Post-review modification

Lilian Vanderveken and Michel Crucifix

Dear editor,

Thank you for allowing us the opportunity to make modifications to our manuscript before publication.

The revision concerns the scaling relationship between rainfall and the destruction timescale of vegetation pulses in Section 3, Effect of Rainfall Perturbation on Vegetation Dynamics: Identifying Critical Timescales.

In the original manuscript, we proposed a single timescale (τ_{pulse}) for both the creation and destruction of vegetation patterns, assuming a linear dependence on rainfall based on dimensional analysis. However, after discussions with my thesis jury, we decided to refine this approach. We now define only the destruction timescale, corresponding to the timing of vegetation pulse collapse. A new dimensional analysis led us to a revised scaling law, showing that the destruction timescale scales cubically with rainfall:

$$\tau_{\text{destruction}} = \frac{\alpha}{c\, g_{max}} \left(\frac{R}{k_2\, g_{max}} \right)^3 \frac{1}{d} = 52\text{d}$$

While the order of magnitude remains consistent with our previous estimate, this cubic dependence was further validated through numerical experiments.

To test the proposed scaling, we conducted additional simulations, where we started from a stable equilibrium consisting of two vegetation pulses at $R = 1.2$ mm/d and abruptly reduced rainfall to values between 0.3 mm/d and 0.8 mm/d. This perturbation led to vegetation collapse, and we determined the destruction timescale by fitting an exponential function to the mean biomass evolution. The results, presented in the right panel of Fig. 1, confirm the cubic relationship between rainfall and the destruction timescale.

We have updated the manuscript to reflect these refinements. The core results and interpretations remain unchanged, but we have revised the text accordingly and removed references to the creation timescale. Additionally, we refined some statements in the discussion to reflect the corrected scaling law. We also modified the rearrengement timescale accordingly to reflect this new scaling.

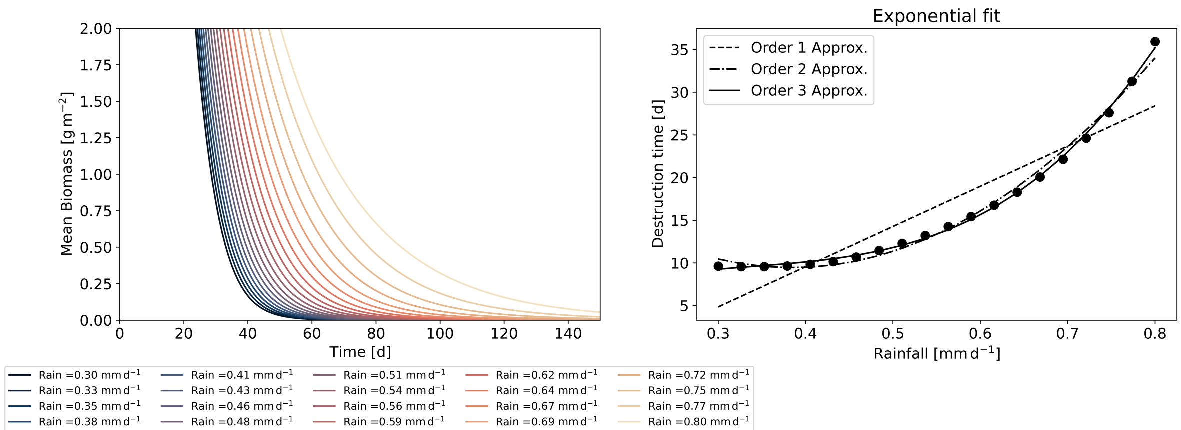

Figure 1: The left panel shows the mean biomass over time for rainfall values ranging from 0.3mm.d^{-1} to 0.8mm.d^{-1}. The right panel displays the destruction timescales obtained from an exponential fit as a function of rainfall, along with polynomial fits of the destruction timescale of orders one, two, and three.

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It is not surprising that the response depends on the rate of change. Ashwin et al. (2012) established general principles of so-called rate-induced tipping (R-tipping) in models based on ordinary differential equations, and rate-dependent responses were also described specifically in models of vegetation patterns (Siteur et al. (2014), Chen et al. (2015), R. Bastiaansen et al. (2020)). However, the value of $\tau_{R_{\mathbf{tip}}}$ is intriguing. Indeed, the timescale associated with the destruction of vegetation pulses is, through dimensional analysis, estimated to be $\tau_{\mathbf{destruction}} = \frac{\alpha}{c\, g_{max}} \left(\frac{R}{k_2\, g_{max}}\right)^3 \frac{1}{d} = 52\mathbf{d}$.

This relationship is linked to the transfer of water to biomass, to vegetation mortality and to rainfall. To test the dependence of the proposed scaling on rainfall, we conducted the following numerical experiments. Starting from a stable equilibrium consisting of two vegetation pulses at $R = 1.2\text{mm.d}^{-1}$, we reduced abruptly the rainfall to values between 0.3mm.d^{-1} and 0.8mm.d^{-1}. This reduction leads to vegetation collapse, driving system toward the bare soil equilibrium. We then determined the destruction timescale by fitting an exponential function to the mean biomass evolution for different rainfall values. The resulting timescales are shown in the right panel of Fig. 1. Our results reveal a cubic relationship between the destruction timescale and rainfall, supporting the validity of our scaling.

line 110- 115:

Inspired by the scaling proposed in Robbin Bastiaansen and Doelman (2019), we reason on the fact that the movement of pulses are determined by diffusion

coefficients. Specifically, we take the advantage of the fact that the ratio between the slow and the fast diffusion coefficients in the reaction-diffusion model drives the creation of the patterns (Murray (2003),Meron (2015)). For the Rietkerk model the fast component is the surface water (O) and the slow components are the biomass (B) and the soil water (W). Hence, we propose the following scaling for the rearrangement time $\tau_{rear} = \frac{\alpha}{c\, g_{max}} \left(\frac{R}{k_2\, g_{max}}\right)^3 \frac{1}{d} \sqrt{\frac{D_O}{D_B}} \sim 1000$d.

To support these clarifications, we have also included a new figure illustrating the updated scaling relationship. Since these modifications primarily concern explanatory text and do not alter the main findings, we believe they improve the clarity and correctness of the manuscript.

References

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