resulting in $G_{rs}^{(1)}$ $(r, s = 1, 2)$, i.e., $\left(G_{21}^{(1)}, G_{22}^{(1)}, G_{11}^{(1)}\right)$. The constellation must be nonplanar to achieve this result. This is verified as follows.

Following Zhou and Shen (2024), in order for the solution to exist, it is expected that the position of all the spacecraft in the constellation must not obey the following formula:

$$a_{11}\left(x_{(\alpha)}^1\right)^2 + a_{12}x_{(\alpha)}^1 x_{(\alpha)}^2 + a_{12}x_{(\alpha)}^2 x_{(\alpha)}^1 + a_{22}\left(x_{(\alpha)}^2\right)^2 = 0, \tag{13}$$

where $a_{rs}$ $(r, s = 1, 2)$ is a set of fixed coefficients. The above equations can be rewritten as follows:

$$a_{11}\left(x_{(\alpha)}^1 / x_{(\alpha)}^2\right)^2 + 2a_{12}\left(x_{(\alpha)}^1 / x_{(\alpha)}^2\right) + a_{22} = 0, \tag{14}$$

which reduces to $x_{(\alpha)}^1 / x_{(\alpha)}^2 =$ constant. It means that all the spacecraft are in the plane parallel to the $x_3$ axis or the motion direction. Therefore, it is necessary to have the constellation not be planar in order to deduce the quadratic magnetic gradients as well as the linear magnetic gradient. The next iterations would also require this condition.

### 2.2.2 First-order iteration

Assuming that

$$S = \frac{1}{N}\sum_{\alpha=1}^{N}\left[f_c^{(1)} + x_{(\alpha)}^i g_i^{(1)} + \frac{1}{2}x_{(\alpha)}^i x_{(\alpha)}^j G_{ij}^{(1)} - f_{(\alpha)}\right]^2, \tag{15}$$

if $\delta S = 0$, then

$$\frac{\partial S}{\partial f_c^{(1)}} = 0, \frac{\partial S}{\partial g_i^{(1)}} = 0. \tag{16}$$

From $\frac{\partial S}{\partial f_c^{(1)}} = 0$, it can be assumed that

$$\frac{1}{N}\sum_{\alpha=1}^{N}\left[f_c^{(1)} + x_{(\alpha)}^i g_i^{(1)} + \frac{1}{2}x_{(\alpha)}^i x_{(\alpha)}^j G_{ij}^{(1)} - f_{(\alpha)}\right] = 0, \tag{17}$$

meaning that

$$f_c^{(1)} = \frac{1}{N}\sum_{\alpha=1}^{N} f_{(\alpha)} - \frac{1}{2N}\sum_{\alpha=1}^{N} x_{(\alpha)}^i x_{(\alpha)}^j G_{ij}^{(1)} = \frac{1}{N}\sum_{\alpha=1}^{N} f_{(\alpha)} - \frac{1}{2}R^{ij}G_{ij}^{(1)}. \tag{18}$$

If $\frac{\partial S}{\partial g_i^{(1)}} = 0$, this can be reduced to TS1

$$\frac{1}{N}\sum_{\alpha=1}^{N}\left[f_c^{(1)} + x_{(\alpha)}^i g_i^{(1)} + \frac{1}{2}x_{(\alpha)}^i x_{(\alpha)}^j G_{ij}^{(1)} - f_{(\alpha)}\right]x_{(\alpha)}^k = 0, \tag{19}$$

i.e., the following applies:

$$\frac{1}{N}\sum_{\alpha=1}^{N} x_{(\alpha)}^k x_{(\alpha)}^i g_i^{(1)} + \frac{1}{2}\frac{1}{N}\sum_{\alpha=1}^{N} x_{(\alpha)}^k x_{(\alpha)}^i x_{(\alpha)}^j G_{ij}^{(1)} - \frac{1}{N}\sum_{\alpha=1}^{N} f_{(\alpha)} x_{(\alpha)}^k = 0. \tag{20}$$

The tensor $R^{kij} = \frac{1}{N}\sum_{\alpha=1}^{N} x_{(\alpha)}^k x_{(\alpha)}^i x_{(\alpha)}^j$ is then defined, resulting in

$$R^{ki} g_i^{(1)} + \frac{1}{2}R^{kij}G_{ij}^{(1)} - \frac{1}{N}\sum_{\alpha=1}^{N} f_{(\alpha)} x_{(\alpha)}^k = 0. \tag{21}$$

Therefore, the first magnetic gradient is

$$g_\ell^{(1)} = -\frac{1}{2}\left(R^{-1}\right)^{k\ell} R^{kij} G_{ij}^{(1)} + \left(R^{-1}\right)^{k\ell} \cdot \frac{1}{N}\sum_{\alpha=1}^{N} f_{(\alpha)} x_{(\alpha)}^k. \tag{22}$$

Using Eq. (3), it is now possible to obtain the corrected apparent velocity $\boldsymbol{V}^{(1)}$ of the magnetic structure and the longitudinal components of the corrected quadratic magnetic gradient $(\partial_3 \nabla \boldsymbol{B})^{(2)}$ $\left((\partial_3 \partial_i \boldsymbol{B})^{(2)}\right)$ as in the zeroth iteration.

The least-squares method is then used to obtain the remaining nine components of the corrected quadratic magnetic gradient.

If

$$S = \frac{1}{N}\sum_{\alpha=1}^{N}\left[f_{(c)}^{(1)} + x_{(\alpha)}^i g_i^{(1)} + \frac{1}{2}x_{(\alpha)}^i x_{(\alpha)}^j G_{ij}^{(2)} - f_{(\alpha)}\right]^2 = \frac{1}{N}\sum_{\alpha=1}^{N}\left[f_c^{(1)} + x_{(\alpha)}^i g_i^{(1)} - f_{(\alpha)} + \left(1 - \frac{1}{2}\delta_{i3}\right)x_{(\alpha)}^i x_{(\alpha)}^3 G_{i3}^{(2)} + \frac{1}{2}x_{(\alpha)}^p x_{(\alpha)}^q G_{pq}^{(2)}\right]^2, \tag{23}$$

then $G_{pq}^{(2)}$ $(p, q = 1, 2)$ can be obtained using the same procedure as that used for the zeroth iteration so that all the components of the corrected quadratic magnetic gradient $(\nabla\nabla\boldsymbol{B})^{(2)}$ are obtained.

Similarly, two or more iterations can be performed until stable linear and second-order magnetic gradients are obtained.

This algorithm requires that the constellation be composed of at least seven spacecraft and that its configuration be nonplanar. Because both the 9S/C HelioSwarm and 7S/C Plasma Observatory satisfy these requirements, the linear and quadratic magnetic gradients can be readily obtained.

The curlometer technique (Dunlop et al., 2002b) is used to calculate the current density based on multiple spacecraft magnetic measurements, with the relative error estimated by