Transformations of electromagnetic fields in different reference frames and their applications

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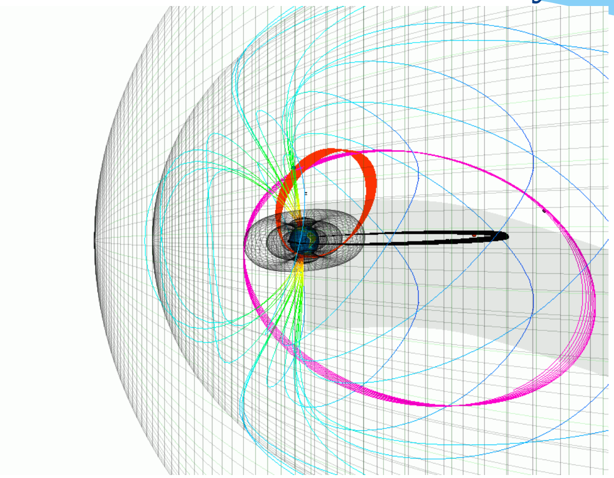
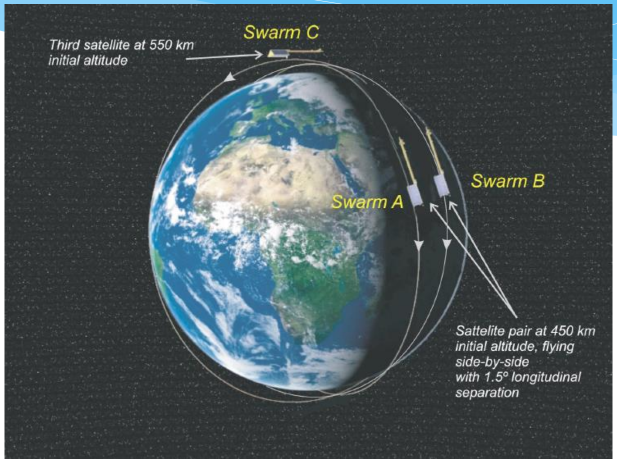
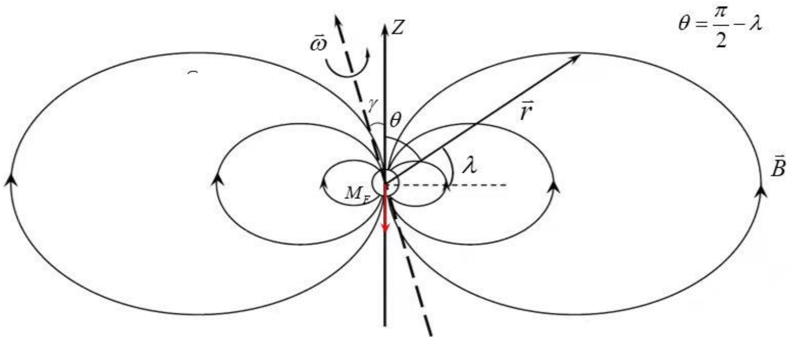
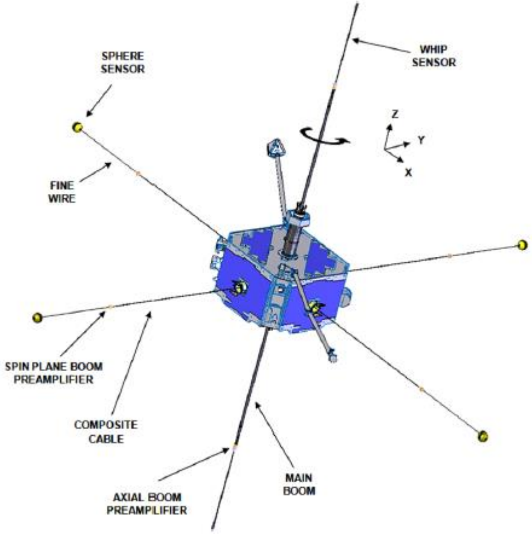
1. Background

The transformation of electromagnetic fields across different frames of reference, whether inertial or non-inertial, is a common challenge encountered in electromagnetic space measurements and analyses.







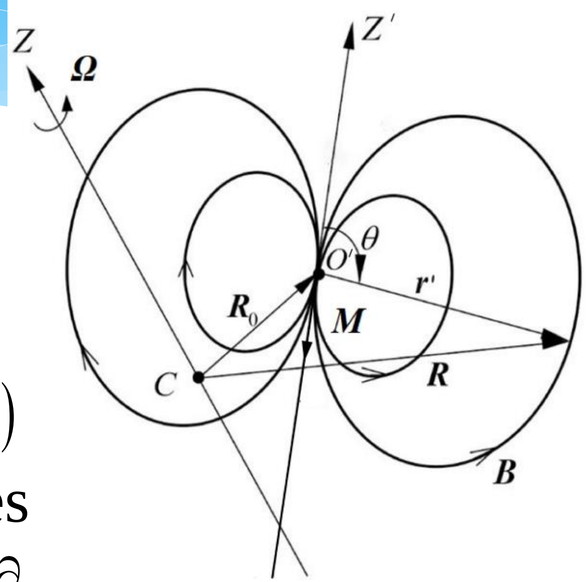


2. Lorentz transformations in low-speed moving reference frames

2. 1 space-time transformation

The impact of the equivalent gravity on the non-inertial reference system is $2\varphi/c^2 \sim u^2/c^2 \ll 1$,

the physical laws for flat spacetime remain valid.



Space-time coordinates in frame K: $(x^\mu) = \left(x^0 = ct, \mathbf{x}\right)$

Space-time coordinates in frame K’: $(x'^\mu) = \left(x'^0 = ct', \mathbf{x}'\right)$

Lorentz transformations for space-time coordinates

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\gamma = \Lambda^\mu_\gamma dx^\gamma \qquad \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu} = \hat{\Lambda}^\nu_\mu \frac{\partial}{\partial x^\nu}$$

Lorentz transformation matrix

$$\Lambda^\mu_\gamma = \frac{\partial x'^\mu}{\partial x^\gamma} = \begin{pmatrix} \gamma & -\gamma\beta_j \\ -\gamma\beta_i & \delta_{ij} + \frac{\gamma^2}{\gamma+1}\beta_i\beta_j \end{pmatrix} \qquad \hat{\Lambda}^\mu_\gamma = \frac{\partial x^\mu}{\partial x'^\gamma} = \begin{pmatrix} \gamma & \gamma\beta_j \\ \gamma\beta_i & \delta_{ij} + \frac{\gamma^2}{\gamma+1}\beta_i\beta_j \end{pmatrix}$$

$$\beta_i = u_i / c$$

$$\gamma = \left(1-\beta^2\right)^{-1/2}$$

Local transformation of the spacetime coordinate

$$d\mathbf{x}' = d\mathbf{x} + \frac{\gamma^2}{(\gamma+1)c^2}(d\mathbf{x}\cdot\mathbf{u})\mathbf{u} - \gamma\mathbf{u}dt$$

$$dt' = \gamma\left(dt - \mathbf{u}\cdot d\mathbf{x}/c^2\right)$$

Superposition law of velocities

$$\mathbf{v}' = \frac{1}{1-\mathbf{v}\cdot\mathbf{u}/c^2}\left[\frac{1}{\gamma}\mathbf{v} - \mathbf{u} + \frac{\gamma}{\gamma+1}\cdot\frac{(\mathbf{v}\cdot\mathbf{u})\mathbf{u}}{c^2}\right]$$

4-vector potential $(\phi/c, \mathbf{A})$， 4-current density， $(\mathrm{c}\rho, \mathbf{j})$

4-wave vector $(\nu/\mathrm{c}, \mathbf{k})$.......

Local Lorentz transformation

$$p'^{\mu} = \Lambda^{\mu}_{\gamma} p^{\gamma}$$

The transformation of the electromagnetic tensor

$$F'^{\mu\nu} = \Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\sigma}F^{\lambda\sigma}$$

The transformations of the electro-magnetic fields for non-inertial reference frames

$$\begin{cases} \boldsymbol{E}' = \gamma\left(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} - \dfrac{\gamma}{\gamma+1}(\boldsymbol{u}\cdot\boldsymbol{E})\boldsymbol{u}/c^2\right) \\ \boldsymbol{B}' = \gamma\left(\boldsymbol{B} - \dfrac{1}{c^2}\boldsymbol{u}\times\boldsymbol{E} - \dfrac{\gamma}{\gamma+1}(\boldsymbol{u}\cdot\boldsymbol{B})\boldsymbol{u}/c^2\right) \end{cases}$$

$$\begin{cases} \phi' = \gamma(\phi - \boldsymbol{u}\cdot\boldsymbol{A}) \\ \boldsymbol{A}' = \boldsymbol{A} - \dfrac{\gamma^2}{c^2}\phi\boldsymbol{u} + \dfrac{\gamma^2}{(\gamma+1)c^2}(\boldsymbol{u}\cdot\boldsymbol{A})\boldsymbol{u} \end{cases}$$

$$\begin{cases} \boldsymbol{P}' = \gamma\left(\boldsymbol{P} - \dfrac{1}{c^2}\boldsymbol{u}\times\boldsymbol{M} - \dfrac{\gamma}{\gamma+1}(\boldsymbol{u}\cdot\boldsymbol{P})\boldsymbol{u}/c^2\right) \\ \boldsymbol{M}' = \gamma\left(\boldsymbol{M} + \boldsymbol{u}\times\boldsymbol{P} - \dfrac{\gamma}{\gamma+1}(\boldsymbol{u}\cdot\boldsymbol{M})\boldsymbol{u}/c^2\right) \end{cases}$$

$$\begin{cases} \rho' = \gamma(\rho - \boldsymbol{u}\cdot\boldsymbol{j}/c^2) \\ \boldsymbol{j}' = \boldsymbol{j} - \gamma\rho\boldsymbol{u} + \dfrac{\gamma^2}{(\gamma+1)c^2}(\boldsymbol{u}\cdot\boldsymbol{j})\boldsymbol{u} \end{cases}$$

$$\begin{cases} \nu' = \gamma(\nu - \boldsymbol{u}\cdot\boldsymbol{k}) \\ \boldsymbol{k}' = \boldsymbol{k} - \dfrac{\gamma}{c}k\boldsymbol{u} + \dfrac{\gamma^2}{(\gamma+1)c^2}(\boldsymbol{u}\cdot\boldsymbol{k})\boldsymbol{u} \end{cases}$$

The errors are of the second order of u/c.

Lorentz transformation matrix under the first-order approximation

$$(\Lambda_\nu^\mu) = \begin{pmatrix} 1 & -\beta_j \\ -\beta_i & \delta_{ij} \end{pmatrix}$$

The transformation of space-time coordinates for low speed cases

$$\begin{cases} d\mathbf{x}' = d\mathbf{x} - \mathbf{u}dt \\ dt' = dt - \frac{1}{c^2}\mathbf{u}\cdot d\mathbf{x} \end{cases}$$

The transformations of the electro-magnetic fields

$$\begin{cases} \phi' = \phi - \mathbf{u}\cdot\mathbf{A} \\ \mathbf{A}' = \mathbf{A} - \frac{1}{c^2}\phi\mathbf{u} \end{cases} \qquad \begin{cases} \mathbf{E}' = \mathbf{E} + \mathbf{u}\times\mathbf{B} \\ \mathbf{B}' = \mathbf{B} - \frac{1}{c^2}\mathbf{u}\times\mathbf{E} \end{cases} \qquad \begin{cases} \boldsymbol{P}' = \boldsymbol{P} - \frac{1}{c^2}\boldsymbol{u}\times\boldsymbol{M} \\ \boldsymbol{M}' = \boldsymbol{M} + \boldsymbol{u}\times\boldsymbol{P} \end{cases}$$

$$\begin{cases} \rho' = \rho - \mathbf{u}\cdot\mathbf{j}/c^2 \\ \mathbf{j}' = \mathbf{j} - \rho\mathbf{u} \end{cases} \qquad \begin{cases} \nu' = \nu - \mathbf{k}\cdot\mathbf{u} \\ \mathbf{k}' = \mathbf{k} - \frac{1}{c}k\mathbf{u} \end{cases}$$

The errors are of the second order of u/c.

3. Transformations for electromagnetic fields in space plasmas

3.1 Galilean transformation for spacetime

Local Galilean transformation

$$\begin{cases} d\mathbf{x}' = d\mathbf{x} - \mathbf{u}dt \\ dt' = dt \end{cases}$$

Spacetime transform matrix

$$(\Lambda_\nu^\mu) = \begin{pmatrix} 1 & 0 \\ -\beta_j & \delta_{ij} \end{pmatrix}$$

Galilean velocity superposition $\mathbf{v}' = \mathbf{v} - \mathbf{u}$

The transformations of spacetime gradients

$$\begin{cases} \dfrac{\partial}{\partial t'} = \dfrac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \\ \nabla_{\mathbf{x}'} = \nabla_{\mathbf{x}} \end{cases} \qquad \begin{cases} \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial t'} - \mathbf{u} \cdot \nabla_{\mathbf{x}'} \\ \nabla_{\mathbf{x}} = \nabla_{\mathbf{x}'} \end{cases}$$

Total derivative $\dfrac{D}{dt} \equiv \dfrac{\partial}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}}$ satisfies $\dfrac{D'}{dt'} = \dfrac{D}{dt}$

3.2 Galilean transformations for an electromagnetic field

Due to the high conductivity of space plasmas,
The transform matrix for electromagnetic field is

$$(\Lambda^{\mu}_{\nu}) = \begin{pmatrix} 1 & -\beta_j \\ 0 & \delta_{ij} \end{pmatrix}$$

The transformations of the electro-magnetic fields

$$\begin{cases} \phi' = \phi - \mathbf{u}\cdot\mathbf{A} \\ \mathbf{A}' = \mathbf{A} \end{cases} \qquad \begin{cases} \mathbf{E}' = \mathbf{E} + \mathbf{u}\times\mathbf{B} \\ \mathbf{B}' = \mathbf{B} \end{cases} \qquad \begin{cases} \boldsymbol{P}' = \boldsymbol{P} - \dfrac{1}{c^2}\boldsymbol{u}\times\boldsymbol{M} \\ \boldsymbol{M}' = \boldsymbol{M} \end{cases}$$

$$\begin{cases} \rho' = \rho - \mathbf{u}\cdot\mathbf{j}/c^2 \\ \mathbf{j}' = \mathbf{j} \end{cases} \qquad \begin{cases} \nu' = \nu - \mathbf{u}\cdot\mathbf{k} \\ \mathbf{k}' = \mathbf{k} \end{cases}$$

For each group of Eqs., the error for the 1st Eq is of the second order of u/c, and that for the 2nd Eq. is of the first order of u/c.

3.3 Galilean transformations for the spatial and temporal gradients of the electromagnetic field

The gradients of magnetic field are invariants for different frames

$$\nabla'\mathbf{B}'=\nabla\mathbf{B} \qquad \nabla'\nabla'\mathbf{B}'=\nabla\nabla\mathbf{B} \qquad \nabla'\nabla'\nabla'\mathbf{B}'=\nabla\nabla\nabla\mathbf{B}$$

Relationships between the spatial and temporal gradients of the electromagnetic field

$$\begin{cases} \partial_t\mathbf{B}=-\mathbf{u}\cdot\nabla\mathbf{B} \\ \partial_t\nabla\mathbf{B}=-\mathbf{u}\cdot\nabla\nabla\mathbf{B} \end{cases}$$

These formulas are commonly used to calculate the structures' relative velocities and partial components of the magnetic gradients.

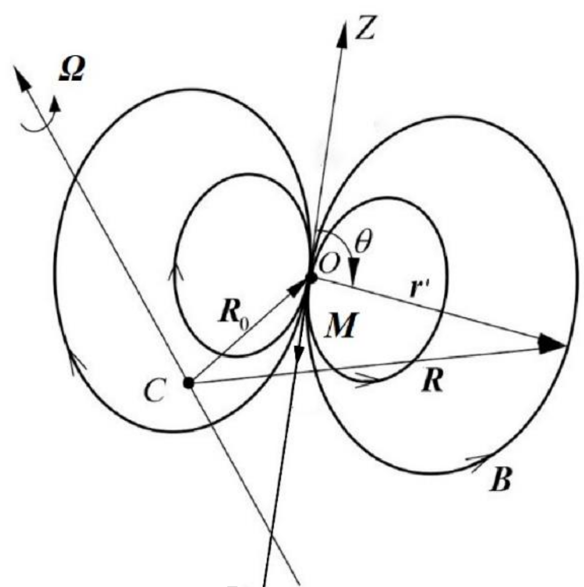
For low-speed space plasmas, the Coulomb gauge is valid:

$$\nabla\cdot\mathbf{A}=\nabla'\cdot\mathbf{A}'=0$$

4. Application 1: Corotational potential of planets with intrinsic magnetic fields

The dipole magnetic field of a planet

$$\mathbf{B}' = \frac{3\mathbf{r}'(\mathbf{r}'\cdot\mathbf{M}) - r'^2\mathbf{M}}{r'^5} \qquad \mathbf{A}' = \frac{\mathbf{M}\times\mathbf{r}'}{r'^3}$$



Corotation electric field observed in the planetary centroid reference frame

$$\mathbf{E}_{cor} = -\mathbf{u}\times\mathbf{B}' = -(\mathbf{\Omega}\times\mathbf{R})\times\mathbf{B}'$$

The corotation potential

$$\phi_{cor} = \mathbf{u}\cdot\mathbf{A}' = \tfrac{1}{r'^3}\left[(\mathbf{M}\cdot\mathbf{\Omega})(\mathbf{R}\cdot\mathbf{r}') - (\mathbf{M}\cdot\mathbf{R})(\mathbf{\Omega}\cdot\mathbf{r}')\right]$$

If the magnetic core coincides with the planetary centroid, and the magnetic moment is aligned with the rotation axis, then

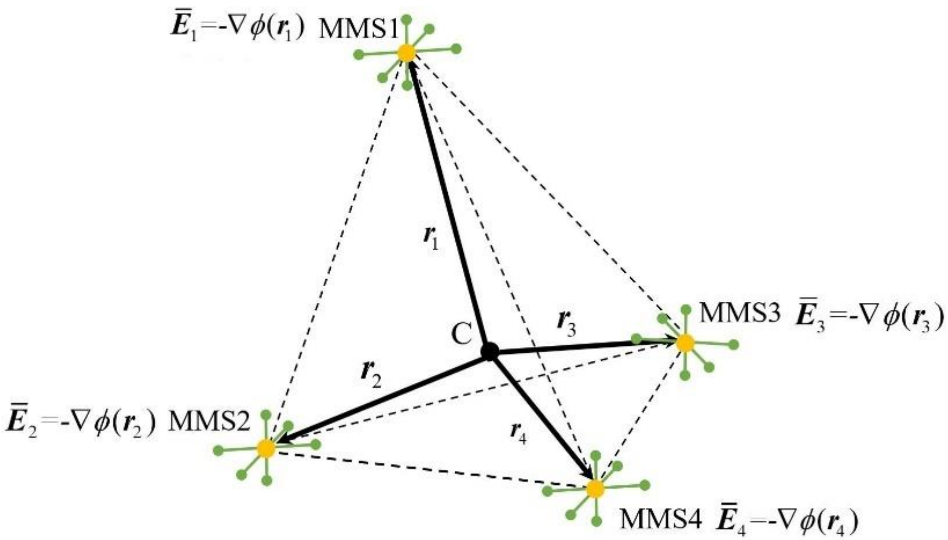
$$\phi_{cor} = \frac{\Omega M \sin^2\theta'}{r'} = \frac{\Omega M}{L R_p}$$

（Note: the divergence is not strictly zero： $\nabla\cdot\mathbf{E}_{cor} = 2\mathbf{B}'\cdot\mathbf{\Omega}$ ）

The error is of the first order of u/c

5. Application 2: Charge density deduced from the electric potential measurements by constellations

The analysis involves transformations among four different reference frames:



(1) Spacecraft rotational reference frames $\tilde{K}$: $\mathbf{E} = -\nabla_{\tilde{\mathbf{r}}}\phi - \partial_{\tilde{t}}\mathbf{A}$

(2) Spacecraft centroid reference frames K_s : $\mathbf{E}_s = -(\nabla_s\phi) - \frac{\partial}{\partial t_s}\mathbf{A}_\mathbf{s} = \bar{\mathbf{E}}_s - \frac{\partial}{\partial t_s}\mathbf{A}_\mathbf{s}$

(3) Constellation centroid reference frame K : $\mathbf{E} = \mathbf{E}_s - \mathbf{v}\times\mathbf{B} = \bar{\mathbf{E}} - \frac{\partial}{\partial t_s}\mathbf{A}_\mathbf{s}$

Charge density: $\rho = \varepsilon_0 \nabla\cdot\mathbf{E} = \varepsilon_0\nabla\cdot\bar{\mathbf{E}}$

(4)Proper reference frame of the structure K' :

$$\rho' = \rho - \frac{1}{c^2}\mathbf{V}\cdot\mathbf{j} = \varepsilon_0\left(\nabla\cdot\bar{\mathbf{E}}\right)_c - \frac{1}{c^2}\mathbf{V}\cdot\mathbf{j}$$

The error is at the first order of u/c.

6. Conclusions

- **(1) The Lorentz transformation is suitable for electromagnetic field transformations between any low-speed reference frames, whether inertial or non-inertial.**
- **(2) Approximate transformations for electromagnetic fields in various reference frames for low-speed, slow-varying space plasmas.**
- **(3) A general formula for the planetary corotation electric field is presented.**
- **(4) verifying the deduction of charge density from MMS electrostatic field measurements**

Thank You for your Attention !

3.4 Eqs of MHD obey Galilean transformations

Equations of MHD contains

$$\frac{D\rho_m}{dt} + \rho_m \nabla \cdot (\boldsymbol{v}) = 0 \qquad \rho_m \frac{D\boldsymbol{v}}{dt} = \boldsymbol{J} \times \boldsymbol{B} - \nabla p$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \cdot \boldsymbol{B} = 0 \qquad \nabla \cdot \boldsymbol{E} = \varepsilon_0^{-1} \rho_e$$

$$\nabla \cdot \boldsymbol{J} = 0 \qquad \boldsymbol{J} = \sigma_0 (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

They are invariant under the following Galilean transformations：

$$\mathbf{v}' = \mathbf{v} - \mathbf{u} \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_{\mathbf{x}} \qquad \nabla_{\mathbf{x}'} = \nabla_{\mathbf{x}} \qquad \frac{D'}{dt'} = \frac{D}{dt}$$

$$\mathbf{B}' = \mathbf{B} \qquad \mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B} \qquad \mathbf{J}' = \mathbf{J} \qquad \rho_e' = \rho_e - \mathbf{u} \cdot \mathbf{J} / c^2$$