Responses Part 2A (R2A): A Brief Note on Turbulence-based Turnover Time

We appreciate the reviewer's comments, which provide an opportunity for further discussions about Lilly's (1972, 1973) use of turnover time to estimate predictability. While we're working on providing comprehensive responses that will be available later, we've offered a quick version that specifically addresses the following comments by Reviewer 2:

> The main flaw in the reasoning of the authors is to assume that the discretisation can be chosen a priori. The inertial range of turbulence implies that the local turnover time depends on a power of the scale. From this follows that changes in the turnover time in scale, which are related to the growth of error and the propagation of uncertainty, cannot be linear with the scale. In other words, if we add a fixed increment to the scale, the increment of the turnover time will not be equal across scales when scaled in local turnover times. On the other hand, if we add an increment of scale that is proportional to the scale, the increment of the turnover time will be proportional to a constant in local turnover times. This is what gives Lorenz's and Lilly's models validity even though they are very crude representations of reality. In this sense, the physical assumptions and the theory come before determining the appropriate scaling for the scale increments, and not the other way round. I do not think the authors have a clear physical justification for a linear scaling of the scale increments. Moreover, many investigations have shown that the growth of perturbations follow a self-similar scaling, none of which are cited in the paper, for instance, a recent one, Boffetta, G., & Musacchio, S, PRL, 2017.

In response to Reviewer 1's comments, we've added a brief note explaining the connection between turnover time and the target energy spectrum described by the -5/3 power law (egusphere-2024-2228-AC2). Below, we delve into the concepts of scale invariance and self-similarity associated with power laws to address the raised concerns.

Scale Invariance

First and foremost, we would like to emphasize that the choice between a uniform or non-uniform grid is simply a matter of selecting sample points to construct a series or an integral in our manuscript. This choice does not alter the power law characteristics of the turnover time. However, a different choice may yield a different result for the series or integral.

For instance, in the inertial subrange where Kolmogorov's -5/3 power law holds, the energy spectrum is expressed as:

$$\mathrm{E(k)} \sim \varepsilon^{\frac{2}{3}} \mathrm{k}^{-\frac{5}{3}}. \qquad (\mathrm{R2A.1})$$

The corresponding turnover time is given by:

$$\tau_{\mathrm{k}} \sim \varepsilon^{-\frac{1}{3}} \mathrm{k}^{-\frac{2}{3}}. \qquad (\mathrm{R2A.2})$$

Equations (R2A.1) and (R2A.2) imply that when the kinetic energy spectrum exhibits a -5/3 power, the corresponding turnover time exhibits a -2/3 power. These equations align with the findings discussed in Boffetta and Musacchio (2017).

Secondly, we apply Equation (R2A.2) to demonstrate the characteristic of the reviewer's comments.

> *"On the other hand, if we add an increment of scale that is proportional to the scale, the increment of the turnover time will be proportional to a constant in local turnover times."*

The comment above suggests the following calculation:

$$\frac{d\tau_k}{\frac{1}{k}dk} = \mathrm{k}\frac{d\tau_k}{dk} \sim \mathrm{k}\varepsilon^{-\frac{1}{3}}\left(-\frac{2}{3}\right)\mathrm{k}^{-\frac{5}{3}} = -\frac{2}{3}\tau_k \qquad (\mathrm{R2A.3})$$

Note that the left-hand side can be written as $d\tau_k/d(\ln(k))$. The above indicates that the derivative of turnover time with respect to ln(x) is proportional to the local turnover time. Please note that the property mentioned in the reviewer's comment remains valid provided the turnover time follows a power function of wavenumber k, with the exception of constant functions. However, scale invariance associated with power laws is commonly illustrated using a linear transformation of wavenumber as discussed below.

In certain turbulent flows, self-similarity is formalized through the concept of "self-preserving" or "self-similar" solutions to the governing equations. A power-law form for the energy spectrum $E(k) = k^{-m}$ is indeed a strong indicator of scale invariance and thus suggests self-similarity. When $E(k) \approx k^{-m}$ for some constant exponent $-m$, it means that if we rescale the wavenumber by some factor, the shape of the spectrum remains the same up to a multiplicative constant. Rescaling k by a constant factor λ simply results in

$$E(\lambda k) = (\lambda k)^{-m} = \lambda^{-m}k^{-m} = \lambda^{-m}E(k) \qquad (\mathrm{R2A.4})$$

This transformation shows that the shape of the spectrum is preserved under a change of scale; the only difference is an overall multiplicative factor λ^{-m}. This invariance under scale transformations is the essence of self-similarity. Scale invariance is most intuitively defined as invariance under uniform "stretching" or "shrinking" of the scale, i.e., multiplying by a constant factor. This keeps the physical interpretation straightforward: zooming in or out by the same factor at every scale.

On the other hand, if one instead tries to rescale k using an exponential function, say something like $k \rightarrow 2^{\lambda k}$ or $k \rightarrow e^{\lambda k}$, as employed in Lilly's formula and discussed in the manuscript, it turns a power law into an exponential form in terms of the new variable. Exponential transformations lose that intuitive interpretation in the original k space. As before, we cautioned that the findings obtained through such transformations (for instance, the selection of a non-uniform grid) should be interpreted with caution.

Error Growth and Lyapunov Exponent

We have presented an analysis of Lyapunov exponents within generalized Lorenz models over a decade (Shen 2014-2019). In recent years, we have also discussed the validation of Lyapunov exponents in determining predictability horizons (e.g., Shen et al. 2022a, b), citing related studies by Aurell, Boffetta et al. (1996).

We acknowledge that Aurell et al. (1996) referenced Ruelle's work on the relationship between two time scales, namely the reciprocal of the largest Lyapunov exponent and turnover time for the smallest eddy (for instance, Figure R2A.1). Nevertheless, we assert that no theoretical studies have conclusively proven a direct one-to-one correspondence between these two time scales across a wide range of wavenumbers.

While Boffetta and Musacchio (2017) employed the concept of error growth, particularly the Lyapunov exponent, to evaluate predictability, this manuscript delves into the application of the integral of the turnover time, as expressed in Eq. (R2A.2), to determine predictability horizons, encompassing both finite and infinite predictability horizons. We will offer supplementary responses in the final responses section if the reviewer can provide specific comments.

In fact, the findings of Boffetta and Musacchio (2017) support our view in the manuscript and Shen et al. (2021), suggesting the discrepancies between Lorenz's and Lilly's formulas and a revised perspective on the dual nature of chaos and order, with distinct predictability.

> *"Our findings suggest that the dimensional estimate of the Lyapunov exponent as the inverse Kolmogorov time does not give an accurate characterization of the chaoticity of a turbulent flow."*

> *"The independence of the FSLE in the scaling range on the value λ observed for infinitesimal errors provides a clear explanation of how in turbulent flows it is possible to*

observe the coexistence of long predictability time at large scales and strong chaoticity at small scales."

In fact, if the above predictability time scales are related to the turnover time scales at the corresponding wavenumbers, we can inquire about the physical significance of the sum of these turnover times.

The Lilly's Series and Integral: $\int^{\infty} \tau_k d\ \ln(k)$

As discussed, the following integral of turnover time with respect to ln(k) has been used in Lilly (1990) and Vallis (2006):

$$\int^{\infty} \tau_k d\ ln\,(k) = \int^{\infty} \tau_k \frac{1}{k} dk \qquad (\mathrm{R2A.5})$$

We contend that while the aforementioned integral may capture certain aspects of turbulence, such as cascades, and consequently weather patterns, it cannot definitively determine the upper limit of predictability for the entire weather system.

For instance, such a choice with a non-uniform grid omits crucial weather systems at wavenumber 10 for baroclinic waves. Furthermore, Lorenz's 1969 model was derived from a partial differential equation (PDE) that preserves 2D barotropic vorticity. The original PDE lacks the necessary forcing and dissipation terms required for studying turbulence (e.g., Figure R2A.2), and it also omits the inclusion of thermodynamic equations essential for understanding weather. Consequently, the results obtained from Lorenz's and Lilly's formulas should be interpreted with caution. As we have consistently emphasized in several of our papers, when Lorenz published his book titled "The Essence of Chaos" in 1993, he did not cite the Lorenz (1969) study or the related works by Lilly.

References:

- Boffetta, G. and S. Musacchio, 2017: Chaos and Predictability of Homogeneous-Isotropic Turbulence. *Physical Review Letters*, PRL 119, 054102.
- Boffetta and Celani (1998): Predictability in chaotic systems and turbulence
- Ruelle, 1979: *Microscopic Fluctuations and Turbulence. Physics Letters, Vol.72A, No.2, pp. 81-82.*
- Shen, B.-W., Pielke, R.A., Sr., Zeng, X., Baik, J.-J., Faghih-Naini, S., Cui, J., Atlas, R. 2021a: Is weather chaotic? Coexistence of chaos and order within a generalized Lorenz model. Bull. Am. Meteorol. Soc. 2021, 2, E148–E158.

- Shen, B.-W., R. A. Pielke Sr., X. Zeng, 2022a: One Saddle Point and Two Types of Sensitivities Within the Lorenz 1963 and 1969 Models. Atmosphere 13, no. 5: 753. https://doi.org/10.3390/atmos13050753
- Shen, B.-W., R. A. Pielke Sr., X. Zeng, J. Cui, S. Faghih-Naini, W. Paxson, A. Kesarkar, X. Zeng, R. Atlas, 2022b: The Dual Nature of Chaos and Order in the Atmosphere. Atmosphere 13, no. 11: 1892. https://doi.org/10.3390/atmos13111892.

size $l_0 \sim (\nu^3/\epsilon)^{1/4}$. (As usual, ν is the kinematic viscosity and ϵ the energy dissipated per unit mass and unit time.) This largest characteristic exponent is thus

$$\omega(l_0) \sim \epsilon^{1/3}/l_0^{2/3} \sim \nu/l_0^2 , \quad (1)$$

and a disturbance $\Delta(t)$ will grow with time like $\Delta(0) \times \exp \omega(l_0)t$. We want to estimate the time it takes for a velocity fluctuation due to the motion of the molecules composing the fluid to become noticeable, i.e., of the order of magnitude of the flow itself. This gives the condition

$$\Delta(0)\exp \omega(l_0)t \sim \nu/l_0 . \quad (2)$$

Figure R2A.1 The relationship between the largest Lyapunov exponent (LE, $\omega(l_0)$) and the eddy size (l_0). (Ruelle 1978). Equation (1) represents the following expressions:

$$LE = \omega(l_0) \sim \frac{1}{l_o^{2/3}} \sim k_o^{2/3}$$

$$T \sim \frac{1}{LE} \sim \frac{1}{\omega(l_0)} \sim k_o^{-2/3}$$

The time scale T, which corresponds to the largest Lyapunov exponent, follows a -2/3 power law and is therefore proportional to the turnover time at the smallest eddy.

3 FINITE SIZE PREDICTABILITY IN TURBULENCE

We now consider fully developed turbulence as a well known example of system with many characteristic scales. Because of the ubiquity of turbulence in nature, the example is also of physical relevance. Following the original picture of Kolmogorov, turbulence is characterized by an wide range of locally interacting scales (inertial range) in which the energy is simply transfer from large to small scale. The energy cascade is maintained stationary by an energy source a large scales (forcing) and an energy sink at small scales (viscous dissipation). The typical transfer time at scale ℓ (eddy turnover time) is dimensionally given by

$$\tau_\ell \simeq \frac{\ell}{u_\ell} \tag{5}$$

Let we now consider the growth of finite errors in a turbulent velocity field. We assume to have two realizations of the turbulent flow $\mathbf{u}(\mathbf{x}, t)$ and $\mathbf{u}'(\mathbf{x}, t)$ at a distance $\delta(t)$. Following the phenomenological ideas of Lorenz [5], the growing rate for δ can be identified with the inverse eddy turnover time τ_ℓ at the scale ℓ at which $u_\ell \sim \delta$. Indeed, at smaller scales ℓ' for which $u_{\ell'} \ll \delta$ we can assume that the error is already saturated, as in the example of section 2; larger scales have longer characteristic time and thus are subleading in the error growth rate. Expressing the eddy turnover time (5) in terms of the velocity difference $u_\ell \simeq \delta$, we obtain through (7) the prediction for the FSLE in turbulence [8]

$$\lambda(\delta) \simeq \epsilon\delta^{-2} \tag{9}$$

Figure R2A.2: The top panel emphasizes the significance of energy sources and sinks in sustaining stationary turbulence. Notably, the partial differential equation (PDE) used to derive the Lorenz 1969 model omits these sources and sinks. The bottom panel asserted that Lorenz proposed the relationship between the growth rate and turnover time. However, to our knowledge, we have not found any of Lorenz's studies that explicitly describe this relationship. (Two panels were adapted from Boffetta and Celani 1998.)