https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

Milankovitch Theory "as an Initial Value Problem"

Mikhail Y. Verbitsky[1,2]

[1]Gen5 Group, LLC, Newton, MA, USA
[2]UCLouvain, Earth and Life Institute, Louvain-la-Neuve, Belgium

Correspondence: Mikhaïl Verbitsky (verbitskys@gmail.com)

Abstract.

The dynamics of large ice sheets is fundamentally defined by the advection of mass and temperature. The timescale of these processes is critically dependent on the surface mass balance. Because of the ice-climate system's nonlinearity, its response to the orbital forcing in terms of engagement of negative and positive feedbacks is not symmetrical. This asymmetry may reduce the effective mass influx, and the resultant advection timescale may become longer, which is equivalent to the increased system's memory of its initial conditions. In this case the Milankovitch theory becomes an initial value problem: Depending on initial conditions, for the same orbital forcing and for the same balance between terrestrial positive and negative feedbacks, the historical glacial rhythmicity could have been dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both.

1. Introduction.

Though modelling of the late Pleistocene ice-age history using space resolving three-dimensional models (e.g., Abe-Ouchi *et al*, 2013) and models of intermediate complexity (e.g., Ganopolski and Calov, 2011) are now computationally affordable and may be insightful, the dynamical paleoclimatology (the term was coined by Barry Saltzman, 2002) remains to be a powerful and may be even a preferable tool to study climate evolution on orbital timescales. Even if we put aside the esthetical attractiveness of dynamical models (in the sense of Paul Dirac's vision that a physical law must indeed possess mathematical beauty), there are important physical considerations that make them essential. The first fundamental concern, that even three-dimensional models are not able to comprehensively address, relates to the fact that the ice-sheet volume is controlled by a small difference of two big values, accumulation and ablation, and therefore "…in their current state, ice sheet models do not have the predictive power to precisely reconstruct ice sheet history" (Gowan, 2023). The second consideration has been rigorously substantiated by Bahr *et al* (2015) in their review and comparison of glaciers' and ice sheets' scaling solutions versus corresponding three-dimensional models: "…if any of the numerical models' parameters are unknown or have a distribution of possible values… there is no a priori reason to expect that a tuned model will be more accurate than a tuned scaling solution". This observation is particularly true when the goal of the mathematical modelling is not a regional climate but a history of global ice volume. For these reasons, in our previous work, we derived a dynamical model of glacial rhythmicity based on scaled mass-, momentum-, and heat-conservation equations of non-Newtonian ice flow combined with the energy-balance model of global climate (Verbitsky *et al*, 2018, VCV18 thereafter) and demonstrated that its dynamical properties can be largely defined by only two similarity parameters: (a) the ratio of the orbital forcing amplitude to the amplitude of the terrestrial mass influx, and (b) the so called *V*-number that is the ratio of terrestrial positive-to-negative feedbacks amplitudes (Verbitsky and Crucifix, 2020). Specifically, when the ratio of the orbital forcing amplitude to the amplitude of the terrestrial mass influx is about 1.5 – 2., and the positive feedbacks in the system are well articulated, i.e., $V \sim 0.7 – 1.$, the system produces the obliquity-period doubling bifurcation.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

Although the astronomical forcing is the result of the celestial mechanics, and the orbital periods such as the precession, obliquity, and eccentricity periods are used to be accepted almost like physical constants, the amplitude of this forcing, as a component of the global ice mass balance, is defined much less precisely. It would be of a considerable interest therefore to study a dynamical system behavior when the ratio of the orbital forcing amplitude to the amplitude of the terrestrial mass influx is somewhat less than 1.5.

The VCV18 model is based on the thin-viscous-boundary-layer approximation of ice flow. Therefore, its dynamics is largely defined by the advection of mass and temperature. The timescale of these processes is critically dependent on the surface mass balance, i.e., accumulation minus ablation. When the orbital forcing, that is an important player of the surface mass balance, is neither strong enough to engage sufficient positive feedbacks, nor small enough to be overwhelmed by the terrestrial mass influx, the resultant effective mass influx affected by negative feedbacks may become smaller, the advection timescale may become longer and this is equivalent to the increased system's memory of its initial conditions. Thus the theory of Pleistocene glacial rhythmicity, i.e., the Milankovitch theory, becomes an initial value problem. Though the title of our paper obviously alludes to Saltzman's (1962) landmark work, we do not necessarily anticipate discovering the deterministic chaos in the VCV18 system. We will demonstrate though that, depending on initial conditions, for the same orbital forcing and for the same balance between terrestrial positive and negative feedbacks, the Pleistocene glacial rhythmicity could have been dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both.

2. The experiments

The model we utilize here has been described and analyzed in the series of publications, i.e., VCV18, Verbitsky and Crucifix (2020, 2023), and Verbitsky (2022):

$$\frac{dS}{dt} = \frac{4}{5}\zeta^{-1}S^{3/4}(\hat{a} - \varepsilon F - \kappa\omega - c\theta) \quad (1)$$

$$\frac{d\theta}{dt} = \zeta^{-1}S^{-1/4}(\hat{a} - \varepsilon F - \kappa\omega)\{\alpha\omega + \beta[S - S_0] - \theta\} \quad (2)$$

$$\frac{d\omega}{dt} = -\gamma[S - S_0] - \frac{\omega}{\tau} \quad (3)$$

Here, S (m^2) is the area of an ice sheet, θ (^{o}C) is the ice-sheet basal temperature, and ω (^{o}C) is the global "rest-of-the-climate" temperature. Equation (1) is the ice mass balance $\frac{d(HS)}{dt} = AS$, where the ice thickness H is determined from the thin-viscous-boundary-layer approximation of ice flow, $H = \zeta S^{1/4}$, ζ is dimensional factor (Verbitsky, 1992) and $A = \hat{a} - \varepsilon F - \kappa\omega - c\theta$ is the surface mass influx. Equation (2) describes vertical ice-temperature advection, and equation (3) is the climate energy-balance equation. More specifically, $\hat{a}$ (m s^{-1}) is the snow precipitation rate; F is adimensional external forcing of the amplitude ε (m s^{-1}); $\kappa\omega$ is "fast" positive feedback of the global climate in ice mass balance; $c\theta$ represents ice-sheet basal sliding combining positive feedback, $\alpha\omega$, and a negative feedback $\beta[S - S_0]$ (both are "slow" due to the vertical temperature advection). The term $-\gamma[S - S_0]$ is albedo forcing for global temperature. Remaining parameters κ (m s^{-1} ^{o}C^{-1}), c (m s^{-1} ^{o}C^{-1}), α (adimensional), β (^{o}C m^{-2}) and γ (^{o}C m^{-2} s^{-1}) are sensitivity coefficients; S_0 (m^2) is a reference glaciation area; and τ (s) is the global-temperature timescale.

For the purpose of this study, all model parameters are fixed at their reference VCV18 values, such that in all experiments $V = 0.75$, i.e. the terrestrial positive feedbacks are well articulated relative to the negative feedbacks. The only values that are going to be changed will be the initial area of an ice sheet $S(0)$ (10^6 km^2) and the adimensional amplitude of the astronomical forcing that, instead of the mid-June insolation at 65^o latitude, used in VCV18, will be modeled as the following:

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

$$F = \varepsilon_p \left(sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right) \right) + \varepsilon_o sin\left(\frac{2\pi t}{41}\right) \quad (4)$$

Here t (kyr) is time, and ε_p and ε_o are adimensional amplitudes of the precession and obliquity correspondingly. The first two terms replicate eccentricity-modulated precession, and the last term represents the obliquity forcing. We will now describe five numerical experiments.

2.1 Pure eccentricity-modulated precession, $F = 1.\left(sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right)$.

The results of the first experiment are presented in Figure 1. As it could be expected (e.g., Abe-Ouchi *et al*, 2013, Ganopolski, 2024), without obliquity, the system response is dominated by the eccentricity period (110 kyr in our case). Most importantly, though the time series are obviously not identical for different initial conditions, the dominant period of the system response is independent of initial conditions. This independence is preserved also for other values of the precession amplitude, $\varepsilon_p < 1$ (these results are not being presented in Figure 1). The memory of the system about its initial conditions does not exceed ~50 kyr.

2.2 Pure obliquity forcing, $F = \varepsilon_o\, sin\left(\frac{2\pi t}{41}\right)$.

The results of the next three experiments are presented in Figures 2 – 4. Consistently with VCV18, when terrestrial positive feedbacks are well articulated (V = 0.75) and the amplitude of the obliquity forcing is strong enough ($\varepsilon_o = 1$, Figure 2), the system exhibits the obliquity-period doubling bifurcation. The dominant period of the system response (~80 kyr) is not sensitive to the initial conditions. Also like in VCV18, when terrestrial positive feedbacks are still strong (V = 0.75) but the amplitude of the obliquity forcing is relatively weak ($\varepsilon_o = 0.5$, Figure 3), the system does not bifurcate. The dominant period of the system response (~40 kyr) is not sensitive to the initial conditions either. The most interesting behavior of the system is presented in Figure 4 ($\varepsilon_o = 0.7$). It can be observed that, when the orbital forcing is of intermediate intensity, the relaxation timescale becomes much longer that is equivalent to the increased system's memory of its initial conditions. Consequently, the dominant period of fluctuations becomes ***sensitive to the initial conditions*** and may be either of the obliquity-period or of the double-obliquity-period value.

This intriguing phenomenon needs a physical explanation. Because of the system's nonlinearity, its response to the orbital forcing in terms of engagement of negative and positive feedbacks is not symmetrical. In the VCV18 model, this may be observed as a shift of the time-mean glaciation area that is dependent on the effective mass influx. For example, for the reference values of model parameters and without astronomical forcing, the equilibrium glaciation area S = 15 10^6 km^2. The obliquity forcing of a relatively small amplitude ($\varepsilon_o = 0.5$, Figure 3) does not engage sufficient positive feedback (it is delayed due to vertical temperature advection) and the dominant negative feedbacks shift mean glaciation area to S = 13.5 10^6 km^2 that in the VCV18 model is equivalent to 50% reduction of mean terrestrial mass influx. The obliquity forcing of a large amplitude ($\varepsilon_o = 1$, Figure 2) administers strong positive feedbacks and shifts mean glaciation area to S = 17.7 10^6 km^2 that is equivalent to almost doubled mean terrestrial mass influx. The obliquity forcing of an intermediate amplitude ($\varepsilon_o = 0.7$, Figure 4) shifts mean glaciation area to 12.3 10^6 km^2 that is equivalent to the significant, ten-fold, reduction of terrestrial mass influx.

The timescale of the advection processes in the "thin" (relative to its horizontal size) ice sheet can be estimated as $\tau = \frac{H}{\tilde{A}}$, where H is the characteristic ice thickness and $\tilde{A}$ is the mean terrestrial mass influx. Since $H \sim S^{1/4}$, the timescale in the described experiments is mostly defined by the effective mass influx, and the ten-fold reduction of it means ten-fold longer vertical-advection timescale and, consequently, ten-fold longer system's memory of its initial conditions. Without astronomical forcing, the relaxation

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

process in the VCV18 dynamical system takes about 100 kyr. The ten-time extension of it implies that the initial conditions of the ice-climate system may be remembered through the entire late Pleistocene.

2.3 Eccentricity-modulated precession and obliquity forcing of intermediate intensity,

$$\boldsymbol{F = 0.7\left(sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right) + 0.7 sin\left(\frac{2\pi t}{41}\right).}$$

When the system is forced by a combination of eccentricity-modulated precession and obliquity, both with intermediate values of their amplitudes, $\varepsilon_p = \varepsilon_o = 0.7$, the system response, depending on initial conditions, may be dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both (Figures 5 and 6).

3. Discussion and Conclusions.

The interpretation of the Milankovitch theory that we present in this study is very simple and it is based on very explicit physics:

(1) The dynamics of large ice sheets is defined by the advection of mass and temperature;
(2) The timescale of ice advection depends mostly on the surface total mass influx;
(3) Because of the ice-climate system's nonlinearity, its response to the orbital forcing in terms of engagement of negative and positive feedbacks is not symmetrical. This may change the effective mass influx and the resultant advection timescale;
(4) The orbital forcing of a significant amplitude ($\varepsilon_o = 1$) may reduce the internal ice-sheet advection timescale to the value comparable with the obliquity period, and this will produce the obliquity-period doubling bifurcation;
(5) The orbital forcing of less intensive amplitude ($\varepsilon_o = 0.7$) may significantly increase the internal ice-sheet advection timescale. It means that the ice-climate system may remember its initial conditions through the entire late Pleistocene, and for the same orbital forcing and for the same balance between terrestrial positive and negative feedbacks, the historical glacial rhythmicity could have been dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both.

Since we have only one historical time series, our theory cannot, indeed, be verified based on the empirical data, and, instead, other models may need to be involved in a comprehensive test. Obviously, this may be easier said than done. A hypothetical model that would be appropriate for such testing should be able to explicitly account for all the above physics. Therefore, all phenomenological models, obtained either from the fitting to the empirical data or by emulating the behavior of more comprehensive models, will be of little help because they may not have physical similarity with the Nature or with the comprehensive models they mimic. On the other hand, three-dimensional and intermediate-complexity models do have, indeed, all the physics needed, but, as we have already discussed in the Introduction, they may not be able to resolve the mass influx that may be responsible for a timescale of about few hundreds of thousands years. Indeed, if a characteristic thickness of ice is a few thousands meters, then we are talking about mass influx of the order of few centimeters per year. We nevertheless hope, that, other than VCV18, simple but physics-based models can be designed to support (or reject) the Milankovitch theory formulated here as an initial value problem.

Competing interests: The author has declared that there are no competing interests.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

References

Abe-Ouchi, A., Saito, F., Kawamura, K., Raymo, M. E., Okuno, J. I., Takahashi, K., and Blatter, H.: Insolation-driven 100,000-year glacial cycles and hysteresis of ice-sheet volume, Nature, 500, 190–194, 2013.

Bahr, D. B., Pfeffer, W. T., and Kaser, G.: A review of volume-area scaling of glaciers, Rev. Geophys., 53,95–140, doi:10.1002/2014RG000470, 2015.

Ganopolski, A.: Toward generalized Milankovitch theory (GMT), Clim. Past, 20, 151–185, https://doi.org/10.5194/cp-20-151-2024, 2024.

Ganopolski, A. and Calov, R.: The role of orbital forcing, carbon dioxide and regolith in 100 kyr glacial cycles, Clim. Past., 7, 1415–1425, https://doi.org/10.5194/cp-7-1415-2011, 2011.

Gowan, E.J.: Bayesian analysis and paleo ice sheet modelling: a commentary on the proposal by Tarasov and Goldstein. https://egusphere.copernicus.org/preprints/2023/egusphere-2022-1410/egusphere-2022-1410-RC2-supplement.pdf, 2023.

Saltzman, B.: Finite amplitude free convection as an initial value problem, Journal of atmospheric sciences, 19, 4, 329-341, 1962.

Saltzman, B.: Dynamical paleoclimatology: generalized theory of global climate change, in: Vol. 80, Academic Press, San Diego, CA, ISBN 0126173311, 2002.

Verbitsky, M.Y.: Equilibrium ice sheet scaling in climate modeling, Climate Dynamics, 7, 105–110, https://doi.org/10.1007/BF00209611, 1992.

Verbitsky, M. Y.: Inarticulate past: similarity properties of the ice–climate system and their implications for paleo-record attribution, Earth Syst. Dynam., 13, 879–884, https://doi.org/10.5194/esd-13-879-2022, 2022.

Verbitsky, M. Y. and Crucifix, M.: π-theorem generalization of the ice-age theory, Earth Syst. Dynam., 11, 281–289, https://doi.org/10.5194/esd-11-281-2020, 2020.

Verbitsky, M. Y. and Crucifix, M.: Do phenomenological dynamical paleoclimate models have physical similarity with Nature? Seemingly, not all of them do, Clim. Past, 19, 1793–1803, https://doi.org/10.5194/cp-19-1793-2023, 2023.

Verbitsky, M. Y., Crucifix, M., and Volobuev, D. M.: A theory of Pleistocene glacial rhythmicity, Earth Syst. Dynam., 9, 1025–1043, https://doi.org/10.5194/esd-9-1025-2018, 2018.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

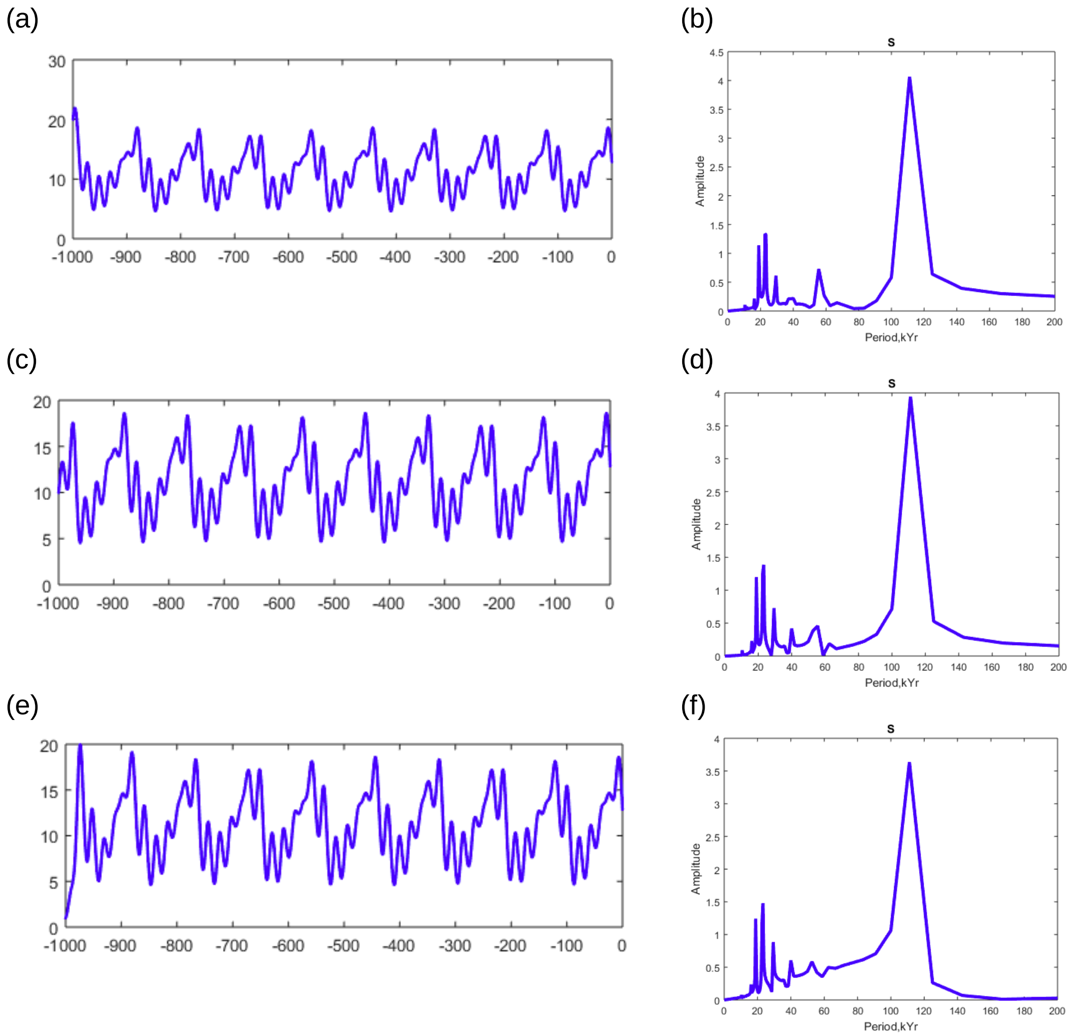

Figure 1. The dynamical system response to pure eccentricity-modulated precession, $F = \sin\left(\frac{2\pi t}{19}\right) + \sin\left(\frac{2\pi t}{23}\right)$.

(a), (c), and **(e)** are the time series of the area of glaciation in (10^6 km^2); **(b), (d),** and **(f)** are the corresponding spectral diagrams; **(a)** and **(b)** are for S(0) = 20 10^6 km^2; **(c)** and **(d)** are for S(0) = 10 10^6 km^2; **(e)** and **(f)** are for S(0) = 1 10^6 km^2.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

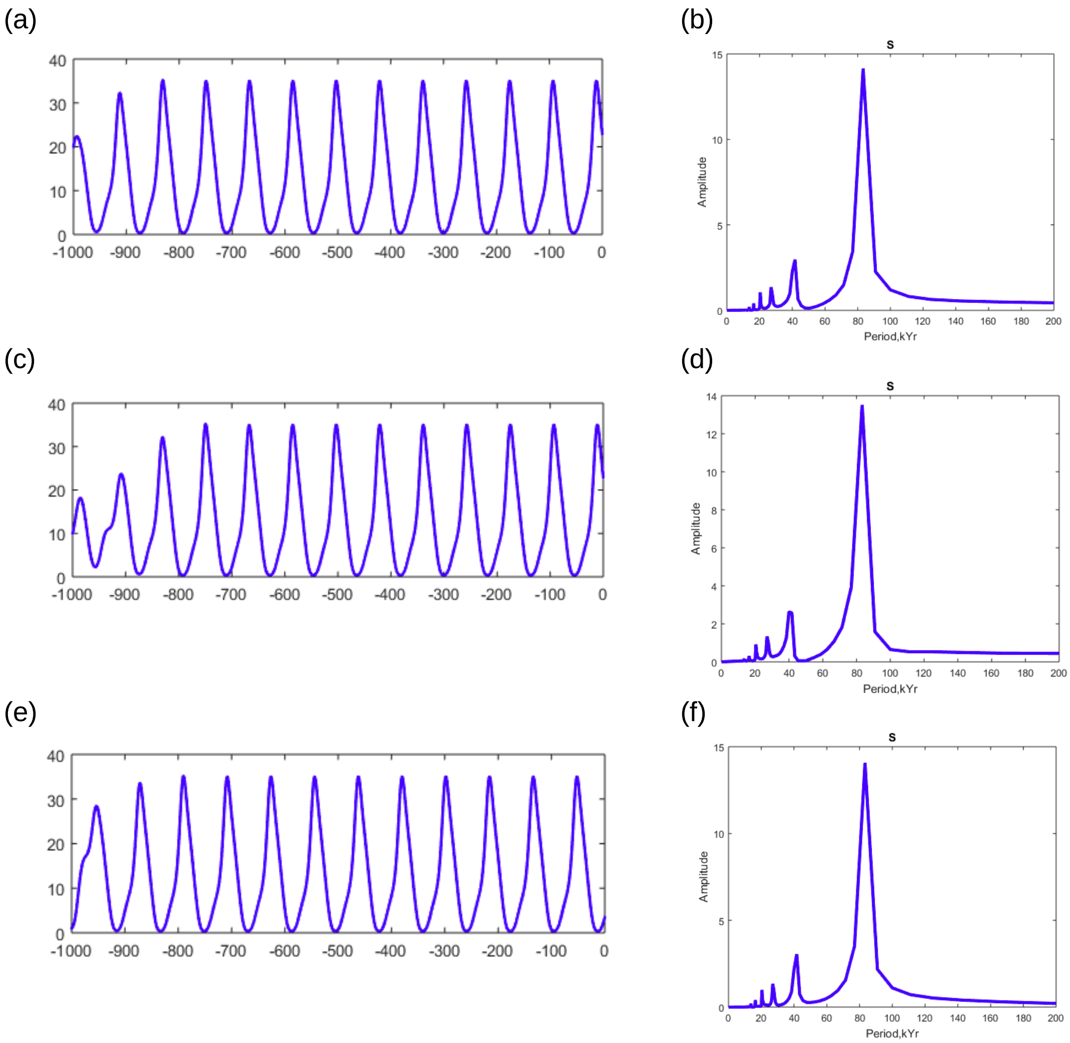

Figure 2. The dynamical system response to pure obliquity forcing, $F = 1.0\, sin\left(\frac{2\pi t}{41}\right)$.

(a), (c), and **(e)** are the time series of the area of glaciation in (10^6 km^2); **(b), (d),** and **(f)** are the corresponding spectral diagrams; **(a)** and **(b)** are for S(0) = 20 10^6 km^2; **(c)** and **(d)** are for S(0) = 10 10^6 km^2; **(e)** and **(f)** are for S(0) = 1 10^6 km^2.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

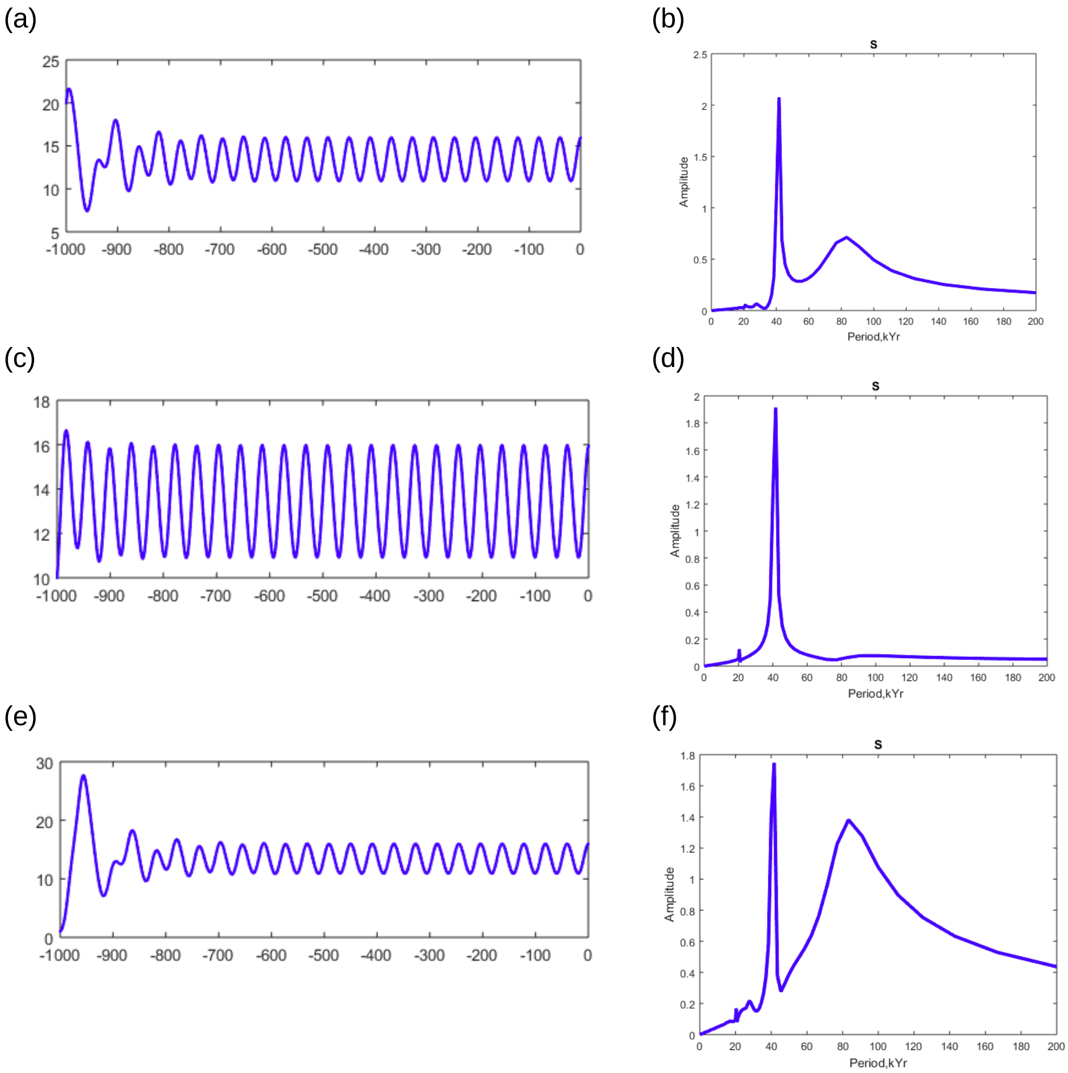

Figure 3. The dynamical system response to pure obliquity forcing, $F = 0.5\, sin\left(\frac{2\pi t}{41}\right)$.

(a), (c), and **(e)** are the time series of the area of glaciation in (10^6 km^2); **(b), (d),** and **(f)** are the corresponding spectral diagrams; **(a)** and **(b)** are for S(0) = 20 10^6 km^2; **(c)** and **(d)** are for S(0) = 10 10^6 km^2; **(e)** and **(f)** are for S(0) = 1 10^6 km^2.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

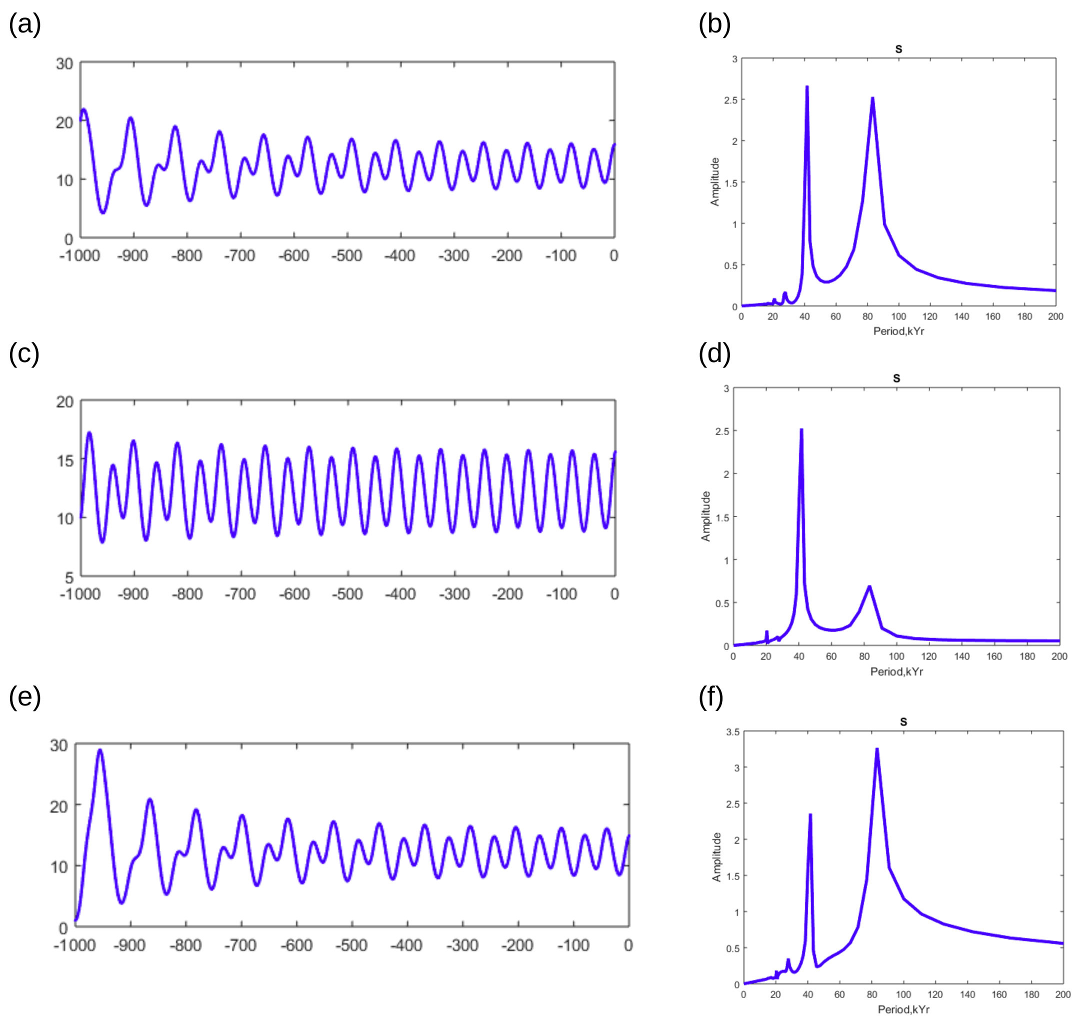

Figure 4. The dynamical system response to pure obliquity forcing, $F = 0.7\ sin\left(\frac{2\pi t}{41}\right)$.

(a), (c), and **(e)** are the time series of the area of glaciation in (10^6 km^2); **(b), (d),** and **(f)** are the corresponding spectral diagrams; **(a)** and **(b)** are for S(0) = 20 10^6 km^2; **(c)** and **(d)** are for S(0) = 10 10^6 km^2; **(e)** and **(f)** are for S(0) = 1 10^6 km^2.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

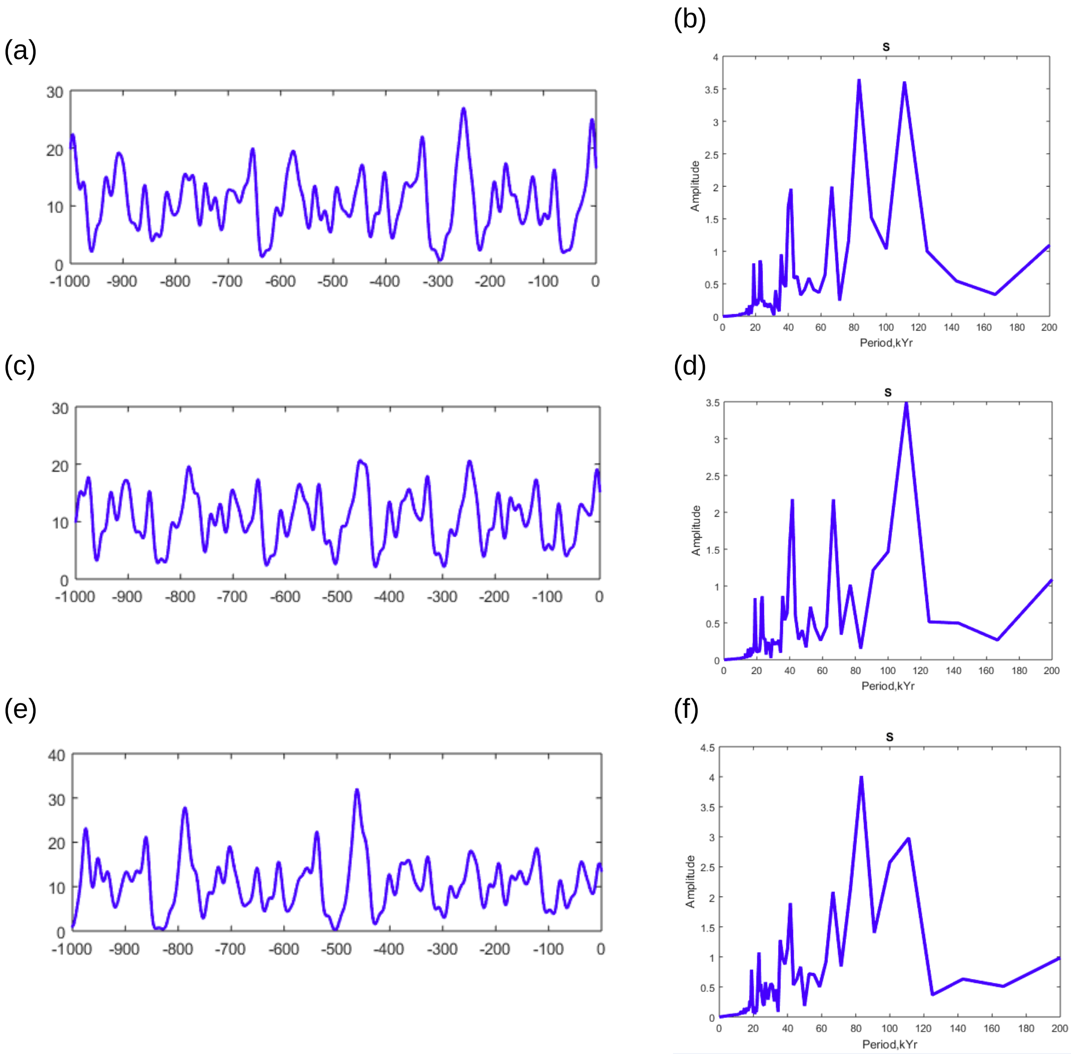

Figure 5. The dynamical system response to orbital forcing, $F = 0.7\left(sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right) + 0.7sin\left(\frac{2\pi t}{41}\right)$.

(a), (c), and **(e)** are the time series of the area of glaciation in (10^6 km^2); **(b), (d),** and **(f)** are the corresponding spectral diagrams; **(a)** and **(b)** are for S(0) = 20 10^6 km^2; **(c)** and **(d)** are for S(0) = 10 10^6 km^2; **(e)** and **(f)** are for S(0) = 1 10^6 km^2.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024

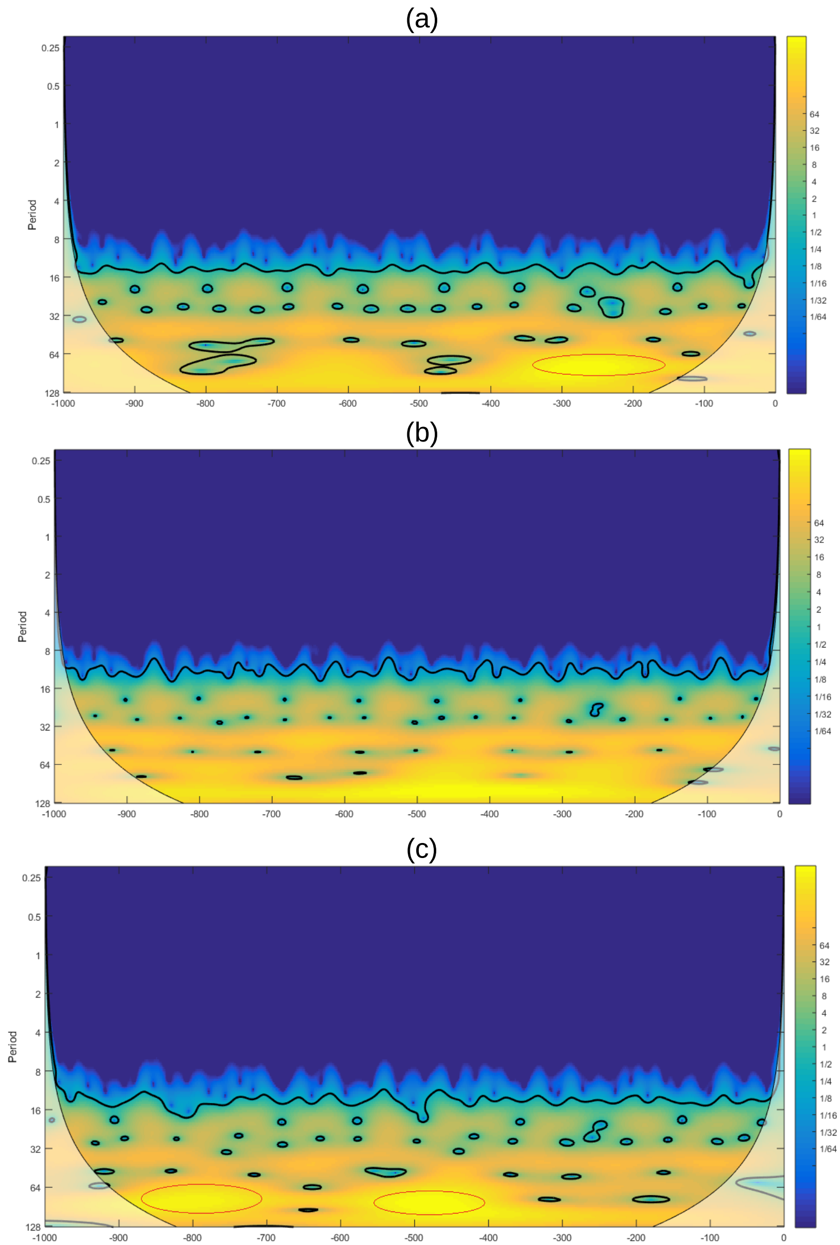

Figure 6. The dynamical system response to orbital forcing, $F = 0.7\left(sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right) + 0.7sin\left(\frac{2\pi t}{41}\right)$.

The evolution of wavelet spectra for the time series of Fig. 5; **(a)** S(0) = 20 10^6 km^2; **(b)** S(0) = 10 10^6 km^2; **(c)** S(0) = 1 10^6 km^2. The red ellipses show the episodes of 80-kyr dominance.