Replies to the first Reviewer

We are very grateful for the referee's valuable comments. We address your suggestions point-by-point below and improve our manuscript presentation throughout. Major changes include the structuring of the introduction and adding more context to make the manuscript accessible to a broader group of audience. We hope that our revised manuscript is now more accessible to the general readers. In this reply, the comments of the referee are marked in black or red colors, and the replies in blue color.

Review of egusphere-2024-1330 (ANGEO)

Quadratic Magnetic Gradients from 7-SC and 9-SC Constellations

by Chao Shen et al.

This paper describes a least-squares gradient computation technique for linear and quadratic magnetic gradients. The technique is applied to two test cases to show its performance. One of the goals is to demonstrate that 7- and 9-spacecraft constellations provide enough measurements to infer those gradients. The paper starts with an introduction that properly references earlier work on gradient computation. It then presents the technique, the test cases, and it ends with a conclusion.

The introduction could be better structured. This can probably be remedied by shifting some material from the description of the technique to the introduction, so that the characteristics of the technique are put in contrast with the earlier work on the subject (see details below). The actual contents of the paper is sound and will undoubtedly be useful for the community. I do have a number of questions/suggestions regarding the method, the test cases, and the presentation (see comments below).

The manuscript would benefit seriously from language editing. I have listed just a few language suggestions (see below).

Major comments

In the abstract and at various places in the text, the authors say that 4 measurements are needed for computing the linear gradient and 10 measurements are needed for the quadratic gradient. This statement is somewhat imprecise. It would be more correct to state instead that 4 simultaneous measurements are needed for the linear spatial gradient and 10 simultaneous measurements for the quadratic spatial gradient components of a scalar field. Perhaps it would also be useful to mention from the start that, when using the least-squares approach, one adds the time derivatives and the mixed space-time derivatives, so that at least 5 measurements are needed for the linear and 15 for the non-linear gradients of a scalar field in general.

Reply:

Thank you for these valuable comments. To address your points, we have made the following modifications.

1. We have now emphasized that we require multi-point 'simultaneous' measurements everywhere in the text.

2. We have emphasized that 4-point simultaneous measurements are needed to resolve the linear spatial gradient of a scalar field in Paragraph 1 of the introduction.

3. We have elaborated the necessity of 10-point simultaneous measurements to resolve the quadratic spatial gradient of a magnetic field in Paragraph 3 of the introduction. Furthermore, we have further generalized potential applications of quadratic spatial gradients, in addition to resolving complex magnetic structures such as flux ropes, to include nonlinear plasma dynamics that would benefit also from the high-order magnetic gradient calculation from measurements *in situ* at the end of the same paragraph.

Generally, the number of the measurement points required for drawing till the r-th gradients in the d dimensional space is $\mathrm{C}_{d+r}^{r}=\frac{(d+r)!}{d!r!}$ (Zhou & Shen, 2024). For the situation considered in this study (to obtain the 1st and 2nd magnetic gradents in 3 dimensional space), r=2, d=3. Thus the spacecraft needed in the constellation is at least

$$\mathrm{C}_{d+r}^{r}=\frac{(3+2)!}{3!2!}=10 .$$

Ref: Zhou, Y. and Shen, C.: Estimating gradients of physical fields in space, Ann. Geophys., 42, 17–28. https://doi.org/10.5194/angeo-42-17-2024, 2024.

4. We have added that, to compute quadratic gradients from 7- or 9-point simultaneous measurements, we consider the transformation of reference frame involving mixed space-time derivatives of the magnetic field at the end of Paragraph 4 of the introduction. We also specified in the methodology that we consider the mixed space-time derivatives to avoid confusion.

In the description of the method, I was expecting that somewhere the condition div B = 0 would have been incorporated. If I understand well, that is not the case; rather that condition is used for evaluating the precision of the technique. Still, inclusion of a div B = 0 constraint would make the technique more precise and robust, as it can remove a possible ill-posedness of the problem for certain spacecraft constellation geometries. Can the authors comment on whether and how such a condition can be included?

Reply:

Here, the transformation relationship constraints are sufficient already for obtaining the complete linear spatial gradient and quadratic spatial magnetic gradient.

Certainly, applying the div B = 0 and grad div B=0 constraints can improve the algorethim, but not very significantly. And also, grad div B=0 can only provide two constraints equations (the gradient of div B along the motion direction can be obtained from the transformation relationship).

Alternatively, we only apply the transformation relationship constraints in this method, While the div B = 0 and grad div B=0 constraints are used as the quantitative measures of the errors of the magnetic gradient calculated in this algorithm. By the way, the Curlometer technique (Dunlop et al., 2002b) to calculate the current density based on multiple spacecraft magnetic measurements has used abs (div B/curl B) to evaluate the error.

Nevertheless, the previous method (NMG, Shen et al., 2021a) for calculating the linear and quadratic magnetic gradients based on 4-spacecraft MMS mission observations has to apply both the div B = 0 and grad div B=0 constraints. Several other methods also utilize the div B = 0 or both the div B = 0 grad div B=0 constraints (Liu et al., 2019; Torbert et al., 2020).

Ref:

Dunlop, M. W., Balogh, A., Glassmeier, K.-H., and Robert, P.: Four-point cluster application of magnetic field analysis tools: The curlometer, J. Geophys. Res., 107, 1384. https://doi.org/10.1029/2001JA005088, 2002b.

Shen, C., Zhang, C., Rong, Z., Pu, Z., Dunlop, M. W., Escoubet, C. P., Russell, C. T., Zeng, G., Ren, N., Burch, J. L., Zhou, Y.: Nonlinear magnetic gradients and complete magnetic geometry from multispacecraft measurements, J. Geophys. Res., 126, JA028846. https://doi.org/10.1029/2020JA028846, 2021a.

For the reader it is confusing that the time derivative is used (line 109) in the explanation of the technique, while time derivatives or mixed space-time derivatives do not appear in the variable count on lines 122ff.

Reply:

Here we use the time derivatives and mixed space-time derivatives to add more constraints in order to obtain a unique solution of quadratic gradients from 7- or 9-point observations. We agree that this can lead to confusion as our work is focused on estimation of spatial gradients only. To avoid the confusion, we now specified everywhere whether it is a 'spatial' or 'temporal' gradient where applicable. Also, in our previous version of the manuscript, our assumption involving the temporal change of the magnetic structure was not clearly stated. To better clarify this point, we now emphasized that we assume that the magnetic structures (e.g., flux ropes, current sheets, boundary layers, magnetic reconnection regions, etc.) are slowly evolving during their passages through the multi-point constellations. This assumption has now explicitly been stated in Paragraph 1 of Section 2.

For actual magnetic observations, the time resolution is very high (~0.01sec), there are plenty of time series data, so it is not difficult to get the time derivative of magnetic field even in high orders. We are concentrated on the spatial gradients of magnetic field in this study.

I think having the paragraph from line 122ff in the introductory section would help in setting the broader problem of balancing the number of unknowns versus the number of available observations.

Reply:

The introduction on the system of equations was introduced briefly in the 2nd paragraph of our original manuscript. To better introduce the setting of our problem, we expand the introduction to include these.

The discussion of the volume tensor states that its determinant should be nonzero. At this point, no mention is made of the condition number, which is – practically speaking – more important than the tensor being non-singular. The statement that "This algorithm requires that the constellation be composed of at least seven spacecraft and that its configuration is non-planar. Because both the 9S/C HelioSwarm and 7S/C Plasma Observatory satisfy these requirements, the linear and quadratic magnetic gradients can be readily obtained" is therefore perhaps a bit optimistic. It is appreciated that in the examples the eigenvalues of the volume tensor are given. Still, that only partially describes the conditioning of the problem.

Reply:

Thanks for the comments. For the random shape of the 7S/C in the tests, the three eigenvalues of the volumetric tensor are $w_1 = 0.1643\times10^{-3} R_E^2$, $w_2 = 0.1104\times10^{-3} R_E^2$, and $w_3 = 0.0341\times10^{-3} R_E^2$. And it is already written in this new version of the manuscript.

It is certain that we have not verified the conditioning mentioned. However, it is strictly correct. From the equation (12) in the subsection 2.2.1, we have speculated that "The constellation must be nonplanar to achieve this result" without a verification. However, this could be made clear based on the results obtained in the previous work on high order gradients of physical fields this year (Zhou & Shen, 2024). The equation (12) is that for the parameters $G_{rs}^{(1)}(r,s=1,2)$, i.e., $\left(G_{11}^{(1)},G_{12}^{(1)},G_{21}^{(1)},G_{22}^{(1)}\right)$. Similar to the analysis in the subsection 4.1 of Zhou & Shen (2024), it is expected that, in order the solution exists the position of all the spacecraft in the contellation must not obey the following formula

$$a_{11}{x_1}^2 + a_{12}x_1x_2 + a_{12}x_2x_1 + a_{22}{x_2}^2 = 0, \tag{1}$$

Where a_{rs} is fixed coefficients. The above equation can be rewritten as

$$a_{11}(x_1/x_2)^2 + 2a_{12}(x_1/x_2) + a_{22} = 0, \tag{2}$$

Which means that all the spacecraft are in the plane parallel to the x3 axis or the motion direction. Therefore it is necessary that the constellation should not be planar in order to deduce the quadratic magnetic gradients as well as the linear magnetic gradient. The next iterations would also require this condition. So this would verify the statement.

We need to make an explanation in the revised manuscript accordingly.

Ref: Zhou, Y. and Shen, C.: Estimating gradients of physical fields in space, Ann. Geophys., 42, 17–28. https://doi.org/10.5194/angeo-42-17-2024, 2024.

Figure 1 presents a very specific shape of the 7 S/C constellation. Such a constellation is nice for conceptually presenting the idea of "nested tetrahedra", but cannot be easily maintained in space in practice. This figure is nowhere referenced nor discussed.

Reply:

Figure 1 just presents a schematic diagram of the Plasma Observatory Constellation. It is not the actual shape of the constellation. In the tests in section 3, the positions of the seven spacecrafts are generated randomly and is illustrated in Figure 2. The shape of the constellation in the tests is not the same as that in Figure 1. It is noted in the caption of Figure 1 specially.

The effect of measurement errors is not included in the calculation. This is assuming a homogeneous set of instruments, but that may not be the case for Plasma Observatory, for instance, where there are different instruments on mother and daughter spacecraft.

Reply:

In this study, the homogeneous measurement has been adopted in both the method and tests. For actual multi-satellite measurements, the measurement could be not homogeneous. The magnetic field data from different detectors would be synchronized by time interpolation. This is explained in the Conclusions section.

As for the measurement error, the influence of the magnetic detector on the output of method is a complicated problem. In the previous manuscript, we only checked the errors originated from the method itself or the truncation errors.

Regarding the error caused by the measurements, we may make an initial estimation. Starting from the formulas (1) and (2) in the Section 2, we have

$$\Delta f_{(\alpha)} = f_{(\alpha)} - f_{\mathrm{c}} = x_{(\alpha)}^{i} g_{i} + \frac{1}{2} x_{(\alpha)}^{i} x_{(\alpha)}^{j} G_{ij} \tag{3}$$

Considering the error of the measurement on the positions of the satellite are generally very small and can be neglected, the above equations are linear. It can be expected that the relative errors of gradients are roughly estimated as

$$\frac{\delta \mathrm{g_i}}{|\mathrm{g_i}|} \sim \frac{\delta f}{|\Delta f|}, \quad \frac{\delta G_{\mathrm{ij}}}{|G_{ij}|} \sim \frac{\delta f}{|\Delta f|} \tag{4}$$

Assuming the typical magnetic strength of structure is B, the characteristic spatial scale of the structure is D, while the separation of the satellite is L, then

$$|\Delta f| \sim \frac{LB}{D} \qquad (5)$$

So that the relative error is about

$$\frac{\delta f}{|\Delta f|} \sim \frac{\delta B}{LB/D} = \frac{\delta B}{B}\frac{D}{L} \qquad (6)$$

E.g., for the magnetotail measurements, B~20nT, D~2000km, L~200km, δB ~0.01nT. Then the relative error is

$$\frac{\delta B}{B}\frac{D}{L} \approx \frac{0.01nT}{20nT} \cdot \frac{2000km}{200km} \sim 0.005 \qquad (7)$$

Therefore, the method may be valid.

Presently, the instruments of HelioSwarm and Plasma Observatory constellations are still not fixed and their errors are not available. So we think that it is proper to make a restrict evaluation on the measurement errors in the future and after the operations of the missions.

Nothing is said about error estimates on the results (in the case where you do not know the exact solution). Does the technique allow you to produce such error estimates? If so, it would be useful to compare these estimates to the actual errors for the two test cases.

Reply:

The error is evaluated in section 4. The divergence and gradient of divergence obtained from algorithm are used to evaluate the error. To offer a uniform standard for evaluation, the divergence and gradient of divergence were non-dimensionalized with the corresponding characteristic quantity.

Minor comments

- Title: Personally, I would try to avoid the "SC" abbreviation in the title. Better change into: "Quadratic Magnetic Gradients from 7- and 9-Spacecraft Constellations"

Reply:

Done. Thanks.

The abbreviation has been deleted in the title and been substituted by full writing accordingly.

line 12: remove “therefore”

Reply:

The correction has been made accordingly.

line 13: from -> from the

Reply:

The correction has been made accordingly.

line 17: The tests -> Tests

Reply:

The correction has been made accordingly.

line 18: verifies -> verified

Reply:

The correction has been made accordingly.

line 23: iteration algorithm -> iterative algorithm

Reply:

The correction has been made accordingly.

line 38: gradient -> gradients

Reply:

The correction has been made accordingly.

line 43: tetrahedral -> a tetrahedral

Reply:

The correction has been made accordingly.

line 43: such the missions -> such missions

Reply:

The correction has been made accordingly.

line 62: consisting -> consisting of

Reply:

The correction has been made accordingly.

line 66: an ESA's new mission -> a new ESA mission

Reply:

The correction has been made accordingly.

line 68: drawn -> inferred

Reply:

The correction has been made accordingly.

line 74ff: I suggest to change punctuation into: “a description of the tests conducted for two typical magnetic structures (a cylindrical force-free flux rope and a dipole magnetic field), which were utilized to check the validity and accuracy of the new algorithm, is given …”

Reply:

Punctuations have been changed accordingly.

line 76: error -> accuracy

Reply:

The correction has been made accordingly.

line 84: references -> reference frames

Reply:

The correction has been made accordingly.

line 84: … of the magnetic field

Reply:

The correction has been made accordingly.

caption of Figure 1: relative to the constellations -> relative to the constellation

Reply:

The correction has been made accordingly.

line 95, 97, 195 and elsewhere: no capital needed at the beginning of the line

Reply:

Corrections have been made accordingly.

line 99: draw -> infer

Reply:

The correction has been made accordingly.

line 225: “The characteristic size of the S/C is twice the square root of the maximum eigenvalue” makes no sense. Size of the S/C constellation?

Reply:

Here we follow the definition of Harvey (1998) on the “characteristic size” in the tool book (Section 12.4.3). The characteristic size is used to indicate the size of the polyhedron and is twice the square root of the maximum eigenvalue of the volumetric tensor.

Ref: Harvey, C. C.: Spatial gradients and the volumetric tensor, in: Analysis methods for multi-spacecraft data, edited by Paschmann, G. and Daly, P. W., European Space Agency Publ. Division, Noordwijk, Netherlands, 315-319, 1998.

Fig 4, 6, 8: light yellow lines are hardly visible

Reply: Color of lines in the figures have been changed accordingly.

explain abbreviations when first used: NASA, ESA

Reply:

Abbreviations have been explained accordingly.