Responses to Reviewer 2's Comments

Note to the Editor and all reviewers: We already posted six responses online and these responses will be summarized here. Following the comments and suggestions from the Editor and reviewers, we have revised the manuscript by

- moving the original Section 3.6 regarding the Lilly's formula for two discretization methods into Appendix B to avoid repeated discussions of the scale factor Jacobian,
- adding a few paragraphs, and
- making editorial changes to improve readability (see the manuscript with tracked changes).

> The authors present a review on the fundamental predictability limits of the atmosphere and put forward arguments to suggest that the classical estimates by Lorenz and Lilly could be inexact. In particular, they claim that the discretization in wave-number space influences these estimates, and that it should be taken into account. As exposed below, I think this reasoning is wrong and should be revised.

We appreciate the reviewer's feedback on the initial version of the manuscript and have worked to address their concerns. After carefully reviewing their comments and providing thoughtful responses, we are confident that our revised manuscript and responses address their concerns and meet their standards.

> The main flaw in the reasoning of the authors is to assume that the discretisation can be chosen a priori. The inertial range of turbulence implies that the local turnover time depends on a power of the scale. From this follows that changes in the turnover time in scale, which are related to the growth of error and the propagation of uncertainty, cannot be linear with the scale. In other words, if we add a fixed increment to the scale, the increment of the turnover time will not be equal across scales when scaled in local turnover times. On the other hand, if we add an increment of scale that is proportional to the scale, the increment of the turnover time will be proportional to a constant in local turnover times. This is what gives Lorenz's and Lilly's models validity even though they are very crude representations of reality. In this sense, the physical assumptions and the theory come before determining the appropriate scaling for the scale increments, and not the other way round. I do not think the authors have a clear physical justification for a linear scaling of the scale increments. Moreover, many investigations have shown that the growth of perturbations follow a self-similar scaling, none of which are cited in the paper, for instance, a recent one, Boffetta, G., & Musacchio, S, PRL, 2017.

To address the concerns raised, we would like to highlight that our study re-examined the validity of Lorenz's formula by analyzing the data in Table 1 and delving into the intricacies of Lilly's turbulence-based formula. In response R1B, we presented the physical relationship between kinetic energy, velocity, and turnover times. In response R2A, we further explored scale invariance and self-similarity. Below, we provide a concise summary of R2A.

In response R2A, we provided detailed information on scale invariance and incorporated the reviewer's suggestion. The reviewer proposed the idea of differentiating and integrating turnover times with respect to ln(k). This concept is now included in the revised manuscript, as follows:

$$\frac{d\tau(k)}{dln(k)} = \frac{d\tau(k)}{\frac{1}{k}dk} = k\frac{d\tau(k)}{dk} \sim kC_0\left(-\frac{2}{3}\right)k^{-\frac{5}{3}} = -\frac{2}{3}\tau(k) \quad (19)$$

$$\int_{k_o}^{k_1} \tau d\ln(k) = C_o\left(\int_{k_o}^{k_1} k^{-5/3} dk\right) = -\frac{3}{2}\left((\tau_k(k_1) - \tau_k(k_0)\right) \quad (20)$$

As discussed in the revised manuscript and R2A, the mathematical expressions above suggest that the derivative and integral of turnover time with respect to ln(k) are proportional to the turnover time itself. This property holds true for functions with power laws, excluding constant functions.

However, our objective is to assess the robustness of Lorenz's formula and analysis. Therefore, we reanalyze Lorenz's data presented in Table 1 and verify whether Lorenz's idea is supported with Lilly's work. To address this, we delved into the discrepancies in physical definitions for turnover times and saturation time differences. Furthermore, we reevaluated the validity of Lilly's integrals for predictability estimates by demonstrating the dependence of convergence on two different discretization methods. Our findings do not support the notion of the validity of Lorenz's geometric series and do not suggest a consistent physical and mathematical framework between Lorenz's and Lilly's formulas.

In response R2A, additional discussions on the Lyapunov exponents and their connection to turnover time (Ruelle, 1979) are presented. Several studies by Boffetta (e.g., Boffetta and Celani 1998; Boffetta and Musacchio, 2017; Aurell, Boffetta et al. 1996) are cited, including the following two quotes:

> *"Our findings suggest that the dimensional estimate of the Lyapunov exponent as the inverse Kolmogorov time does not give an accurate characterization of the chaoticity of a turbulent flow."*
>
> *"The independence of the FSLE in the scaling range on the value λ observed for infinitesimal errors provides a clear explanation of how in turbulent flows it is possible to observe the coexistence of long predictability time at large scales and strong chaoticity at small scales."*

The above quotes in R2A and the subsequent discussions in R2B emphasized the significance of incorporating novel concepts, such as bistability and duality, into the estimation of predictability limits. Consequently, two studies by Boffetta (e.g., Boffetta, 2023; Aurell, Boffetta et al. 1996) are cited in the revised manuscript.

> By the way, the same argument as that of Lorenz's and Lilly's for the scale increments is used in the classical theory of the turbulence cascade to justify that energy can travel to the small scales in a finite time, even at infinitely large Reynolds, when the ratio of large over small length-scales is formally infinite. Probably, a linear approach to this problem would yield an infinite cascade time, in clear contradiction with the empirical observation that dissipation occurs even at vanishing viscosity.

As long as a friction layer persists, the highest wavenumber within the inertia subrange must be finite. Therefore, the integral of turnover times with respect to either k (for a uniform grid) or ln(k) (for a non-uniform grid) should be finite. For example, Equation 20 in the main text (also displayed above) produces finite values for finite wavenumbers regardless of whether the kinetic energy spectrum follows a -3/5 or -3 power law.

We firmly believe in the significance of classical turbulence theory. We acknowledge the valuable contributions made by Lorenz and Lilly's studies in the 1960s and 1970s. Classical turbulence theory is based on the Navier-Stokes equations, excluding Coriolis and buoyancy forces. In contrast, Lorenz's 1969 model was derived from a partial differential equation (PDE) that preserves 2D barotropic vorticity. However, the original PDE lacks the necessary forcing and dissipation terms necessary for studying turbulence (as illustrated in Figure R2A.2) and omits thermodynamic equations crucial for understanding weather.

On the other hand, as discussed in Section 2.1 and response R2B**,** recent advancements in turbulence studies have introduced novel concepts, such as bistability and duality for direct and inverse cascades, in more realistic systems. These concepts should be considered to acknowledge the complexities of weather and climate, leading to more accurate estimates of predictability limits.

Links for the Posted Responses:

- Shen, Pielke Sr., and Zeng, 2024: Responses Part 2A (R2A): "A Brief Note on Turbulence-based Turnover Time" (this is different from R1B). https://doi.org/10.5194/egusphere-2024-2228-AC4
- Shen, Pielke Sr., and Zeng, 2024: Responses Part 2B (R2B): "A Brief Note on Bistability, Duality, and Dimensional Transitions in Recent Turbulence Studies" https://doi.org/10.5194/egusphere-2024-2228-AC6

Relevant Responses:

- Shen, Pielke Sr., and Zeng, 2024: Responses to Editor: Additional discussions of Zhang et al. and the validity of the revised Logistic equation. https://doi.org/10.5194/egusphere-2024-2228-AC5
- Shen, Pielke Sr., and Zeng, 2024: Responses Part 1A (R1A): "A reevaluation of Figure 3 in Zhang et al. (2019)". https://doi.org/10.5194/egusphere-2024-2228-AC1
- Shen, Pielke Sr., and Zeng, 2024: Responses Part 1B (R1B): "A Brief Note on Turbulence-based Turnover Time." https://doi.org/10.5194/egusphere-2024-2228-AC2
- Shen, Pielke Sr., and Zeng, 2024: Responses Part 1C (R1C): "Qualitative Predictability Estimates Using Lilly's Formula and Comparative Insights" https://doi.org/10.5194/egusphere-2024-2228-AC3