Responses Part 1B (R1B): A Brief Note on Turbulence-based Turnover Time

This brief report specifically addresses the following comments by Reviewer 1:

> From what I understand, the significant part of the paper begins with the introduction of the eddy turnover time $\tau(k)$ (Eq. 5). That quantity is introduced with a reference to Vallis (2006), without appropriate explanation as to its physical significance nor on how it has been determined. The only indication in the paper is that $\tau(k)$ is *the time for a parcel with velocity v_k to move a distance of 1/k, with v_k being the velocity associated with wavenumber k*. (ll. 298-299). More information would be necessary, be that only to refresh the reader's memory. I simply note that, since v_k is defined as the velocity associated with wavenumber k, the variations of v_k with k contain the same basic information as the spectrum of kinetic energy, which is considered later in the paper. That should be mentioned explicitly.

We appreciate the reviewer's comments for enhancing physical discussions about the turnover time used by Lilly (1972, 1973) to estimate predictability. These discussions rely on Kolmogorov turbulence theory instead of Lilly's numerical models. Below, we review the discussions in Vallis (2006) to introduce the concepts of turnover time and its relationship to the target energy spectrum characterized by the -5/3 power law. Subsequently, the turnover time is expressed as a function of wavenumbers following the -2/3 power law. In the final section, we discuss possible reasons why an integral of turnover time can be used to qualitatively estimate predictability limits, which can be finite or infinite. We also illustrate the impact of uniform and non-uniform grids on the predictability estimates.

Energy Spectrum $\mathrm{E(k)}$

The energy spectrum $\mathrm{E(k)}$ describes how the kinetic energy of turbulence is distributed across wavenumbers k, where $\mathrm{k} \sim \frac{1}{\mathrm{L}}$ corresponds to the inverse of a characteristic length scale L.

- The total kinetic energy per unit mass in the system is given by integrating over all wavenumbers:

$$\text{Total energy} = \int \mathrm{E(k)\, dk}. \qquad \text{(R1B. 1)}$$

Physical Interpretation of $\mathrm{E(k)}$

The quantity $\mathrm{E(k)\, dk}$ represents the amount of energy per unit mass contained in the range of wavenumbers between k and $\mathrm{k + dk}$.

- For a specific k, the energy density per unit mass at that scale is proportional to $\mathrm{E(k)}$.

Velocity Scale: $v_k^2 \sim kE(k)$

The velocity v_k at scale k (or eddy size L) is a measure of the characteristic turbulent velocity at that scale. The kinetic energy per unit mass associated with eddies of scale $L \sim \frac{1}{k}$ is proportional to v_k^2, yielding:

$$v_k^2 \sim kE(k). \qquad \text{(R1B. 2)}$$

The above equation indicates that the energy in the eddies at scale k is distributed across a "shell" of wavenumbers near k (i.e., a small range of wavenumbers near k).

Turnover Time (τ_k) and Velocity (v_k)

The turnover time in turbulence refers to the characteristic *time for a parcel with velocity* v_k *to move a distance of 1/k, with* v_k *being the velocity associated with wavenumber k.* It is a measure of energy cascading to smaller scales. Mathematically, it is written as follows:

$$\tau_k \approx \frac{\frac{1}{k}}{v_k} = \frac{L}{v_k}, \qquad \text{(R1B. 3)}$$

From Eq. (R1B.2) and Eq. (R1B.3), we have

$$\tau_k \approx \frac{1}{kv_k} = \frac{1}{k^{\frac{3}{2}}\mathrm{E(k)}^{\frac{1}{2}}}. \qquad \text{(R1B. 4)}$$

The above equation connects the turnover time to energy spectrum E(k) at specific scales, providing a foundation for applying the Lilly's integral formula. In the inertial subrange where Kolmogorov's $-5/3$ power law, the energy spectrum is written as:

$$\mathrm{E(k)} \sim \varepsilon^{\frac{2}{3}}\mathrm{k}^{-\frac{5}{3}}. \qquad \text{(R1B. 5)}$$

Plugging Eq. (R1B.5) into Eq. (R1B.4) yields:

$$\tau_\mathrm{k} \sim \varepsilon^{-\frac{1}{3}}\mathrm{k}^{-\frac{2}{3}}. \qquad \text{(R1B. 6)}$$

Equations (R1B.5.) and (R1B.6) suggest that when the kinetic energy spectrum exhibits a -5/3 power, the corresponding turnover time exhibits a -2/3 power.

The Lilly's Series and Integral: $\int^\infty \tau_k d \ln{(k)}$

As shown in the top panel of Figure R1B-1, Lilly (1972) first suggested that the sum of a sequence of turnover time over the selected wavenumbers ($2^n k_L$, where k_L represent the reference wavenumber) could provide predictability estimate. Later, the series was re-formulated as an integral in Lilly (1973) and became the following in Lilly (1990):

$$\int^{\infty} \tau_k \frac{1}{k} dk = \int^{\infty} \tau_k d\, ln\,(k) \qquad (\text{R1B}.7)$$

Equation (R1B.4) is a representation of Equation (2.1) from Lilly (1990), while Equation (R1B.7) is indeed Equation (2.2) from the same source. The expression on the right-hand side of Equation (R1B.7) was also adapted by Vallis (2006). Mathematically, Equation (R1B.7), which includes the contribution of all wavenumbers, including $k \to \infty$, is known as an improper integral. This integral can either yield a finite or an infinite value. The physical interpretation of the above integral and its significance in terms of its finite value are discussed in the subsequent subsection.

Possible Physical Interpretation of Lilly's Series and Integral

Although Lilly's series and integral, derived from turnover time, have been utilized to estimate predictability (e.g., Lilly 1972, 1973, 1990; Vallis 2006), these studies lack convincing detailed physical interpretations of these series and integrals. To address this gap, we present our analysis below.

The turnover time at a specific scale denotes the duration of energy transfer from that scale to a different scale within a system. Consequently, it becomes evident that the physical processes underlying the turnover time differ from those associated with the saturation time, which is linked to the growth of initial energy and the instability of the system. Therefore, our manuscript underscores the significance of investigating the physical and mathematical connection between the turnover time employed by Lilly and the saturation time difference used by Lorenz.

On the other hand, we can explore what we can learn from the Lilly's formula alone, without comparing it to Lorenz's formula, in this response. From Figure R1B-1, it is reasonable to interpret the sum or integral of the turnover time with respect to (non-dimensional) wavenumber as the total time for energy cascade across all scales through a chain of processes. Consequently, the integral could potentially signify the total transfer time of an initial noise throughout the entire fully turbulent system. Therefore, the integral may serve as another definition of predictability horizons. However, readers should be reminded of the following:

- To what extent does the predictability limit of stationary turbulence reflect the predictability limit of weather?
- Does the integral with a varying scale factor of 1/k in Eq. (R1B.7) represent a specific or general route for energy cascade observed in weather?

To address the first equation, it is acknowledged that predictability estimates in stationary and decaying turbulence differ (e.g., Metais and Lesieur 1986). While it's possible to argue that both cases yield comparable results, our response is that turbulence models that may not accurately simulate real weather systems have adjustable time scales that can effectively reach the predictability limit of two weeks. Furthermore, when using a specific case like Madden-Julian Oscillations (MJOs), can turbulence models be employed to estimate the predictability of MJOs?

To address the second question, our manuscript first presented that Lilly's series over a non-uniform grid ($k = 2^n k_L$) and Lilly's integral with a varying scale factor of 1/k are consistent. A follow-up question is the physical meaning of the scale factor 1/k in the integral. After extensive searching for comprehensive discussions, we found none in the original studies. However, in an email from Vallis in 2022, it was suggested that the inclusion of the scale factor 1/k may be due to self-similarity:

> "*The error cascades from 4k to 2k, and then 2k to k, and so on, because of self-similarity in the cascade*."

The above describes the inverse cascade. In the forward cascade, energy transfers from $k \rightarrow 2k \rightarrow 4k \rightarrow \cdots 2^{n-1}k_L$, and so on. While we agree that Vallis's physical explanation supports the scale factor of 1/k in Eq. (R1B.7), we argue in the manuscript that real weather systems may not always exhibit exact self-similarity. For instance, Vallis's assumption of self-similarity excludes non-power-of-2 wavenumbers such as 3, 5, and 10. As discussed in the manuscript, wavenumber 10 systems (e.g., baroclinic waves) play important roles in real weather phenomena.

In the manuscript, we discussed that the factor of 1/k is crucial for producing a finite value in Eq. (R1B.7), thereby establishing the concept of a finite predictability. However, our manuscript also demonstrated that when considering a different set of wavenumbers, the sum of turnover time over a uniform grid (i.e., an integral with respect to k, associated with a different scaling factor $1/k_L$) can result in an infinite value. This underscores the limitations of Lilly's formula with a scaling factor 1/k in Eq. (R1B.7) when applied to predictability limits in complex weather systems.

Lilly (1972)	$\tau(k) = A^{-1/2} k^{(n-3)/2}.$ (2.2)	Eqs. (2.2)-(2.3) of Lilly (1972)
	$\sum_{n=0}^{p} \tau(2^n k) = \frac{1-2^{(n-3)p/2}}{1-2^{(n-3)/2}} \tau(k).$ (2.3)	
Lilly (1973)	$\sum_{n=0}^{\infty} \tau(2^n k) = \frac{\alpha^{-1/2} \varepsilon^{-1/3} k^{-2/3}}{1 - 2^{-2/3}} \sim 2.7\alpha^{-1/2} \varepsilon^{-1/3} k^{-2/3} = 2.7\tau(k)$. (9.3)	Eq. (9.3) of Lilly (1973)
Lilly (1990)	$T_e(k) = 1/[kV(k)] = [k^3 E(k)]^{-1/2}.$ (2.1)	Eqs. (2.1) and (2.2) of Lilly (1990)
	$T_p(k) = \int_k^{k_T} (T_e/k)\, dk = \int_k^{k_T} [k^5 E(k)]^{-1/2}\, dk.$ (2.2)	
	Lorenz assumed $E(k) \sim k^{-5/3}$ for wavelengths much less than the earth's circumference, leading to the relation $T_p \sim 1{\cdot}5T_e$. The scaling velocity $V(k)$ of thunderstorm	

Figure 3: The evolution of Lilly's formulas for determining a predictability horizon.

Figure R1B-1: This figure is from Figure 3 of the manuscript. In the bottom panel, Eqs. (2.1) and (2.2) of Lilly (1990) are represented as Eqs. (R1B.4) and (R1B.7), respectively.

References:

- Metais O. and M. Lesieur, 1986: Statistical Predictability of Decaying Turbulence. J. Atmos. Sci., 43 (9), 857-870.