Replies to the Reviewer#2

Thank the reviewer for the valuable comments and suggestions, which benefit significantly the improvements of the paper. We address your suggestions point-by-point below and modify our manuscript presentation throughout. In this reply, the comments of the referee are marked in black or red colors, and our replies in blue color.

Comments on the manuscript entitled

Quadratic Magnetic Gradients from 7-SC and 9-SC Constellations

submitted by Chao Shen, Gang Zeng, and Rungploypan Kieokaew.

General comments

The manuscript is concerned with a novel method to estimate the first and second spatial derivatives of stationary magnetic structures from multi-point measurements in constellations consisting of N spacecraft where N=7 (Plasma Observatory) or N=9 (HelioSwarm). In addition to the set of 3N simultaneous magnetic field measurements, also discrete representations of the 3N first time derivatives are utilised in the method, amounting to an effective number of 6N input data that are used for estimating the 33 model parameters (30 parameters in the second-order Taylor expansion, and 3 parameters for the velocity of the stationary structure). The paper presents the model equations and an iterative algorithm for estimating the parameters. The method is demonstrated using two magnetic field models. Deviations of the model predictions from their analytical counterparts are discussed.

While the study presented here can be considered a proof of concept that introduces the general framework and demonstrates the processing flow of the proposed method, a number of open issues and limitations need to be addressed and critically discussed, e.g., the concept of stationarity in the context of magnetohydrodynamics, the different types of errors, and the numerical stability of the inversion/reconstruction method.

Stationarity in the context of magnetohydrodynamics:

The method utilises discrete time derivative measurements through advection-type equations (3): $\frac{\partial \mathbf{B}}{\partial t} = -\mathbf{V} \cdot \nabla \mathbf{B}$. In magnetohydrodynamics, however, the local time derivative of the magnetic field $\mathbf{B}$ is connected to the velocity $\mathbf{V}$ through Faraday's law and an appropriate Ohm's law, which in the ideal case (collision-free plasmas in geospace and the heliosphere) equates the local time derivative with the curl of the cross product of velocity and magnetic field: $\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B})$ (hydromagnetic theorem), implying the invariance of magnetic flux through a surface transported with the plasma flow. The authors are asked to explain in which sense their notion of stationarity differs from the canonical interpretation (frozen-in magnetic flux) in space plasma physics.

Reply:

Thanks to the referee, who raised an interesting question we had not expected.

For the ideal MHD fields, the following magnetic convection equation is valid

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{u} \times \vec{B}), \qquad (1)$$

or

$$\frac{\partial \vec{B}}{\partial t} = \vec{u} \cdot \nabla \vec{B} + \vec{B} \cdot \nabla \vec{u} - \vec{B}(\nabla \cdot \vec{u}) \qquad (2)$$

Where $\vec{u}$ is the bulk velocity of the MHD plasmas.

As presented in the text of manuscript, another relationship as below is valid:

$$\frac{\partial \vec{B}}{\partial t} = -(\vec{v} \cdot \nabla)\vec{B} \qquad (3)$$

Where $\vec{v}$ is the apparent velocity of the magnetic structure.

$\vec{u}$ and $\vec{v}$ are possibly the same. However, for some situations, $\vec{u}$ and $\vec{v}$ are different. For example, when we observe the shock wave front at the shock frame, we see that the shock is at rest with zero apparent velocity, while the upstream and downstream plasmas are moving at their bulk velocities.

For some cases, it is possible that $\vec{u} = \vec{v}$, e.g., in magnetic clouds. We may check this kind of situation when $\vec{u} = \vec{v}$. Combining the two Eqs. (2) and (3) reducing to

$$(\vec{B} \cdot \nabla)\vec{v} - \vec{B}\nabla \cdot \vec{v} = 0 \qquad (4)$$

which limits the fluid velocity.

Furthermore, if the MHD plasma is incompressible, then

$$\nabla \cdot \vec{v} = 0 \qquad (5)$$

so that,

$$(\vec{B}\cdot\nabla)\vec{v}=0 \tag{6}$$

which means the velocity of the plasma is constant along the magnetic field lines.

The only situation we could expect to satisfy both Eqs. (5) and (6) is

$$\vec{v}=\mathrm{const}ant \tag{7}$$

everywhere. This implies that the MHD fluids are in an equilibrium state. Therefore, it is most likely that $\vec{u}$ cannot be equal to $\vec{v}$ globally except that the MHD plasmas are at equilibrium.

It can be emphasized that the Eq. (3) is valid for various space plasmas, not limited to MHD plasma. The constraints to the Eq. (3) are that the plasmas are highly conductive and have a very low velocity ($v/c<<1$, where c is the speed of light in vacuum), and the physical processes are slowly evolving at low frequencies.

Discretisation errors:

As pointed out by the authors in lines 223-227, the separation of spacecraft in the array introduces up to three different spatial discretisation scales. It should be added that a fourth spatial scale comes into play through the finite difference representation of local time derivatives, namely, the product of the intrinsic time scale (time resolution) with the velocity of the magnetic structure in the spacecraft frame.

Reply：

Commonly the separation of the spacecraft in one constellation is several 100km to several 1000km. For the time dimension, the time resolution of the magnetic measurement Δt is about 0.01 sec, i.e. $\Delta t = 0.01\mathrm{sec}$ .Considering that the magnetic structure is moving at a velocity V<500 km/s, the spatial resolution along the motion direction is about $v\Delta t < 5km$,which is much less than the S/C seperation. Therefore, it can be excepted that the error brought will be much less.

Iteration errors:

When in Section 3 the convergence properties of the iterative method are discussed, a particular type of error considered there is the mismatch of the actual limit of the procedure and the approximation reached after a finite number of iterations. This error may be termed iteration error. It is not associated with the finite resolution of the spacecraft array or the time series and thus needs to be considered separately.

Reply：

In this research, we have used an iteration procedure to solve the problem. The transformation relationship (3) is used as the constraints. It is noted that both the two Eqs. in the formula (3) in the text are nonlinear with a 2nd-order term on the left-hand side. However, the iteration procedures have made the problem a linear one. This will help reduce the calculation error and make the calculation more stable. Nevertheless, it is not easy to get the formula of the error of the iterations. In this study, we have performed two tests on the typical magnetic fields to illustrate the feasibility of this method and check the errors. The detailed evaluation on the iteration accuracy of this algorithm can be made in the future when the real mission data are available.

Random errors:

Due to imperfect (noisy) input data (measurement inaccuracies), the estimated parameters (first and second derivatives) will be subject to random errors, in addition to the discretisation errors and iteration errors mentioned above. In the current version of the manuscript, with demonstrations using noise-free model magnetic fields only, neither random errors are considered, nor the stability of the estimation (inversion) procedure (parameter reconstruction from noisy input data) which is likely to be associated with the set of different spatial discretisation scales. Since the inverse problem is weakly nonlinear, a condition number could be constructed for the linearised problem in the iterative procedure, or Monte Carlo simulations could be utilised to assess the impact of random errors. If such an approach is considered beyond the scope of this paper, the authors should at least critically discuss the implications of random errors, and outline the directions for future work.

Reply：

The noise or disturbances in the data can come from the measurement error, but they could mainly be caused by the plasma waves. This can make the calculation of the high order magnetic gradients very difficult (see Shen et al. 2021). In analyzing the actual observation data, we could use filtering methods to delete the high frequency components so as to smooth the raw data and avoid the negative effect of the data disturbance.

Magnetic field divergence:

To quantitatively assess the limitations of this high-dimensional reconstruction problem with 33 model parameters, it is not sufficient to consider only a scalar quantity such as an estimate of the divergence of the magnetic field. Furthermore, in its original form, the divergence is normalised by the curl of the magnetic field (lines 201-203), while the latter quantity is zero for one of the two test cases (dipole field) in Section 3. To see if a dimensionless version of the divergence differs significantly from zero, meaningful reference values need to be chosen.

Reply：

1. In this approach, the magnetic Gaussian Low ($\nabla \cdot \vec{B} = 0$ along with $\nabla\left(\nabla \cdot \vec{B}\right) = 0$) has been used as the measures of the errors of the first order and second order magnetic gradients for the actual data analysis. Really it is not perfect because it can not include partial components of the magnetic gradients (the formula $\nabla \cdot \vec{B} = 0$ contains 3 of the total 9 components of $\nabla \vec{B}$ while $\left|\nabla\left(\nabla \cdot \vec{B}\right)\right| = 0$ contains 9 of the total 18 components of $\nabla\nabla\vec{B}$). The advantage to use them as the measures of the errors of the magnetic gradients is that they are very reliable and also simple. We still have not found other better ways for evaluating the accuracy of the algorithm because the actual values of the magnetic gradients are unknown for comparation when analyzing the real observation data. So that it is a feasible and practical way for calculating the errors of the magnetic gradients.
2. It is true that the method for calculating the error in Curlometer method is invalid when there is no electric current. Here we use the normalized forms to avoid this problem.
3. It is sure that the characteristic magnetic field and spatial scale of the structures must be properly chosen during the actual data analyses thus the errors resulted can represent the accuracy of the calculations.

Terminology:

It is very unusual to refer to the tensor of second partial derivatives as the "quadratic gradient". It is strongly recommended to adjust the terminology.

Canonical options are: "Hessian" or "Hessian matrix" (2nd derivatives of a scalar field) or "Hessian tensor".

Reply：

We are also very concerned of the names of the second partial derivatives. Hessian tensor is a possible name for it, but too unfamiliar to the average readers. Liu et al. (2019) have used the second order gradient for it, which are somewhat too long if frequently used. Torbert et al. (2020) have used quadratic coefficient for it. In the 1998 data analysis book, Chanteur (the first to stress this question) has used the term quadratic for it. Therefore, we think that the cautious and proper way may be calling it quadratic gradient, and giving a note of Hessian tensor for it at the beginning (in the Introduction section). It is noted that the second order magnetic gradient is composed of 18 components, while Hessian tensor contains 6 components because it commonly means the second partial derivatives of a scalar.

Specific comments

Abstract and Key Points:

- The statements

"The tests for the situations of magnetic flux ropes and dipole magnetic field have verifies the validity and accuracy of this approach."

and

"Magnetic flux ropes and dipole magnetic field testing verifies the validity and accuracy of the approach."

are too strong (and also difficult to understand in the first place). A proof of concept is presented, but a complete assessment of the accuracy would require studying all error types and the stability of the model inversion procedure.

Reply：

As the referee pointed out, not all error types have been considered. This manuscript only evaluates the truncation errors. It is the limitation of our study. The strong statements have been modified accordingly.

The first reviewer also raised the problem on the measurement errors, please refer to the reply to the referee #1.

Theoretical evaluating on iteration stability is a tough work, which cannot be solved completely in a short time. In this initial study, we are concentrated on the feasibility of the algorithm. Nevertheless, two tests made in this research have confirmed the reliability of the method because the iterations for both tests can arrive at convergence and the total errors are rather small.

introduction:

- Line 54: The statement "To obtain high-order gradients in the magnetic field ..." is ambiguous as it could also refer to different orders of accuracy in discrete representations of the gradient. Instead, one could write "To estimate second derivatives of the magnetic field ..."

Reply：

The correction has been made accordingly.

Method:

- Line 81: The statement "Calculation of the linear and quadratic gradients of a magnetic field generally requires magnetic measurements from at least ten spacecraft" should be made precise and briefly explained (3+9+18=30 parameters in the Taylor expansion up to second order, 3N magnetic field measurements in an array with N spacecraft).

Reply：

The problem of balancing the number of unknowns versus the number of available observations is briefly explained in section 2.1.2 of the manuscript. A brief explanation has been also inserted in the referee mentioned place.

- Lines 105/106: The statement "The errors in formula (3) are on the order V/c." is unclear. What kind of errors? Meaning of the variable c? Non-relativistic limit?

Reply：The errors mean the truncation errors of the formula (3) compared with the accurate one. Here V is the apparent speed of the magnetic structure and c is the speed of light in vacuum, which are explained in the text now. In the non-relativistic limit ($V/c \ll 1$, it is generally valid in space plasmas) we can derive the simple formula (3). The accurate formula is complicated for the analysis. An explanation on this issue is in the talk of the first author in the EGU meeting this year and the ppt is attached for the reference.

- Line 114: first-order or zero-order?

Reply：

As referee pointed out, it should be zero-order. The correction has been made accordingly.

- Line 121: In the statement "The iterations are performed repeatedly until satisfactory results are achieved.", quantify what is meant by "satisfactory results" (which error measure/threshold).

Reply：

As Figure 4 and 8 in the manuscript shown, the lines become flat with increasing number of iterations, which means that the iterative results are convergent. In the tests, the number of iterations is set to 100, and the results with 100 iterations are regarded as "satisfactory results". The sentence has been modified as “The iterations are performed repeatedly until results are converge, which means satisfactory results are achieved.”

- Line 142: In the statement "The temporal variation rate ... is readily obtained using time-series magnetic observation.", explain how the temporal variation rate is approximated (finite differencing? time resolution?).

Reply：

In the test, central difference has been used. And the first value of series data has been obtained by first order forward difference, and the last value has been obtained by first order backward difference. Explanation has been made after the sentence accordingly.

For actual magnetic observations, the time resolution is very high (~0.01sec), there are plenty of time series data, so it is not difficult to get the time derivative of magnetic field even in high orders.

Comparison of new method with analytical modelling:

- General comment: With spacecraft separations on the order 0.01 RE, and model magnetic fields varying on spatial scales on the order RE, the magnetic configurations vary only gradually on the spacecraft array scale, so these are not particularly challenging tests of the proposed method. In geospace, magnetic field structures can vary on much smaller scales. Furthermore, the model magnetic field configurations are simplified and highly symmetrical structures with a very small number of parameters so that only a minor subset of the 33 degrees of freedom can be assessed. The specifics and the limitations of the chosen test cases should thus be critically discussed.

Reply：

The linear gradient of the magnetic field has 9 components, while the quadratic gradients comprise 18 independent components due to the symmetry of quadratic gradients. For the flux rope case, only 3 components of linear gradient and 5 components of quadratic gradients have been assessed. But for dipole field case, 4 components of linear gradient and 10 components of quadratic gradients have been assessed. The number of assessed parameters has reached half. We have chosen so symmetrical model magnetic field in order to easily compare the simulation results with the accurate analytic calculations. Nevertheless, these are still somewhat complete tests because the zero components of the magnetic gradients are calculated with the algorithm as well and checked. Accordingly, further evaluations on the algorithm could be made with the modeled magnetosphere with less symmetry in the future.

- Lines 226/227: With the given value of the first eigenvalue $w_1 = 0.1643\, R_E^2$, the characteristic size L should be $L = 2\sqrt{w_1} = 0.8106\, R_E$.

Reply：

In the manuscript, the value of characteristic size L is correct, but wrong values of the eigenvalues are given. $w_1 = 0.1643\times 10^{-3} R_E^2$, $w_2 = 0.1104\times 10^{-3} R_E^2$, and $w_3 = 0.0341\times 10^{-3} R_E^2$. Corrections have been made accordingly.

- Lines 254-258: Only total errors after a given number of iterations are discussed. It would be more interesting to get separate assessments of iteration errors and discretisation errors.

Reply：

It is not very easy to separate iteration errors and discretisation errors. The separate assessments of errors should be considered in future work.

- Lines 277/278 and line 321: In the statement "The relative error approaches

50%; however, the absolute error is low." it is not clear which reference is used (low/small compared to what?)

Reply：

The term “low” is compared to zero. Explanation has been made accordingly. “however, the absolute error is just 0.143, which is approaching zero.”

Errors:

- General comment: As explained above, this section is very incomplete regarding the various types of errors. In particular, the current version of the manuscript lacks a critical discussion of random errors and the stability of the parameter estimation (inversion) procedure.

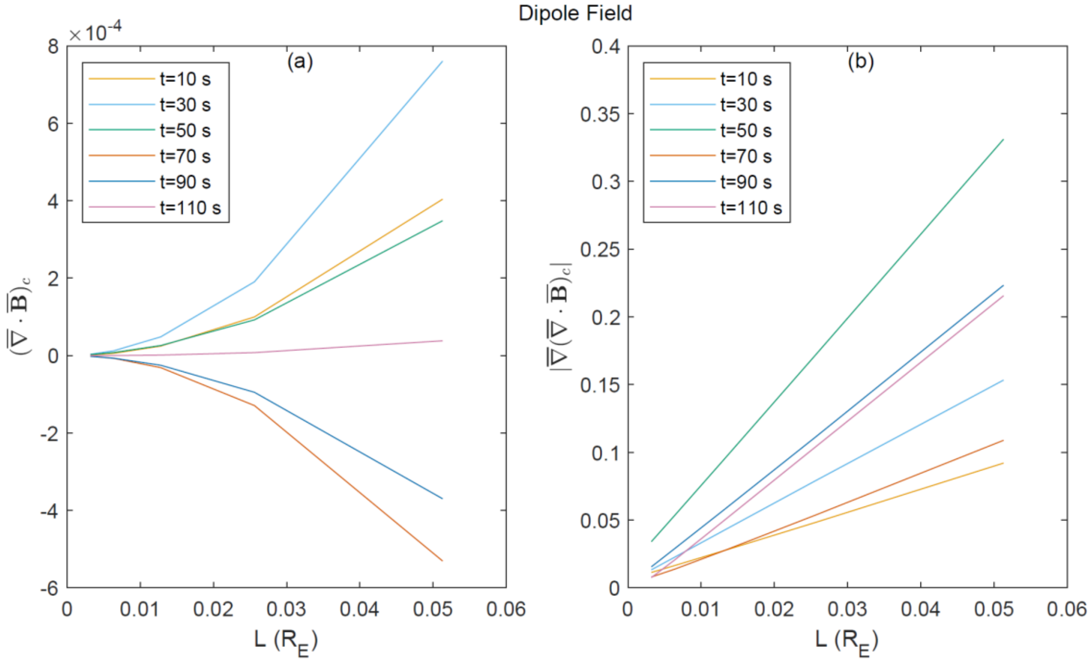
Reply：

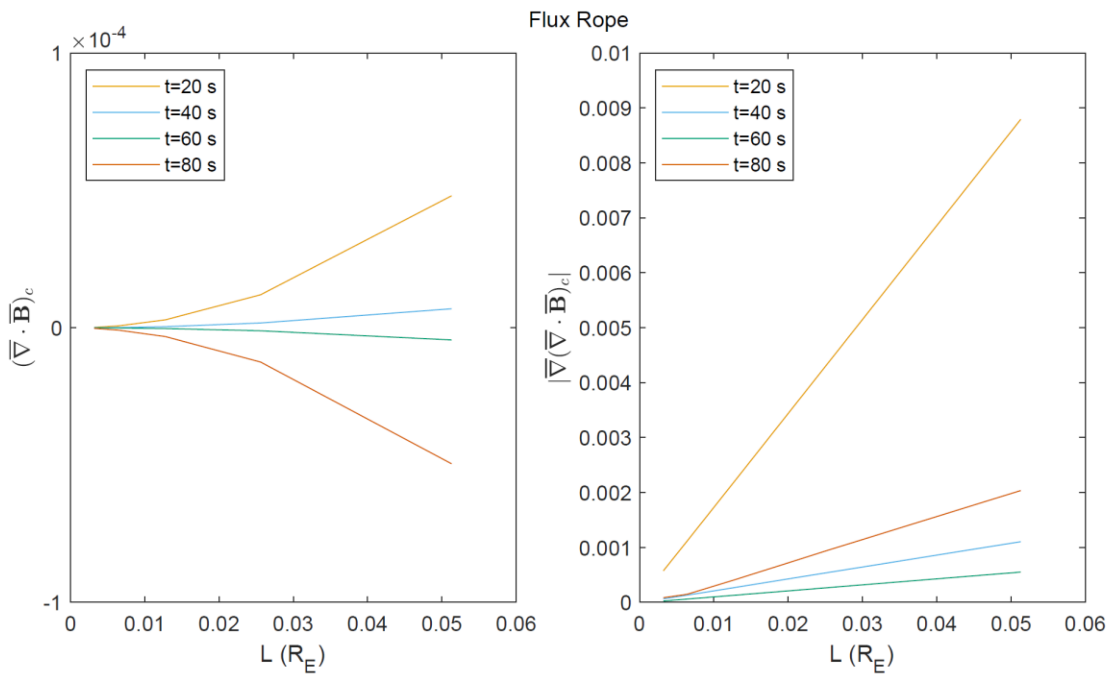
Regarding the stability of the parameter estimation (inversion) procedure, refer to the above response.

Regarding the random errors: The noise or disturbances in the data can come from the measurement error, but they could mainly be caused by the plasma waves. This can make the calculation of the high order magnetic gradients very difficult (see Shen et al. 2021). In analyzing the actual observation data, we could use filtering methods to delete the high frequency components so as to smooth the raw data and avoid the negative effect of the data disturbance at utmost.

- Figures 12 and 13: It may be worth mentioning that the errors of the first derivative decrease quadratically with the scale L (second-order accuracy with regard to discretisation errors) whereas the errors of the second derivatives decrease linearly with L (first-order accuracy with regard to discretisation errors).

Reply:





The above two figures show the trend of dimensionless divergence and gradient of divergence with characteristic size L for dipole field and flux rope case, respectively. As referee's suggestion, the errors of the first derivative decrease quadratically with the scale L whereas the errors of the second derivatives decrease linearly with L. This conclusion has been added in the manuscript accordingly.

Conclusions:

- General comment: In line with the previous comments, this section should be rewritten to reflect the actual limitations of this study and the method, and explain where further work is required.

Reply:

Thanks for the reminding! In the future, the random errors, measurement errors, iteration errors and discretisation errors could be evaluated in details, especially when the mission payloads are fixed and the real mission data are available. The statement has been added in Conclusions (Section 5) accordingly.