Responses Part 1C (R1C): Qualitative Predictability Estimates Using Lilly's Formula and Comparative Insights

This brief report specifically addresses the following comments by Reviewer 1:

> Actually, the point I have found of most interest in the paper is the fact that Lorenz, although he used in Lorenz (1969d) a linear nonturbulent model, found a predictability time of about the same magnitude as Lilly, who used a nonlinear turbulent model. That fact, which is certainly of great interest, is not further discussed in the paper, but I accept it could be considered as going beyond its scope.

We appreciate the reviewer's comments that helped us delve deeper into the specific contributions of Lorenz's and Lilly's formulas to predictability estimates. This report emphasizes that in simple models or formulas, a model time may not align with real-world time. Therefore, qualitative predictability estimates should be a primary focus when applying Lorenz's and Lilly's formulas.

Based on Table R1C-1, we first highlight that Lorenz's formula was derived from simulations using his simplified system with a non-uniform grid. This system produced a saturation time of 16.8 days for large-scale flows.

The saturation time sequence derived from the Lorenz (1969) model has been interpreted as forming a convergent geometric series, suggesting a finite predictability limit. As a result, the 16.8-day limit has been frequently cited as a theoretical threshold for predictability.

However, our re-analysis of Lorenz's saturation times finds little evidence to support the existence of such a geometric series. Furthermore, we argue that the 16.8-day limit, based on a simplified model in which one model time unit represents six real-world days, is not a robust upper bound. Interestingly, Lorenz himself reported a longer value of 20.6 days in 1972, using a comparable approach and a kinetic energy (KE) spectrum featuring a spectral gap (see Shen et al. 2024).

Historically, Lilly's formula was designed to show that the turnover times in turbulence can create a series similar to the one Lorenz attempted to construct. However, it's important to note that the characteristics (such as time scales) differ between turbulence and idealized weather conditions, making it difficult to obtain precise quantitative estimates.

Lilly's formula did not provide a specific characteristic time scale for quantitatively determining the predictability limit. While Lilly's formula may be qualitatively applied to determine the finite or infinite predictability of various KE spectra, this study also reveals the dependence of finite or infinite predictability on the spectral grid spacing.

Based on Tables R1C-1 and R1C-2, we present a concise overview of key studies in atmospheric predictability. These studies explore various approaches, metrics used to estimate predictability limits, and the significant conclusions drawn from their findings.

1. Charney et al. (1966):

- Approach: Used a General Circulation Model (GCM) to study predictability.
- Estimate Metrics: The doubling time of perturbations was determined to be 5 days over a 30-day period, spanning from Day 234 to Day 264.
- Major Conclusions: Established the well-known "two-week predictability limit" for atmospheric processes (Lorenz 1993).

2. Lorenz (1969d):

- Approach: Utilized a system of simplified second-order Ordinary Differential Equations (ODEs) where <u>one model time unit equated to 6 real-world days</u>. Additionally, empirical formulas were used.
- Estimate Metrics: Evaluated saturation times when perturbations grew to their maximum amplitude, determined by the kinetic energy (KE) spectrum, and calculated "saturation time differences."
- Major Conclusions: Derived a 16.8-day predictability limit for large-scale processes from model simulations. Emphasized the relationship between predictability and the KE spectrum using the empirical formula, referred to as the Lorenz formula in this study.

3. Lilly (1972, 1973, 1990):

- Approach: Based on turbulence theory.
- Estimate Metrics: Assessed predictability using a series and integral of turnover times in turbulence.
- Major Conclusions: Demonstrated that finite or infinite predictability depends on the characteristics of the KE power spectrum.

4. Vallis (2006):

- Approach: Considered two frameworks: local cascade theory and direct interaction theory.
- Estimate Metrics: Turnover times and growth rates were used to infer time scales of predictability.
- Major Conclusions: The local cascade theory highlights predictability limits linked to energy cascading laws, where:
    - The -5/3 power law indicated finite predictability.
    - The -3 power law suggested infinite predictability.

  The local cascade theory's approach is consistent with Lilly's formula. On the other hand, the direct interaction theory suggests:
    - Both -5/3 and -3 power laws indicated infinite predictability.

5. Shen et al. (2024) (This study):

- Approach: Revisited Lorenz's and Lilly's formulas while using uniform grid spacing in the series and integrals.
- Estimate Metrics: Focused on how convergence depended on uniform and non-uniform spectral grid spacing in addition to the KE power spectrum.
- Major Conclusions: Illustrated the dependence of convergent series and integrals on spectral grid spacing and the kinetic energy power spectrum. Highlighted the ability of the Lorenz and Lilly's formulas to provide qualitative predictability estimates.

This report offers a concise overview of the selected major advancements in atmospheric predictability research. It covers a range of real-world physical models, such as General Circulation Models (Charney et al. 1966), simplified Ordinary Differential Equation (ODE) systems (Lorenz 1969d), and turbulence-based theories (Lilly 1972, 1973, 1990). Key metrics, including doubling times, saturation times, turnover times, and e-folding times, are crucial in determining the finite or infinite predictability limits of these models. Recent studies, including this one, caution against interpreting findings solely based on Lorenz and Lilly's formulas, which are dependent on spectral grid spacing. Instead, we advocate for a thorough review of predictability challenges by employing advanced theoretical models, physics-based real-world models, and AI-driven prediction models (Shen et al. 2024; Shen 2024).

Table R1C-1: A summary of predictability studies. For more details on the findings by Vallis (2006), refer to Table R1C-2.

Study	Approach	Estimate Metrics	Major Conclusions
Charney et al. (1966)	General Circulation Model (GCM)	Doubling time of 5 days	Two-week predictability limit
Lorenz (1969d)	Simplified 2nd order ODEs	Saturation times (maximum perturbation growth based on KE spectrum)	16.8-day limit for large-scale processes
	Empirical formulas	A series of "saturation time differences"	KE spectrum dependence
Lilly (1972, 1973, 1990)	Turbulence theory	Series and integral of turnover times	Finite/infinite predictability dependence on KE spectrum
Vallis (2006)	Local cascade theory	Integral of turnover times	-5/3 law: finite predictability
			-3 law: infinite predictability
	Direct interaction theory	(Exponential) growth rate, which is the inverse of the turnover time at the large-scale	-5/3: infinite predictability
			-3 law: infinite predictability
Shen et al. (2024) (this study)	Revisit of Lorenz and Lilly's formulas with uniform grid spacing	Series and integral of turnover times using both uniform and non-uniform grids	Dependence of convergence on spectral grid spacing and KE spectrum

Table R1C-2: Qualitative predictability estimated using the local cascade theory and the direct interaction theory. In this context, η is the energy injection rate, ϵ denotes the energy dissipation rate, k_0 signifies the largest scale wavenumber, and k_1 denotes the smallest scale wavenumber (which corresponds to the largest wavenumber). The column "Predictability" is determined as k_1 approaches infinity. The column titled "Equation #" provides the equation number as per Vallis (2006).

Approach	Estimated Time	Predictability	Equation #
Local cascade theory	$T_{2d} \sim \eta^{-1/3} \ln\left(\frac{k_1}{k_0}\right)$	infinite	8.82a
	$T_{3d} \sim \epsilon^{-1/3} k_0^{-2/3}$	finite	8.82b
Direct interaction theory	$T'_{2d} \sim \eta^{-1/3} \ln\left(\frac{k_1}{k_0}\right)$	infinite	8.86
	$T'_{3d} \sim k_0^{-2/3} \epsilon^{-1/3} \ln\left(\frac{k_1}{k_0}\right).$	infinite	8.89

References

- Shen, B.-W.*, R. A. Pielke Sr., X. Zeng, and X. Zeng, 2024: Exploring the Origin of the Two-Week Predictability Limit: A Revisit of Lorenz's Predictability Studies in the 1960s. Atmosphere 2024, 15(7), 837; https://doi.org/10.3390/atmos15070837
- Shen, B.-W.*, 2024: Exploring Downscaling in High-Dimensional Lorenz Models Using the Transformer Decoder. Mach. Learn. Knowl. Extr. 2024, 6, 2161-2182. https://doi.org/10.3390/make6040107