a general expression for the time derivative of $A(t,x)$.

```
eq := simplify(eqA - eqM * A(t, x));
dA_dt := solve(eq, diff(A(t, x), t));
```

The equation eq is derived by subtracting the mass conservation equation, multiplied by $A(t,x)$, from eqA. This results in an equation for the time derivative of $A(t,x)$, which is then solved to obtain dA_dt. Next, we calculate the total derivative of $A(t,x)$, including the convective term:

```
DA_dt := collect(simplify(dA_dt + diff(A(t, x), x)*Vx(t, x)), Q);
```

The variable DA_dt represents the total (material) derivative of $A(t,x)$, which includes both the time derivative and the convective term $\frac{\partial A}{\partial x} \cdot v_x(t,x)$. The resulting expression is then collected and simplified with respect to the source terms Q:

mass, momentum, entropy, and energy are expressed in the Eulerian frame as follows:

$$\text{DA\_dt}\frac{\partial(\rho\phi)}{\partial t}+\nabla_j\left(\rho\phi\boldsymbol{v}_j+q_\rho^j\right)=\underline{=\frac{-A(t,x)Q_\rho-\frac{\partial q_x(t,x)}{\partial x}+Q_A}{\rho(t,x)}}Q_p, \tag{B1}$$

B2 ~~Thermodynamic Admissibility in Fluid-Saturated Porous Media~~

~~Simplifying Assumptions~~

~~To simplify the model under specific assumptions, we set several parameters to zero:~~

```
Qrhof := 0; # No mass source or sink in the fluid phase
RDarcy := 0; # Removes contribution of Qrhof from fluid momentum balance
Pcor := 0; # Allows reaction to change porosity
De[ph] := 0; # Turns off intraphase mass diffusion
eta[f] := 0; # if=0 then no full Stokes for pore scale fluid flow = only Darcy's law
lam[ph]:= 0; # Turns off intraphase heat diffusion
```

~~Flux Definitions and Constitutive Relations~~

~~Effective properties~~

~~We define the effective properties of the solid phase using mixture rules: Effective Thermal Conductivity: Starting from the total thermal conductivity:~~

$$\underline{\lambda_t=(1-\phi)\lambda_s}\frac{\partial(\rho\phi\boldsymbol{v}_i)}{\partial t}+\nabla_j\left(\underline{\rho\phi\lambda_f}\boldsymbol{v}_i\boldsymbol{v}_j+q_{\boldsymbol{v}}^{ij}\right)=Q_{v_i}, \tag{B2}$$

~~Solving for λ_s:~~

$$\underline{\lambda_s}\frac{\partial(\rho\phi\boldsymbol{s})}{\partial t}+\nabla_j\left(\rho\phi\boldsymbol{s}\boldsymbol{v}_j+q_{\boldsymbol{s}}^j\right)=\underline{\frac{\lambda_t-\lambda_f\phi}{1-\phi}}Q_s, \tag{B3}$$

~~Effective Mass Diffusion Coefficient:~~

$$\underline{D_c^{(s)}}\frac{\partial(\rho\phi\boldsymbol{e})}{\partial t}+\nabla_j\left(\rho\phi\boldsymbol{e}\boldsymbol{v}_j+q_{\boldsymbol{e}}^j\right)=\underline{\frac{D_c^{(t)}-D_c^{(f)}\phi}{1-\phi}}Q_e, \tag{B4}$$

~~Effective Viscosity:~~

$$\underline{\eta_s=\frac{\eta_t-\eta_f\phi}{1-\phi}}$$

~~Kinematic Relations~~

```
dphi_dt := diff(phi(t, x), t) + V(x) * diff(phi(t, x), x);
```

The rate of change of porosity ϕ is given by:

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + v\frac{\partial \phi}{\partial x}$$

where v is the velocity, and $\frac{d\phi}{dt}$ where $\boldsymbol{v}_j$, $\boldsymbol{s}$, and $\boldsymbol{e}$ denote the velocity, specific entropy, and specific total energy per unit mass, respectively. The term ρ denotes (phase-specific) density, ϕ the phase volume fraction (e.g., porosity for the fluid). The terms ∇_j represents the material derivative of porosity. The fluid velocityv^f relates to the solid velocity v^s and the Darcy flux q^D:

$$v^f = v^s + \frac{q^D}{\phi}.$$

partial derivative with respect to spatial coordinates, while q^j_ρ, $q^{ij}_{\boldsymbol{v}}$, $q^j_{\boldsymbol{s}}$, and $q^j_{\boldsymbol{e}}$ correspond to the fluxes of mass, momentum, entropy, and energy, respectively. The terms Q_ρ, Q_{v_i}, Q_s, and Q_e represent the corresponding production rates due to irreversible processes (Yarushina and Podladchikov, 2015).

Fluxes and Source Terms

Here we define fluxes for heat, momentum, and solute transport based on non-equilibrium thermodynamics:

```
qs := -lam[ph]*phi(t, x)*diff(T(x), x) / T(x); # Fourier's law for heat flux
qv := -eta[ph]*phi(t, x)*diff(V(x), x) + phi(t, x)*P(x); # Stokes' law for viscosity
qc := -Dc[ph]*phi(t, x)*diff(mu(x), x); # Fick's law for diffusion
qu := T(x)*qs + V(x)*qv + mu(x)*qc; # Energy flux
```

- ‘qs‘: Heat flux defined according to Fourier’s law, with thermal conductivity $\lambda[ph]$.- ‘qv‘: Viscous flux based on Newtonian viscosity, incorporating pressure $P(x)$.- ‘qc‘: Solute flux following Fick’s law of diffusion, with chemical potential gradient $\mu(x)$.- ‘qu‘: Total energy flux, a combination of heat, mechanical, and chemical contributions.

- Heat Flux (q_s) . According to Fourier’s law:

  $$q_s = -\lambda_{\text{ph}}\phi\frac{\partial T}{\partial x}\cdot\frac{1}{T},$$

  where λ_{ph} is the phase-dependent thermal conductivity.

- Momentum Flux (q_v). Using Newtonian viscosity (Stokes flow approximation):

  $$q_v = -\eta_{\text{ph}}\phi\frac{\partial V}{\partial x} + \phi P,$$

  where η_{ph} is the phase-dependent viscosity.

- Mass Flux (q_c). Following Fick’s law for diffusion:

  $$q_c = -D_c^{(\text{ph})}\phi\frac{\partial \mu}{\partial x},$$

  where $D_c^{(\text{ph})}$ is the phase-dependent mass diffusion coefficient.

- Energy Flux (q_u). Combining the above fluxes:

$$q_u = Tq_s + vq_v + \mu q_c$$

Balance Equations

```
qdrho_dt := (-(diff(V(x),x)*phi(t,x)+dphi_dt)*rho(t,x)      # conservation law in non-divergent form
                                        +Qrho)/phi(t,x):
dU_dt    := (-diff(qu,x) + Qu - U    *Qrho)/rho(t,x)/phi(t,x): # Energy - eq balance energy
dV_dt    := (-diff(qv,x) + Qv - V(x)*Qrho)/rho(t,x)/phi(t,x): # Newton 2nd law
dC_dt    := (-diff(qc,x) + Qc - C(x)*Qrho)/rho(t,x)/phi(t,x): # balance mass of solute
dS_dt    := (-diff(qs,x) + Qs - S(x)*Qrho)/rho(t,x)/phi(t,x): # balance of entropy - increasing
```

- Mass Balance (Non-Divergent Form). The rate of change of density ρ is:

$$\frac{d\rho}{dt} = \frac{-\left(\phi\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial t} + v\frac{\partial \phi}{\partial x}\right)\rho + Q_\rho}{\phi},$$

where Q_ρ is the mass source term.

- Energy Balance

$$\frac{dU}{dt} = \frac{-\frac{\partial q_u}{\partial x} + Q_u - UQ_\rho}{\rho\phi},$$

where Q_u is the energy source term.

- Momentum Balance (Newton's Second Law)

$$\frac{dv}{dt} = \frac{-\frac{\partial q_v}{\partial x} + Q_v - vQ_\rho}{\rho\phi},$$

where Q_v is the momentum source term.

- Concentration Balance

$$\frac{dC}{dt} = \frac{-\frac{\partial q_c}{\partial x} + Q_c - CQ_\rho}{\rho\phi},$$

where Q_c is the concentration source term.

- Entropy Balance

$$\frac{dS}{dt} = \frac{-\frac{\partial q_s}{\partial x} + Q_s - SQ_\rho}{\rho\phi},$$

where Q_s is the entropy source term.

Deriving Entropy Production

```
LET := dU_dt = T(x)*dS_dt
          + P(x)*drho_dt/rho(t, x)^2
          + V(x)*dV_dt
          + mu(x)*dC_dt
          + tau[phi]*dphie_dt/rho[ph](t,x)/(phi(t,x));
TQs := simplify(T(x)*solve(LET, Qs));
```

- 'LET': The local thermodynamic equilibrium condition, which includes terms for internal energy, entropy, volume, kinetic energy, chemical potential, and porosity change.- 'TQs': The entropy production term, simplified from the LTE condition to ensure non-negative production.

Local Entropy Production

The local entropy production is derived from the energy balance and is given by:

$$\frac{dU}{dt} = T\frac{dS}{dt} + \frac{p}{\rho^2}\frac{d\rho}{dt} + v\frac{dv}{dt} + \mu\frac{dC}{dt} + \frac{\tau_\phi}{\rho\,\phi}\frac{d\phi^e}{dt},$$

where:

- τ_ϕ is the thermodynamic variable (pressure) conjugated to porosity change. Note, that τ_ϕ is not defined yet.
- $\frac{d\phi^e}{dt}$ is the reversible part of the porosity rate change.

Physical Interpretation of Terms:

- $T\frac{dS}{dt}$: Heat stored in internal energy U.
- $\frac{p}{\rho^2}\frac{d\rho}{dt} = -p\frac{d(1/\rho)}{dt}$: Work stored in elastic energy (Hooke's Law). Note that $\frac{dp}{K} = \frac{d\rho}{\rho}$, where K is the bulk modulus, $dp = p - p_{\text{ref}}$, and p_{ref} is the reference pressure.
- $v\frac{dv}{dt}$: Newtonian mechanics (kinetic energy, e.g., $v\frac{dv}{dt} = \frac{1}{2}\frac{dv^2}{dt}$).
- $\mu\frac{dC}{dt}$: Energy due to changes in composition (chemical potential), which is zero in the present derivation.
- $\frac{\tau_\phi}{\rho\,\phi}\frac{d\phi^e}{dt}$: Poroelastic effects: reversible part of the energy change due to the changes in porosity.

Entropy Production (TQ_s)

B1.1 Entropy Production (TQ_s)

Solving the local entropy production equation for Q_s and multiplying both sides by the absolute temperature T, we have: obtain:

$$TQ_s = \eta\phi\left(\frac{dv}{dx}\right)^2 + \frac{\lambda\phi}{T}\left(\frac{dT}{dx}\right)^2 + pv\frac{d\phi}{dx} + \mu Q_\rho C - vQ_v - Q_\rho G_{\text{Gibbs}} + Q_u + p\frac{d\phi}{dt} - \tau_\phi\frac{d\phi^e}{dt} \tag{B5}$$

This expression represents the entropy production, which must be non-negative according to the second law of thermodynamics. Notably, this formulation assumes local thermodynamic equilibrium separately for the solid and fluid phases. This is a weaker assumption than Biot's original model (Biot, 1962), which postulated a single internal energy potential for the entire two-phase system in the linear poroelastic regime (Yarushina and Podladchikov, 2015).

Phase Properties and Kinematic Substitutions

B2 Thermodynamic Constraints on Fluxes and Productions

We consider both fluid and solid phases, assigning specific properties to each.

```
Fluid := {ph=f,rho(t,x)=rho[f](t,x),V(x)=Vf    ,P(x)=Pf(x)       ,G(x)=Gf    ,Qv= Qvf,Qrho= Qrhof,Qc= Qcf,Qu= Quf,tau[phi]=0
Solid := {ph=s,rho(t,x)=rho[s](t,x),V(x)=Vs(x),P(x)=Pf(x)-dP(x),G(x)=Gf-dG,Qv=-Qvf,
Qrho=-Qrhof,Qc=-Qcf,Qu=-Quf,phi(t,x)=1-phi(t,x)}:
sbs:={diff(  phi(t,x),t) = dphife_dt+dphifvis_dt - Vs(x)*diff(  phi(t,x),x)
     ,diff(rho[s](t,x),t) = drhos_dt - Vs(x)*diff(rho[s](t,x),x)
     ,diff(rho[f](t,x),t) = drhof_dt -    Vf*diff(rho[f](t,x),x)
     ,diff(Vs(x),x)       = divVs};
```

We introduce substitutions for derivatives to simplify the expressions:

- Porosity Rate Change (note that porosity is divided into reversible (elastic) and irreversible (viscous) parts)

$$\frac{\partial\phi}{\partial t} = \frac{d\phi}{dt} - v^s\frac{\partial\phi}{\partial x}$$

- Solid Density Rate Change

$$\frac{\partial\rho_s}{\partial t} = \frac{d\rho_s}{dt} - v^s\frac{\partial\rho_s}{\partial x}$$

- Fluid Density Rate Change

$$\frac{\partial\rho_f}{\partial t} = \frac{d\rho_f}{dt} - v^f\frac{\partial\rho_f}{\partial x}$$

- Solid Velocity Divergence

$$\frac{\partial v^s}{\partial x} = \operatorname{div} v^s$$

Total Entropy Production

The entropy production for both fluid and solid phasesis computed:

```
TQs_total := subs(Fluid, TQs) + subs(Solid, TQs);
```

Substituting the phase properties and kinematic relations into the expression for TQ_s, we obtain CIT requires that the total entropy production :

$$TQ_{s,\text{total}} = TQ_s^{(f)} + TQ_s^{(s)}$$

Thermodynamic Variable Conjugated to Porosity Changes:

The thermodynamic variable τ_ϕ conjugated to porosity changes is now defined as: of the system remains non-negative. This condition applies both to the intra-phase and inter-phase entropy production within a porous medium. Mathematically, this is expressed as:

$$\tau_\phi \sum_{\text{phases}} Q_s = \Delta p \equiv p \sum_{\text{phases}} Q_s - p_f . \quad Q_s^{\text{intra}} + Q_s^{\text{inter}} \geq 0. \tag{B6}$$

Additional Relations:

Fluid Momentum Flux (Q_{vf}). Given that $Q_{\rho f} = 0$ and $R_{\text{Darcy}} = 0$:

$$Q_{vf} = \frac{\partial \phi}{\partial x} p_f - \eta_{\text{dV}} q^D$$

Porosity Rate Change in Fluid Phase ($\frac{d\phi_f}{dt}$) Since $Q_{\rho f} = 0$ and $P_{\text{cor}} = 0$:

$$\frac{d\phi_f}{dt} = \frac{d\phi^{(e)}}{dt} + k_\phi \Delta P$$

Gibbs Free Energy Change (ΔG). With $P_{\text{cor}} = 0$: Here, Q_s^{intra} represents the intra-phase entropy production within each phase (e.g., due to viscosity, heat conduction, or internal diffusion), while Q_s^{inter} represents the entropy production arising from inter-phase interactions (e.g., interactions between the solid skeleton and the fluid phase). To satisfy CIT, each contribution must be non-negative:

$$\Delta G_{\text{Gibbs}} = \Delta G_{\text{2Gibbs}} - \frac{v^f \, q^D}{\phi} \quad Q_s^{\text{intra}} \geq 0, \quad Q_s^{\text{inter}} \geq 0. \tag{B7}$$

Mass Source Term in Fluid Phase ($Q_{\rho f}$)

B3 Extended Thermodynamic Admissibility

Building on the principles of Classical Irreversible Thermodynamics (CIT) (Lebon et al., 2008) and the nonlinear viscoelastoplastic framework of Yarushina and Podladchikov (2015), the derivation of the extended Biot poroelastic equations must satisfy the conditions of thermodynamic admissibility. Specifically, the entropy production Q_s must remain non-negative, and the constitutive relations must be formulated such that they are consistent with the second law of thermodynamics for all admissible thermodynamic paths. ~~Given by:~~

$Q_{\rho f} = -k_\rho \Delta G_{\text{2Gibbs}}$

~~But since $Q_{\rho f} = 0$, it implies $\Delta G_{\text{2Gibbs}} = 0$ or $k_\rho = 0$.~~

~~Total Entropy Production~~

```
1: TQs_total := collect(expand(simplify(subs(sbs, eval(TQs_total)))), {dphie_dt});
```

From equation 37, and taking into account the requirement that entropy production must be non-negative, the inelastic porosity equation takes the form (Yarushina and Podladchikov, 2015):

$$\frac{d^s \phi_f}{dt} - \frac{d^s \phi_f^e}{dt} = -\frac{p_e}{\eta_\phi}, \tag{B8}$$

where η_ϕ stands for the effective bulk viscosity. After simplifying and collecting terms (see Appendix B), the total entropy production becomes:

$$TQ_{s,\text{total}} = \frac{1}{\eta_\phi}\left(\frac{p_e}{(1-\phi)}\frac{p_e}{(1-\phi_f)}\right)^2 + \eta_t\left(\nabla\cdot v^s\right)^2 + \frac{(q^D)^2\eta_{\text{dV}}}{\phi}\frac{(q^D)^2\eta_{\text{dV}}}{\phi_f} + \frac{\lambda_t}{T}\left(\frac{\partial T}{\partial x}\right)^2. \tag{B9}$$

~~As a result, entropy production is non-negative if material parameters are non-negative, which proves the thermodynamic admissibility of the two-phase system.~~

~~Explanation of Terms:~~

- $\frac{1}{\eta_\phi}\left(\frac{p_e}{(1-\phi)}\right)^2\frac{1}{\eta_\phi}\left(\frac{p_e}{(1-\phi_f)}\right)^2$: Entropy production due to ~~poroelastic deformation (poroelastic coefficient k_ϕ and pressure difference~~poroviscous deformation (effective viscosity η_ϕ and effective pressure $p_e = \bar{p} - p_f$).
- ~~$\eta_t(\text{div}\, v^s)^2$~~$\eta_t(\nabla\cdot v^s)^2$: Entropy production due to viscous dissipation in the solid phase.
- $\frac{(q^D)^2\eta_{\text{dV}}}{\phi}\frac{(q^D)^2\eta_{\text{dV}}}{\phi_f}$: Entropy production due to viscous dissipation in fluid flow (Darcy flow).
- $\frac{\lambda_t}{T}\left(\frac{\partial T}{\partial x}\right)^2$: Entropy production due to heat conduction (Fourier's law).

B4 Darcy's Law and Fluid Flow

Darcy's law is derived for fluid flow and evaluated for the fluid phase:

```
Mom_f := 0 = subs(Fluid, dV_dt):
Mom_s := 0 = subs(Solid, dV_dt):
qDx := simplify(solve(Mom_f, qD(x)));
```

- 'qDx': Expression for Darcy's flux, relating it to the pressure gradient. From the fluid momentum balance $\frac{dv^f}{dt} = 0$, we derive Darcy's law for the fluid flux q^D. Starting from the momentum balance for the fluid phase:

$$0 = \frac{-\frac{\partial q^v}{\partial x} + Q_{vf}}{\rho_f \phi}$$

Using the expression for q^v and substituting Q_{vf}:

$$0 = \frac{-\frac{\partial}{\partial x}\left(-\eta_{\text{ph}}\phi\frac{\partial v^f}{\partial x} + \phi p_f\right) + \left(\frac{\partial \phi}{\partial x}p_f - \eta_{\text{dv}}\, q^D\right)}{\rho_f \phi}$$

Simplifying and solving for q^D:

$$q^D = -\frac{1}{\eta_{\text{dv}}}\frac{\partial p_f}{\partial x}$$

This indicates that the fluid flux is driven by the pressure gradient and is proportional to the permeability (inverse of viscosity), which is Darcy's law.

Solid Velocity Divergence

Using the mass balance equations and the substitutions, we derive the divergence of the solid velocity. The mass conservation for the solid, accounting for porosity changes, is given by:

$$\frac{\partial}{\partial t}\left(\rho_s(1-\phi)\right) + \frac{\partial}{\partial x}\left(\rho_s(1-\phi)v^s\right) = 0.$$

Expanding the derivatives, we obtain:

$$(1-\phi)\frac{\partial \rho_s}{\partial t} - \rho_s\frac{\partial \phi}{\partial t} + \rho_s(1-\phi)\frac{\partial v^s}{\partial x} + v^s\frac{\partial}{\partial x}\left(\rho_s(1-\phi)\right) = 0.$$

We further expand the derivative of the last term:

$$(1-\phi)\frac{\partial \rho_s}{\partial t} - \rho_s\frac{\partial \phi}{\partial t} + \rho_s(1-\phi)\frac{\partial v^s}{\partial x} + v^s(1-\phi)\frac{\partial \rho_s}{\partial x} - v^s\rho_s\frac{\partial \phi}{\partial x} = 0.$$

~~Grouping terms and recognizing the material derivative $\frac{d^s}{dt} = \frac{\partial}{\partial t} + v^s \frac{\partial}{\partial x}$:~~

$$(1-\phi)\left(\frac{\partial \rho_s}{\partial t} + v^s \frac{\partial \rho_s}{\partial x}\right) - \rho_s \left(\frac{\partial \phi}{\partial t} + v^s \frac{\partial \phi}{\partial x}\right) + \rho_s (1-\phi)\frac{\partial v^s}{\partial x} = 0.$$

~~Using the material derivatives:~~

$$(1-\phi)\frac{d^s \rho_s}{dt} - \rho_s \frac{d\phi}{dt} + \rho_s (1-\phi)\frac{\partial v^s}{\partial x} = 0.$$

~~We can infer the solid velocity divergence:~~

$$\operatorname{div} v^s \equiv \frac{\partial v^s}{\partial x} = -\frac{1}{\rho_s}\frac{d^s \rho_s}{dt} + \frac{1}{1-\phi}\frac{d^s \phi}{dt}.$$

~~By using equation (B8) we can further simplify the expression:~~

$$\operatorname{div} v^s = -\frac{1}{\rho_s}\frac{d\rho_s}{dt} - \frac{1}{1-\phi}\frac{d\phi^e}{dt} - \frac{\Delta p}{\eta_\phi (1-\phi)}.$$

~~Each term in the expression (??) for $\operatorname{div} v^s$ has a physical interpretation:~~

~~Solid Density Changes:~~

$$-\frac{1}{\rho_s}\frac{d^s \rho_s}{dt}$$

~~This term accounts for the volumetric changes due to variations in the solid density, such as thermal expansion or compression under pressure.~~

~~Reversible Porosity Changes:~~

$$-\frac{1}{1-\phi}\frac{d\phi^e}{dt}$$

~~Accounts for the reversible part of the porosity change.~~

~~Irreversible Porosity Changes:~~

$$-\frac{\Delta p}{\eta_\phi (1-\phi)}$$

~~Incorporates the effect of pressure changes through the poroelastic coefficient η_ϕ~~**The non-negative nature of each term ensures the overall positivity of entropy production, thereby confirming the thermodynamic validity of the system of extended Biot's poroviscoelastic equations**.

A more detailed derivation is given below (see also the discussions provided by Yarushina and Podladchikov (2015)). Additionally, symbolic Maple routines used to reproduce and validate the theoretical results presented in this article are available in a permanent DOI repository (Zenodo) (Alkhimenkov and Podladchikov, 2024).

~~**Summary**~~

~~The derived expressions ensure thermodynamic admissibility by demonstrating that the total entropy production $TQ_{s,\text{total}}$ is non-negative, satisfying the second law of thermodynamics. Each term in the entropy production has a clear physical interpretation, representing the irreversible processes contributing to entropy increase in the system.~~

Author contribution

YA designed the original study, contributed to the development the symbolic routines, and wrote the manuscript. YP contributed to the early work on the derivation of original Biot's and Gassmann's equations, assisted with the study design, developed the early versions of symbolic routines, helped interpret the results, edited the manuscript, and supervised the work.

Competing interests

The contact author has declared that none of the authors has any competing interests.

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