## Response to reviewer 3 (egusphere-2024-1477 manuscript)

Thank you for your careful review. We appreciate your thoughtful comments to improve our paper. We copied your comments in the blue text and have provided our responses in the black text. We have revised the manuscript according to your suggestions. Our point-by-point responses to the reviewer's comments are provided below. We hope that these improvements satisfactorily address the issues pointed out by you.

## General comments

1. The authors make the claim that for global LES modeling (100km grid spacing), high-order DG methods will be important in this context. I dont object to this argument (and dont request any changes), but I will mention that I dont find the arguments persuasive. If the arguments are correct, I think DG methods would be more common in regional models, which often run in the LES regime.

We agree that currently there are less studies of atmospheric LES using DGM. This may be because the numerical behavior was not well investigated in the LES regime. However, recent studies (Sridhar et al., 2022; Kawai and Tomita, 2023; Souza et al., 2024) indicate the possibility of DG dynamical cores to the atmospheric LES. Furthermore, in the CFD community, DGM seems to be regarded as a promising method for turbulent simulations using explicit and implicit LES in terms of high-order accuracy, flexibility of complex geometry, and scalability of parallel computations. These features would be benefit for the future high-resolution atmospheric simulations with O(10-100 m) grid spacing where the complex structure of small-scale topography need to be treated. Thus, we expect that global dynamical cores based on the high-order element-based method will become more common in the LES regime.

[References]


- Kawai, Y. & Tomita, H. (2023): Numerical Accuracy Necessary for Large-Eddy Simulation of Planetary Boundary Layer Turbulence Using the Discontinuous Galerkin Method. *Monthly Weather Review*, 151(6), 1479-1508. https://doi.org/10.1175/MWR-D-22-0245.1
- Sridhar, A., Tissaoui, Y., Marras, S., Shen, Z., Kawczynski, C., Byrne, S., ... & Schneider, T. (2022). Large-eddy simulations with ClimateMachine v0.2.0: a new open-source code for atmospheric simulations on GPUs and CPUs. *Geoscientific Model Development*, 15(15), 6259-6284. https://doi.org/10.5194/gmd-15-6259-2022
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e2022MS003527. https://doi.org/10.1029/2022MS003527

2. One issue not address in this paper is the timestep. DG methods with the values of p proposed here will be quite expensive. A good comparison showing how expensive high order DG can be compared to finite volumes is given in Brdar et al, https://doi.org/10.1007/s00162-012-0264-z which compares the DG based DUNE model with the finite volume (operational weather forecast model), COSMO. See also my comment below in the conclusions about numerical efficiency.

Thank you for informing us about an important work, Bardar et al. (2011), who discussed the computational time to reach a given error tolerance for DG and conventional FV dynamical cores, the DUNE and the COSMO. First, please notice that the temporal scheme is quite different between the two dynamical cores. DUNE adopted a fully explicit Runge-Kutta (RK) method for the inviscid terms. On the other hand, the COSMO adopted a sophisticated time-splitting approach in which the slow processes are integrated with an explicit RK method, while the fast processes are integrated with a small timestep horizontally by a forward-backward scheme and vertically by an implicit Crank-Nicholson scheme. Thus, it is difficult to directly evaluate the computational overhead due to the timestep restriction with DGM. (We would like to emphasize that, as described in the conclusion of Bardar et al. (2011), the different treatment of temporal scheme was not the focus in their experiments. We think that their comparison of the behavior of numerical convergence between the two dynamical cores is very valuable.)

On the other hand, it is well known that the timestep restriction with the explicit Runge-Kutta DGM is more severe compared to that in the grid-point methods (e.g., Cockburn and Shu, 2001). When we use a polynomial order $p$ for the spatial discretization, an approximate allowable timestep for an explicit $p$+1 stage RK method with $p$+1 order is given in the form

$$\Delta t \leq \frac{1}{\lambda_{max}} \frac{h_e}{2p+1}$$

where $p$ is the polynomial order, $h_e$ is the element size, $\lambda_{max}$ is the maximum eigenvalue of Jacobian matrix with the advection terms. This means that we need to set the time step for the DGM such that it is approximately smaller by a factor of 1/2 compared to the grid-point methods with an approximately same DOF, as you have pointed out. However, it is possible that the computational overhead can be ignored in several situations: i) the spatial errors for high-order DGM rapidly reach to a given error tolerance due to the fast numerical convergence compared to conventional low-order methods with totally second-order accuracy. By using a coarser grid, we can significantly reduce the computational cost in three-dimensional problems. Bardar et al. (2011) also pointed out such situation. ii) In massively parallel computations, we consider the situation where there is little DOF per computational node. For conventional high-order grid point methods, the communication of halo data can occupy most of the execution time.

To accelerate DG dynamical cores in other situations discussed above, we agree that relaxing the severe timestep restriction with DG is an important topic. In fact, previous studies have attempted to extend the allowable timestep, for example, by optimizing the stability region of RK methods (e.g., Jahdali et al., 2022) or by using co-volume grids (Warburton, 2008). However, we would like to leave it as a future work.

Based on your comment, we have mentioned the issue of the timestep restriction in lines 673-677 of the revised manuscript as

"Furthermore, a severe timestep restriction for explicit temporal schemes is one of the unsolved issues in high-order DGM. We expect that the computational overheads would be ignored in several cases; A coarser spatial resolution can be used due to the high-order numerical convergence or the small communication cost in DGM is taken advantage of. However, to accelerate DG dynamical cores in all situations, developing sophisticated temporal treatments is an important future work."

[References]


- Cockburn, B. & Shu, C. W. (2001). Runge–Kutta discontinuous Galerkin methods for convection-dominated problems. *Journal of scientific computing*, 16, 173-261. https://doi.org/10.1023/A:1012873910884
- Al Jahdali, R., Dalcin, L., Boukharfane, R., Nolasco, I. R., Keyes, D. E., & Parsani, M. (2022): Optimized explicit Runge–Kutta schemes for high-order collocated discontinuous Galerkin methods for compressible fluid dynamics. *Computers & Mathematics with Applications*, 118, 1-17. https://doi.org/10.1016/j.camwa.2022.05.006
- Warburton, T., & Hagstrom, T. (2008): Taming the CFL number for discontinuous Galerkin methods on structured meshes. *SIAM Journal on Numerical Analysis*, 46(6), 3151-3180. https://doi.org/10.1137/06067260


3. While reading the text, it was clear that the authors use different settings (timestepping, filtering, and Smagorinsky diffusion) for the different test cases. This can be good practice during the development process in order to test specific characteristics of the dycore. But it is also useful to present results with the dycore configured as it would be used in practice. As the authors mention in their conclusions, they have not yet run the model with realistic topography (which is well known to create a lot of problems with high order element methods), and thus the "operational" configuration of SCALE-DG, especially with regards to how much filtering/diffusion will ultimately be needed, may not be known. One suggestion would be to also include all test results with the same configuration used for the planetary boundary layer turbulence test. If the authors consider that beyond the scope of this paper, I would request to

add a table summarizing all the settings used for each test.

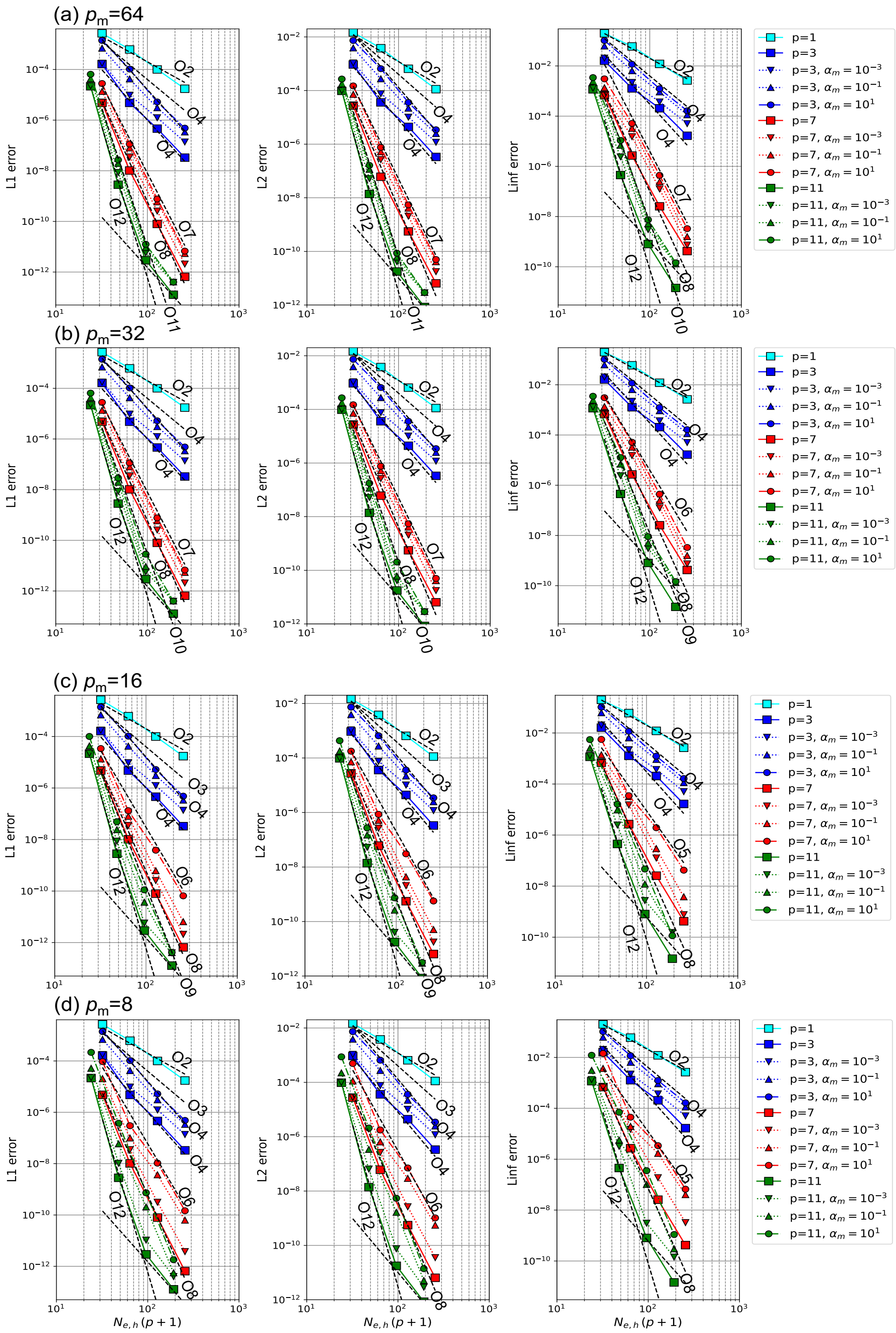


Fig. R1: Impact of the order $p_m$ and the coefficient $\alpha_m$ in the modal filter on the numerical convergence in a linear advection test: (a) $p_m$=64, (b) $p_m$=32, (c) $p_m$=16, and (d) $p_m$=8. In each $p_m$, we changed $\alpha_m$ as 0 (without the filter), $10^{-3}$, $10^{-1}$, and $10^{1}$. Please note that the results for $\alpha_{m=}0$ are identical to those obtained for $\varphi_0$=0 in Fig. 1 of our paper.

We agree that a series of numerical experiments using the turbulent model will provide useful information about numerical and physical dissipation mechanisms necessary for operational runs wherein the realistic topography is included. But we would like to leave comprehensive numerical experiments as a future work.

Instead, in a linear advection test, we discussed how much the strength of modal filter can degrade high-order numerical convergence in Sect. 3.1 of the revised manuscript. Fig. R1 shows the impact of order $p_m$ and the decay coefficient $\alpha_m$ in the modal filter on the numerical convergence. Based on these results, when the scale-selective strong modal filters which immediately remove two-grid scale structure, it is possible to decrease the original convergence rate by 1~3 for $p$=7, 11. For $p$=3, although the degradation of convergence rate appears less obvious, the errors without the modal filter were much larger. Thus, for $p$=3, the effect of the increased error due to the filter may be more pronounced in the representation of the flow fields. It is difficult to determine filter levels required for numerical stability in realistic simulations a priori because they depend on various factors including nonlinearity, spatial resolution, turbulence parametrization, and smoothing of topography. However, we expect that the information about the sensitivity of filters is useful for readers who want to be careful about how much the strong filters can contaminate the quality of flow fields represented by high-order dynamical cores.

In addition, we have summarized all settings of the dissipation mechanism for each test case in Table. 6 of the revised manuscript.

## Specific comments:

4. line 36: (or line 55) "... some researchers have successfully developed global nonhydrostatic atmospheric dynamical cores based… element-based methods". The authors mention some research codes, but neglect recent and larger efforts using high-order element based methods from major modeling centers.   These include E3SM: ( Caldwell et al., JAMES 2021 e2021MS002544, Donahue et al, JAMES 2024 e2024MS004314 ), the Korean KIM model (Hong et al, 2018, https://link.springer.com/article/10.1007/s13143-018-0028-9), and NRL's NEPTUNE NEPTUNE Model, Kelly et al, 2024, https://arxiv.org/abs/2405.06076.

Thank you very much for informing us that we missed several important works with global nonhydrostatic dynamical cores based on the element-based method. In the introduction, we should refer to HOMME-NH and a spectral-element nonhydrostatic dynamical core in Korean Integrated Model. Based on other reviewer's comment, we have mentioned NUMA in the revised manuscript. Thus, we would like to referred to NEPTUNE which utilizes and extends the numerical methods prototyped in NUMA.

In lines 68-75 of the revised manuscript, we have added new statements as

"In the Nonhydrostatic Unified Model of the Atmosphere (NUMA; Kelly and Giraldo, 2012; Giraldo et al., 2013), which is applicable for both limited-area and global atmospheric simulations, the continuous and discontinuous Galerkin methods are adopted for the spatial discretization. The numeric prototyped in the NUMA is utilized and extended to a global spectral-element dynamical core in the Navy Environmental Prediction System Utilizing a Nonhydrostatic Engine (NEPTUNE) for both horizontal and vertical discretization (e.g., Zaron et al., 2022). SEM is also used for the nonhydrostatic High Order Method Modeling Environment (HOMME-NH; Dennis et al., 2005, 2012; Taylor et al., 2020) included in the Energy Exascale Earth System Model (E3SM), and for the nonhydrostatic dynamical core in the Korean Integrated Model (KIM) system (Hong et al., 2018)."

[References]


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- Zaron, E. D., Chua, B. S., Reinecke, P. A., Michalakes, J., Doyle, J. D., & Xu, L. (2022): The tangent-linear and adjoint models of the NEPTUNE dynamical core. *Tellus A: Dynamic Meteorology and Oceanography*, *74*(1). DOI: 10.16993/tellusa.146

5. line 75: "Few such studies for global nonhydrostatic dynamical cores are available although the numerical convergence characteristics of DGM was investigated for regional dynamical core..." For the citations of regional DG dynamical cores, the authors should also cite: Brdar et al, https://doi.org/10.1007/s00162-012-0264-z. This part of the introduction focuses only on three dimensional models, and gives the impression there is limited work on DG for global atmospheric modeling. There are quite a few papers looking at DG on the cubed-sphere grid for the shallow water equations, such as Nair MWR 2005, Ullrich GMD 2014, and the very recent entropy stable formulations: Ricardo et al, 2024, https://www.sciencedirect.com/science/article/pii/S0021999124000123. I also think that the NUMA model from Giraldo et al. (cited in this text for their regional configuration) has a global version that runs both DG and CG, but I dont have a reference for that. A key model that needs to be mentioned is NEPTUNE, which is a global high order element based method that uses CG and DG, making it quite similar to SCALE-DG. NEPTUNE is one of the few models I know that is using 3D higher order elements (as proposed here). (See NEPTUNE references in Kelly et al, 2024, https://arxiv.org/abs/2405.06076)

Thank you for your suggestion. As a previous study that investigated the numerical convergence with DGM in regional nonhydrostatic dynamical cores, we have added Bardar et al. (2013) in line 115 of the revised manuscript.

To mention previous studies with the element-based methods for the global shallow water equations, we have cited Nair (2005, MWR) and Ullrich (2014, GMD) in lines 64-66 of the revised manuscript as

"…developed global nonhydrostatic dynamical core based on high-order grid point and element-based methods. The essence of the numerical methods can be found in the horizontal discretization of the global shallow water equations; For example, Ullrich et al. (2010) for a high-order finite volume method (FVM), while Nair et al. (2005a) and Ullrich (2014) for high-order element-based methods."

In addition, we have cited Ricardo et al. (2024) in lines 79-80 of the revised manuscript as

"A similar method was successfully applied to a global shallow water model in Ricardo et al. (2024) and to a global nonhydrostatic dynamical core…"

We have added a new statement referring to NUMA in lines 68-71 of the revised manuscript as

"In the Nonhydrostatic Unified Model of the Atmosphere (NUMA; Kelly and Giraldo, 2012; Giraldo et al., 2013), which is applicable for both limited-area and global atmospheric simulations, …"

As for NEPTUNE, we have mentioned it in lines 70-73 of the revised manuscript as "The numerical method prototype used in NUMA is utilized and extended to a global spectral-element dynamical core in the Navy Environmental Prediction System Utilizing a Nonhydrostatic Engine (NEPTUNE) for both horizontal and vertical discretization (e.g., Zaron et al., 2022)."

[References]


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- Ricardo, K., Lee, D., & Duru, K. (2024): Conservation and stability in a discontinuous Galerkin method for the vector invariant spherical shallow water equations. *Journal of Computational Physics*, *500*, 112763. https://doi.org/10.1016/j.jcp.2024.112763
- Ullrich, P. A. (2014): A global finite-element shallow-water model supporting continuous and discontinuous elements. *Geoscientific Model Development*, *7*(6), 3017-3035. https://doi.org/10.5194/gmd-7-3017-2014


6. line 84: "However, they did not consider the vector Laplacian operator for the vector quantities (for 85 example, momentum). This might be because the rigorous form of vector Laplacian is so complex that it may not be worth the computational cost required numerical stabilization." The authors should note that vector viscosity for both DG and CG was developed in: Ullrich 2014, https://gmd.copernicus.org/articles/7/3017/2014/gmd-7-3017-2014.pdf as well (for CG) in Guba et al., https://gmd.copernicus.org/articles/7/2803/2014/gmd-7-2803-2014.pdf

Thank you very much for informing us about the references with vector Laplacian operator in the element-based methods. We agree with you that the two papers should be mentioned.

In lines 96-100 of the revised manuscript, we have added new statements as "On the other hand, Ullrich (2014) presented a discretization strategy for the vector Laplacian operator with the continuous and discontinuous Galerkin methods. This approach can distinguish the divergence damping and vorticity damping with constant viscous coefficients. Guba et al. (2014) proposed a strategy of hyperviscosity with variable viscous coefficients in SEM where the vector Laplacian operator acts on the Cartesian component of the vector fields. For our purpose of introducing the turbulent model, ..."

[References]


- Guba, O., Taylor, M. A., Ullrich, P. A., Overfelt, J. R., & Levy, M. N. (2014). The spectral element method (SEM) on variable-resolution grids: Evaluating grid sensitivity and resolution-aware numerical viscosity. *Geoscientific Model Development*, *7*(6), 2803-2816. https://doi.org/10.5194/gmd-7-2803-2014

7. line 470 " In addition, the effective resolution is apparently higher than that of the low-order global dynamical core." See my general comment above. The authors are running different dissipation/fiter settings for each test case. As the authors mention, SCALE-DG needs stronger filtering when running more realistic test cases presented later. It is also very likely that even more dissipation will be needed when realistic topography is added. For the baroclinic instability test case, (as well as Held-Suarez) models are recommend to run with their operational diffusion

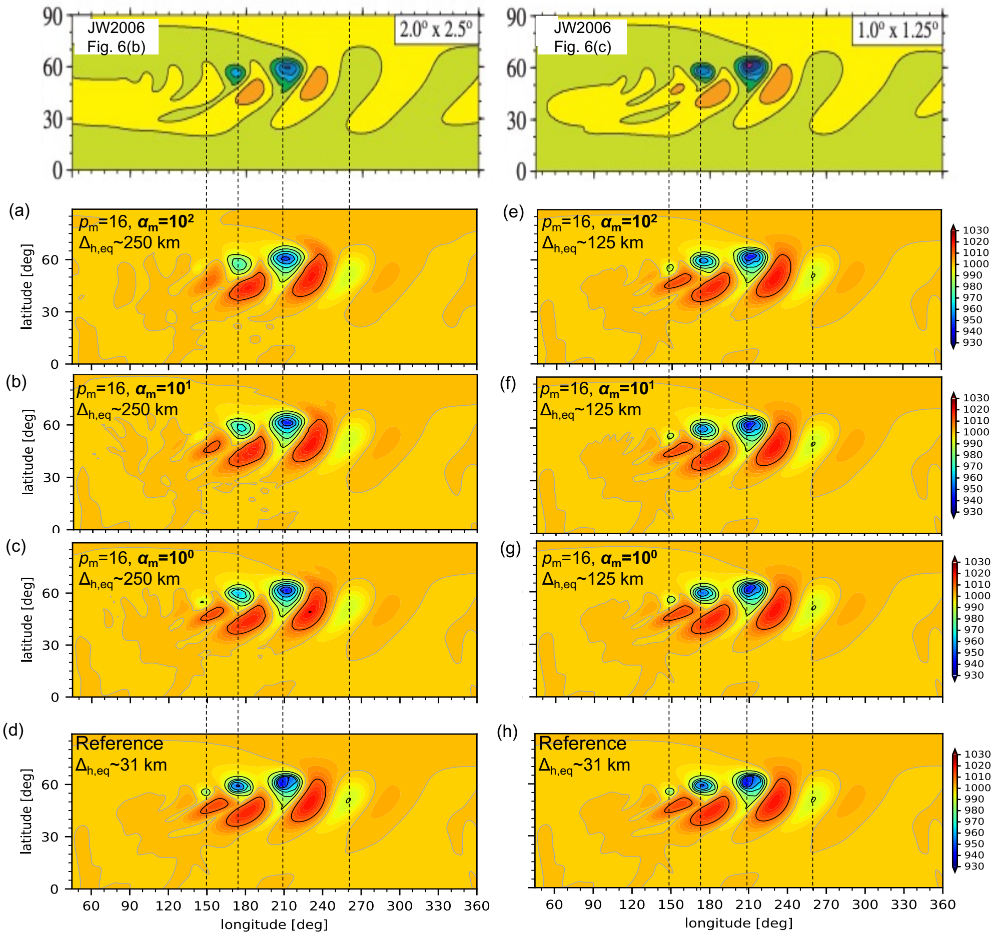


Fig. R2: Impact of the decay coefficient $\alpha_m$ of modal filter with $p_m$=16 on the surface pressure [hPa] at day 9 in the baroclinic wave test changed $\alpha_m$ as $10^2$, $10^1$, and $10^0$. This figure focused on the results where $\Delta_{h,eq}$=250, 125 km using p=7. For the comparison, we presented the results obtained from the FV dynamical core shown in Fig. 6 of Jablonowski and Williamson (2006) at the top panels. The lowest panels show our result obtained from $\Delta_{h,eq}$=31 km using *p*=7 as a reference solution.

settings, which is the case for the FV results. Thus, I would qualify this statement, and note that this might be due to the SCALE-DG's high order discretization, but it could also be due to using filtering levels that would not be practical in realistic problems.

We would like to inform you that high-order modal filters with large decay coefficients such as $p_{\mathrm{m,h}}$=16 and $\alpha_{m,h}$=O(1) for $p$=7 were used in the baroclinic wave and the Held-Suarez tests where small-scale flow structures develop. For the filtering levels, the flow structures at the short wavelength range near two grid scale are immediately dissipated after one timestep in HEVI temporal scheme. The total numerical dissipation of upwind numerical flux and modal filter near two grid scales is never weaker than the inherent numerical diffusion with the monotonic third-order piecewise parabolic method in the FV dynamical core or the explicit hyperdiffusion in the GME shown in Jablonowski and Williamson (2006).

To check the sensitivity of modal filter, we conducted additional experiments of the baroclinic wave test where the decay coefficient in the modal filters changed as $\alpha_{m,h}=10^0$, $10^1$, $10^2$ while $p_{\mathrm{m}}$ was fixed to 16. Figure R2 shows the impact of the decay coefficient on the surface pressure at day 9. Based on Figs. R2(a)-(c), the development of low pressure systems weakened as the decay coefficient increased in the coarsest resolution of $\Delta_{h,\mathrm{eq}}$=250 km. For the case of $\alpha_{m,h}=10^2$, the extent of numerical dissipation was comparable to that in the FV dynamical core. However, as shown Figs. R2(e)-(h), the sensitivity of decay coefficient in the modal filter was not significant for the amplitude and phase of high- and low-pressure systems with the increase in the spatial resolution. This is because we adopted the scale-selective filter with $p_{\mathrm{m}}$=16.

It is unclear that how much we need to strengthen the filtering level to treat realistic steep terrain in high-order DGM. It must depend on how much the topography is coarsened compared to the inherent effective resolution with DGM. As a preliminary investigation, we recently conducted a Held-Suarez test with the realistic topography smoothed by approximately 4 grid lengths. Based on the numerical experiments, a stable long integration seems to be maintained if the decay coefficients of filter $\alpha_{m,h}$ used in the baroclinic wave and the original Held-Suarez test increase by a few factors of 2~4. If we assume the required decay coefficient is at most $\alpha_{m,h}$=O(10) for $p_{\mathrm{m}}$=16 when introducing the realistic topography, we expect such filter level will not fully change high-order DG solutions into low-order solutions based on Fig. R2.

As a future work, we need to further investigate how the strength of modal filters used in realistic atmospheric simulations (with complex surface geometry and the forcing of physics processes) can contaminate the quality of numerical solutions with the high-order DGM. Thank you for your comment.

9. line 567: "and high computational efficiency in recent parallel supercomputers, over grid-point methods." I doubt this statement is true - given the small timestep required by high order DG (see comment above). One might be able to make the case that the methods achieve higher FLOP counts, but most people would interpret computationally efficiency in terms of time-to-solution.

As mentioned in our reply to Comment 2, if the same DOF and temporal method are used, allowable timesteps for the DGM would be shorter by a factor of 2 compared to the grid-point methods. The overhead would be low in several situations where we can utilize the advantage of DGM associated with the high-order numerical convergence or small communication cost in massively parallel computers. In this case, to reach a given error tolerance, the time-to-solution for the DGM can be shorter than the conventional grid-point methods. However, we need to further investigate whether this holds true for various situations. Therefore, following your suggestion, we determined to describe a high FLOPS count in the DGM here. In line 630 of the revised manuscript, we have modified the statement as

"… the high floating-point operations per second (FLOPS) count in recent parallel supercomputers, over …"

10. Terminology: The authors use the phrase ""eight grids" and "10~20 grids" several times. It's clear what they mean, but this is an unusual phrasing and I think technically incorrect because they are referring to the grid spacing or grid cell width, not the grid itself. I'd suggest changing to $8 ¥Delta x$.

Thank you for suggesting an improvement of our terminology. We understood that the term "grids" is incorrect because what we would like to mean here is the wavelength corresponding to the several grid spacing. In the revised manuscript, we have determined to use "grid length" based on an advice of other reviewer.