Responses Part 2B (R2B): A Brief Note on Bistability, Duality, and Dimensional Transitions in Recent Turbulence Studies

We value the reviewer's feedback, which facilitates deeper discussions on recent advancements in turbulence research. While we're currently working on providing comprehensive responses that will be available later, we've offered a concise version that specifically addresses the relatively new concepts of bistability, duality, and dimensional transitions that were not covered in both Lilly's and Lorenz's studies.

Classical turbulence studies, as discussed in the main text and previous responses, cannot be directly applied to support the findings of Lorenz (1969d)'s studies or Lorenz's formula. The Lorenz 1969 model was based on a conservative partial differential equation that lacked crucial physical processes such as dissipations. Furthermore, recent studies (e.g., Pouquet and Marino, 2013; De Wit et al. 2022; Boffetta 2023) have shed light on the complexities of turbulence, weather, and climate, prompting us to interpret the findings of early studies from the 1960s and 1970s with caution.

The key concepts of bistability, duality, and dimensional transitions are discussed below.

Bistability

Bistability and bimodality have been observed in various dynamical systems, including the classical and generalized Lorenz models, as well as other Lorenz systems (e.g., Shen 2019, 2025, Shen et al. 2021). Bistability is widely understood as the coexistence of two distinct attractors within a single system and the same parameters. In a turbulence-related study by De Wit et al. (2022), bimodality is employed to describe the characteristic that two attractors are connected by specific trajectories. De Wit et al. demonstrated that the quasi-2D condensate state and the regular 3D state can coexist within a certain range of Reynolds number (Re) and box sizes (Figure R2B.1).

On the other hand, Shen et al. (2021, 2022) presented the concept of coexisting attractors in the generalized Lorenz model to offer a revised perspective on the nature of weather and climate. They suggested that "*The atmosphere possesses chaos and order with distinct predictability; it includes, as examples, emerging organized systems (such as tornadoes) and time varying forcing from recurrent seasons*".

Duality in Turbulence

As explained in the main text, classical studies on turbulence have highlighted the significant characteristics of direct and inverse energy cascades in both 3D and 2D turbulence. Pouquet and Marino (2013) extended this understanding by demonstrating the presence of a dual energy cascade, characterized by constant flux in both direct and inverse directions, within rotating stratified turbulence. Their research indicates the potential coexistence of idealized large-scale dynamics, governed by quasigeostrophic motions in the ocean and atmosphere, alongside the generation of small scales essential for transport processes.

Dimensional Transitions

A recent study by Boffetta (2023) offers a review of the transition between two-dimensional (2D) to three-dimensional (3D) turbulence. The study also delves into the significant factors that influence the relative strength of the direct and inverse cascades.

For instance, as illustrated in Figures 1, 4, and 5 of Boffetta (2023), the growth rate of kinetic energy in large-scale flows (contributed by the inverse cascade) exhibits a dependence on layer thickness, rotation, and density stratification, respectively. Figure R2B.2, derived from his Figure 1, demonstrates that the normalized growth rate decreases as the layer thickness increases.

Furthermore, Boffetta (2023) highlights the findings of numerical simulations conducted using the 3D Boussinesq equation in the presence of both rotation and a stable stratification. These simulations revealed the coexistence of a small-scale energy flux and a large-scale dynamics dominated by quasigeostrophic motion.

References:

- Boffetta, G. 2023: Dimensional Transitions in Turbulence: The Effects of Rotation and Stratification. *Atmosphere* 2023, 14, 1688. https://doi.org/10.3390/atmos14111688
- De Wit, X.M.; Van Kan, A.; Alexakis, A., 2022: Bistability of the large-scale dynamics in quasi-two-dimensional turbulence. *J. Fluid Mech*. 2022, 939, R2.
- Pouquet A., and R. Marino, 2013: Geophysical Turbulence and the Duality of the Energy Flow Across Scales. *Physical Review Letters. 111, 234501.*
- Shen, B.-W., 2025: Attractor Coexistence, Butterfly Effects, and Chaos Theory (ABC): A Review of Lorenz Models and a Generalized Lorenz Model. The 16th Chaos

International Conference CHAOS 2023. Springer Proceedings in Complexity. Springer, Cham. Proofs available from ResearchGate:
https://www.researchgate.net/publication/372194700 (accepted 2023; to appear in Feb 2025)

- Shen, B.-W**.**, 2019a: Aggregated Negative Feedback in a Generalized Lorenz Model. International Journal of Bifurcation and Chaos, Vol. 29, No. 3 (2019) 1950037 (20 pages). https://doi.org/10.1142/S0218127419500378
- Shen, B.-W., R. A. Pielke Sr., X. Zeng, J.-J. Baik, S. Faghih-Naini, J. Cui, and R. Atlas, 2021: Is Weather Chaotic? Coexistence of Chaos and Order within a Generalized Lorenz Model. *Bulletin of the American Meteorological Society*, *102*(1), E148-E158. https://doi.org/10.1175/BAMS-D-19-0165.1
- Shen, B.-W., R. A. Pielke Sr., X. Zeng, J. Cui, S. Faghih-Naini, W. Paxson, A. Kesarkar, X. Zeng, R. Atlas, 2022: The Dual Nature of Chaos and Order in the Atmosphere. Atmosphere 13, no. 11: 1892. https://doi.org/10.3390/atmos13111892.

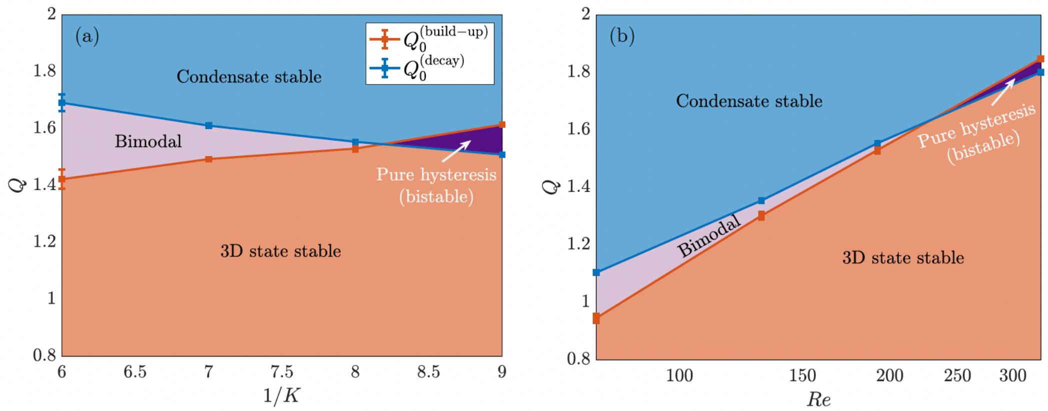

Figure 4: Phase diagrams of the condensate for varying box size at $Re = 192$ (a) and varying Re at $1/K = 8$ (b). Red and blue lines denote the asymptotes of waiting times for build-up and decay events, respectively.

Figure R2B.1: Phase diagrams of the condensate for varying box sizes at a constant Re = 192 (a) and varying Re at a fixed box size of $1/K = 8$ (b). Red and blue lines represent the asymptotes of waiting times for build-up and decay events, respectively. (De Wit et al. 2022).

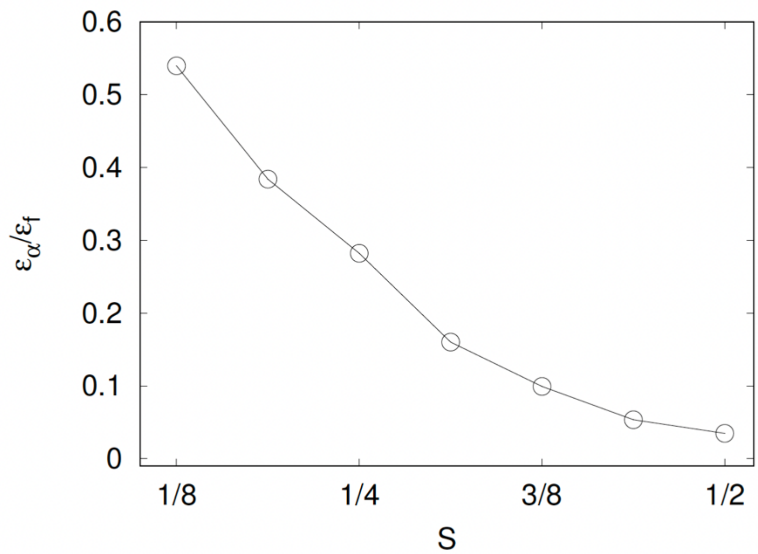

Figure 1. Growth rate of kinetic energy ε_α normalized with the energy input ε_f as a function of the thickness number $S = L_z/L_f$. Simulations at resolution $N_x = N_y = 512$, $N_z = 64S$ data from [17].

Figure R2B.2: Growth rate of the normalized kinetic energy in the vertical axis, plotted against the thickness number (S) in the horizontal axis. (Boffetta 2023).