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Milankovitch Theory "as an Initial Value Problem"

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Abstract. The dynamics of large ice sheets is fundamentally defined by the advection of mass and temperature. The timescale of these processes is critically dependent on the surface mass balance. Because of the ice-climate system's nonlinearity, its response to the orbital forcing in terms of engagement of negative and positive feedbacks is not symmetrical. This asymmetry may reduce the effective mass influx, and the resultant advection timescale may become longer, which is equivalent to a longer system's memory of its initial conditions. In this case the Milankovitch theory becomes an initial value problem: Depending on initial conditions, for the same orbital forcing and for the same balance between terrestrial positive and negative feedbacks, the historical glacial rhythmicity could have been dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both. In fact, empirical records demonstrate that the dominant period of the Late Pleistocene ice ages evolved from ~80-kyr to ~100-kyr rhythmicity. The quantitative similarity of this dominant-period trajectory and the one, made by the long-memory model, suggests that the records of the Late Pleistocene glacial rhythmicity could have been produced by a long-memory initial-value-dependent climate system, or, in other words, the slopes in empirical dominant-period trajectories are signatures of a long memory.

The scaling law of the dominant-period trajectory provides a theoretical insight into the discovered phenomenon. It reveals that this trajectory is dependent on the memory duration that is sensitive to initial conditions. The sensitivity of the memory duration to initial values emerges as the result of system's incomplete similarity in two similarity parameters colliding into one conglomerate similarity parameter that is the ratio of the advection timescale and the orbital period. The critical dependence of this similarity parameter on poorly defined accumulation-minus-ablation mass balance as well as its dependence on initial values makes ice ages to be hardly predictable and disambiguation of paleo-records to be extremely challenging. The quasi-eccentricity periods produced by the long-memory system in response to pure obliquity forcing make a remarkable example of this challenge because in the time series they may be naively attributed to the eccentricity modulated precession forcing.

1. Introduction.

In memory of Barry Saltzman

Though modelling of the Late Pleistocene ice-age history using space resolving three-dimensional models (e.g., Abe-Ouchi *et al*, 2013) and models of intermediate complexity (e.g., Ganopolski and Calov, 2011) are now computationally affordable and may be insightful, the dynamical paleoclimatology (the term was coined by Barry Saltzman, 2002) remains to be a powerful and may be even a preferable tool to study climate evolution on orbital timescales. Even if we put aside the esthetical attractiveness of dynamical models (in the sense of Paul Dirac's vision that a physical law must indeed possess mathematical beauty), there are important physical considerations that make them essential. The first fundamental concern, that even three-dimensional models are not able to comprehensively address, relates to the fact that the ice-sheet volume is controlled by a small difference of two big values, accumulation and ablation, and therefore "…in their current state, ice sheet models do not have the predictive power to precisely reconstruct ice sheet history" (Gowan, 2023). The second consideration has been rigorously

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substantiated by Bahr *et al* (2015) in their review and comparison of glaciers' and ice sheets' scaling solutions versus corresponding three-dimensional models: "…if any of the numerical models' parameters are unknown or have a distribution of possible values… there is no a priori reason to expect that a tuned model will be more accurate than a tuned scaling solution". This observation is particularly true when the goal of the mathematical modelling is not a regional climate but a history of global ice volume.

In this paper, we will discuss to what extent this accumulation-minus-ablation mass influx affects ice-sheet sensitivity to initial values. To introduce the language of our study, we begin with a very simple illustration. Though this linear example by any means is not a complete analogy of highly nonlinear ice-climate system, it will give us an opportunity to introduce concepts of the dynamical system memory, the memory duration, the memory-duration sensitivity to initial values, as well as the key similarity parameter that defines this sensitivity. With this caveat in mind, let us assume that ice volume x evolves according to the following mass balance:

$$\frac{dx}{dt} = a - \frac{x}{\tau} \qquad (1)$$

with the initial condition $x = x_0$ when $t = 0$. Here a is the terrestrial mass influx (accumulation minus ablation) and τ is the ice-sheet dynamical timescale. The adimensional solution of this equation is:

$$\frac{x}{a\tau} = \left(\frac{x_0}{a\tau} - 1\right) e^{-\frac{t}{\tau}} + 1 \qquad (2)$$

Several observations will be useful for our further discussion:

(a) The term $\left(\frac{x_0}{a\tau} - 1\right) e^{-\frac{t}{\tau}}$ is the decaying ***memory*** of the dynamical system (1) about its initial conditions;

(b) The ***memory duration*** is defined by the dynamical properties of the system, i.e., the timescale τ, that can also be considered as a measure of intensity of (in this particular case, dominant negative) feedbacks, and by the ***adimensional similarity parameter*** $C = \frac{x_0}{a\tau}$ that is the ratio of the vertical advection timescale $\frac{x_0}{a}$ and τ;

(c) The short-memory systems ($\tau \to 0$) are initial-value independent;

(d) In a long-memory system, the time series are initial-value dependent. The memory duration though, may be either sensitive to initial values or initial-values independent. If the mass influx is strong, then the advection timescale $\frac{x_0}{a}$ is short relative to τ, i.e., $C \ll 1$, and the memory duration is independent on initial values. A weak mass influx makes advection timescale $\frac{x_0}{a}$ longer such that $C \sim 1$ and the memory duration becomes initial-value sensitive;

(e) A significant portion of a time series (2) produced by a long-memory system (in this particular case we are talking about the steady-state $a\tau$) may be reproduced by a short-memory system, if its mass balance a is adjusted accordingly. Therefore, any claim that the nature is not sensitive to initial conditions because a short-memory model has successfully reproduced a sample time series, should be taken cautiously unless our knowledge about mass balance and internal dynamics is unambiguous. As we already know, this is not the case, and restoration of equation (1) having only partial knowledge of the time series (2) may constitute an attribution challenge.

The dynamics of a real ice sheet, approximated as the thin viscous boundary layer of ice media, is largely defined by the advection of mass and temperature. The timescale of these processes is not fixed, but instead it is critically dependent on the surface mass balance, that is an outcome of a sophisticated interplay of orbital forcing and internal dynamics. In our following presentation, we will largely follow the logic we just entertained with the example (1-2).

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First, we will describe our model and demonstrate that when the orbital forcing is neither strong enough to engage sufficient positive feedbacks, nor small enough to be overwhelmed by the terrestrial mass influx, the resultant effective mass influx affected by negative feedbacks may become smaller, the advection timescale may become longer and this is equivalent to longer and initial-values dependent memory duration. Thus the theory of the Pleistocene glacial rhythmicity, i.e., the Milankovitch theory, becomes an initial value problem. Though the title of our paper obviously alludes to Saltzman's (1962) landmark work, we do not necessarily anticipate identifying the deterministic chaos in the physical model of an ice sheet. We will demonstrate though that, depending on initial conditions, for the same periodicity of the orbital forcing, the Pleistocene glacial rhythmicity could have been dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both. We will also show that a particular trajectory of the dominant period produced by a long-memory model may be reasonably close to the empirical data and therefore the empirical records of the Late Pleistocene glacial rhythmicity could have been produced by a long-memory initial-value dependent climate system.

To provide a theoretical insight into the discovered phenomenon, we will then derive the scaling law of the dominant-period trajectory. The law will reveal that this trajectory is dependent on memory duration that is sensitive to initial conditions. The sensitivity of the memory duration to initial values emerges as the result of system's incomplete similarity in two similarity parameters colliding into one conglomerate similarity parameter that is the ratio of the advection timescale and the orbital period. The critical dependence of this similarity parameter on poorly defined accumulation-minus-ablation mass balance as well as its dependence on initial values makes ice ages to be hardly predictable and disambiguation of paleo-records to be extremely challenging. We will illustrate this challenge with a remarkable phenomenon of quasi-eccentricity periods produced by the long-memory system in response to pure obliquity forcing.

2. The observations

In our previous work, we derived a dynamical model of glacial rhythmicity based on scaled mass-, momentum-, and heat-conservation equations of non-Newtonian ice flow combined with the energy-balance model of global climate (Verbitsky *et al*, 2018, VCV18 thereafter, Verbitsky and Crucifix, 2020, 2023, and Verbitsky, 2022):

$$\frac{dS}{dt} = \frac{4}{5}\zeta^{-1}S^{3/4}(\hat{a} - \varepsilon F - \kappa\omega - c\theta) \quad (3)$$

$$\frac{d\theta}{dt} = \zeta^{-1}S^{-1/4}(\hat{a} - \varepsilon F - \kappa\omega)\{\alpha\omega + \beta[S - S_0] - \theta\} \quad (4)$$

$$\frac{d\omega}{dt} = -\gamma[S - S_0] - \frac{\omega}{\tau_\omega} \quad (5)$$

Here, S (m^2) is the area of an ice sheet, θ (°C) is the ice-sheet basal temperature, and ω (°C) is the global "rest-of-the-climate" temperature. Equation (3) is the ice mass balance $\frac{d(HS)}{dt} = AS$, where the ice thickness H is determined from the thin-viscous-boundary-layer approximation of ice flow, $H = \zeta S^{1/4}$, ζ is dimensional ($m^{1/2}$) factor (Verbitsky, 1992) and $A = \hat{a} - \varepsilon F - \kappa\omega - c\theta$ is the surface mass influx. Equation (4) describes vertical ice-temperature advection, and equation (5) is the climate energy-balance equation. More specifically, $\hat{a}$ ($m\ s^{-1}$) is the snow precipitation rate; F is adimensional external forcing of the amplitude ε ($m\ s^{-1}$); $\kappa\omega$ is "fast" positive feedback of the global climate in ice mass balance; $c\theta$ represents ice-sheet basal sliding combining positive feedback, $\alpha\omega$, and a negative feedback $\beta[S - S_0]$ (both are "slow" due to the vertical temperature advection). The term $-\gamma[S - S_0]$ is albedo forcing for global temperature. Remaining parameters κ ($m\ s^{-1}\ {}^{o}C^{-1}$), c ($m\ s^{-1}\ {}^{o}C^{-1}$), α (adimensional), β (${}^{o}C\ m^{-2}$) and γ (${}^{o}C\ m^{-2}\ s^{-1}$) are sensitivity coefficients; S_0 (m^2) is a reference glaciation area; and τ_ω(s) is the global-temperature timescale.

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We have demonstrated (Verbitsky and Crucifix, 2020) that the dynamical properties of the system (3) – (5) can be largely defined by only two similarity parameters: (a) the ratio of the orbital forcing amplitude to the amplitude of the terrestrial mass influx, $\frac{\varepsilon}{\hat{a}}$, and (b) the so called *V*-number that is the ratio of terrestrial positive-to-negative feedbacks amplitudes.

$$\frac{P}{T} = \Phi_o\left(\frac{\varepsilon}{\hat{a}}, V\right) \qquad (6)$$

Here *P* is the dominant period of the system response, and *T* is the forcing period. Specifically, when the ratio of the orbital forcing amplitude to the amplitude of the terrestrial mass influx is about 1.5 – 2., and the positive feedbacks in the system are well articulated, i.e., $V \sim 0.7 – 1.$, the system produces the obliquity-period doubling bifurcation. Although the astronomical forcing is the result of the celestial mechanics, and the orbital periods such as the precession, obliquity, and eccentricity periods are used to be accepted almost like physical constants, the amplitude of this forcing, as a component of the global ice mass balance, is defined much less precisely. It would be of a considerable interest therefore to study a dynamical system behavior when the orbital forcing amplitude is weakened.

For the purpose of this study, all model parameters are fixed at their reference VCV18 values, such that in all experiments $V = 0.75$, i.e., the terrestrial positive feedbacks are well articulated relative to the negative feedbacks. The only values that are going to be changed will be the initial area of an ice sheet $S(0)$ (10^6 km^2) and the adimensional amplitude of the astronomical forcing that, instead of the mid-June insolation at 65^o latitude, used in VCV18, will be modeled as the following:

$$F = \varepsilon_p\left[sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right] + \varepsilon_o sin\left(\frac{2\pi t}{41}\right) \qquad (7)$$

Here *t* (kyr) is time, and ε_p and ε_o are adimensional amplitudes of the precession and obliquity correspondingly. The first two terms replicate eccentricity-modulated precession, and the last term represents the obliquity forcing. We will now describe five numerical experiments.

2.1 Pure eccentricity-modulated precession, $F = 1.\left[sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right]$.

The results of the first experiment are presented in Figure 1.

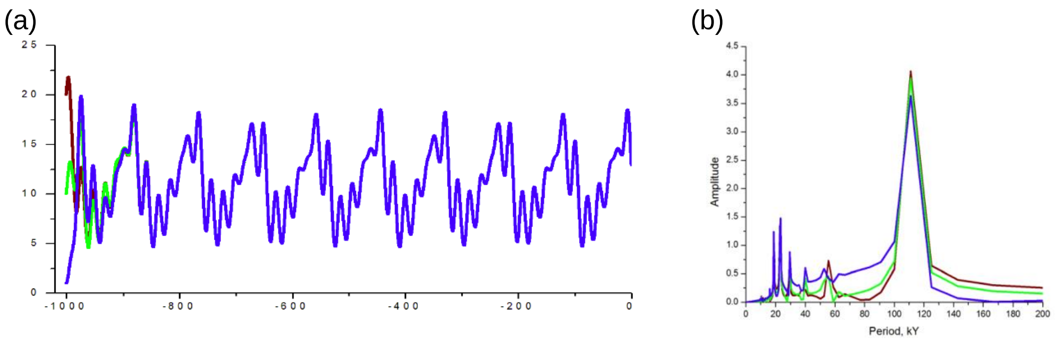

Figure 1. The dynamical system response to pure eccentricity-modulated precession $\boldsymbol{F = 1.\left[sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right]}$. Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** is the corresponding spectral diagram; **brown** is for $S(0)$ = 20 10^6 km^2; **green** is for $S(0)$ = 10 10^6 km^2; **blue** is for $S(0)$ = 1 10^6 km^2.

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Without obliquity, the system response is dominated by the eccentricity period (110 kyr in our case). Most importantly, though the time series are obviously not identical for different initial conditions, the dominant period of the system response is independent of initial conditions. The memory duration of the system about its initial conditions does not exceed ~100 kyr.

2.2 Pure obliquity forcing, $F = \varepsilon_o \, sin\left(\frac{2\pi t}{41}\right)$.

The results of the next three experiments are presented in Figures 2 – 4.

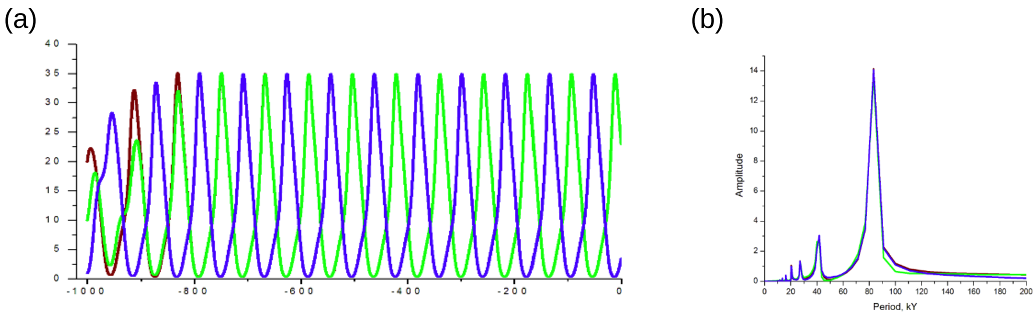

Figure 2. The dynamical system response to pure obliquity forcing, $F = 1.0 \, sin\left(\frac{2\pi t}{41}\right)$. Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** is the corresponding spectral diagram; **brown** is for $S(0)$ = 20 10^6 km^2; **green** is for $S(0)$ = 10 10^6 km^2; **blue** is for $S(0)$ = 1 10^6 km^2.

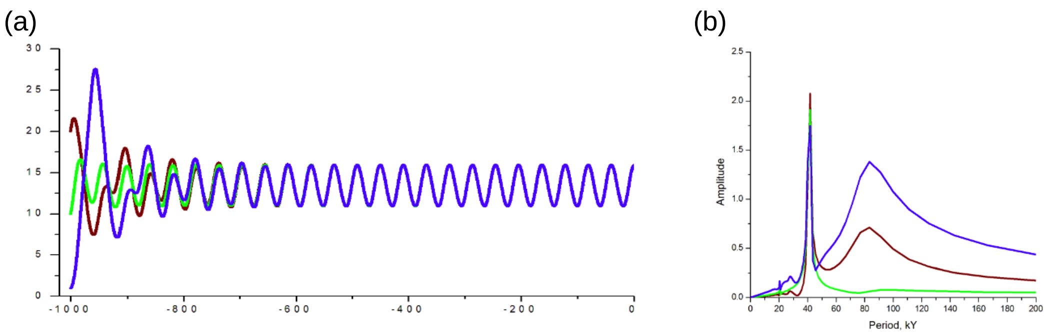

Figure 3. The dynamical system response to pure obliquity forcing, $F = 0.5 \, sin\left(\frac{2\pi t}{41}\right)$. Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** is the corresponding spectral diagram; **brown** is for $S(0)$ = 20 10^6 km^2; **green** is for $S(0)$ = 10 10^6 km^2; **blue** is for $S(0)$ = 1 10^6 km^2.

Consistently with VCV18, when terrestrial positive feedbacks are well articulated (V = 0.75) and the amplitude of the obliquity forcing is strong enough ($\varepsilon_o = 1$, Figure 2), the system exhibits the obliquity-period doubling bifurcation. The dominant period of the system response (~80 kyr) is not sensitive to the initial conditions. Also like in VCV18, when terrestrial positive feedbacks are still strong (V = 0.75) but

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the amplitude of the obliquity forcing is relatively weak ($\varepsilon_o = 0.5$, Figure 3), the system does not bifurcate. The dominant period of the system response (~40 kyr) is not sensitive to the initial conditions either.

The most interesting behavior of the system is presented in Figure 4 ($\varepsilon_o = 0.7$). It can be observed that, when the orbital forcing is of intermediate intensity, the relaxation timescale may become much longer that is equivalent to longer system's memory of its initial conditions. Remarkably, ***the memory duration and the dominant period of fluctuations become sensitive to the initial conditions***, and the dominant period may be either of the obliquity-period or of the double-obliquity-period value.

This intriguing phenomenon needs a physical explanation. Though more rigorous reasoning will be presented below with the scaling law, some preliminary simple considerations may be also appropriate. Because of the system's nonlinearity, its response to the orbital forcing in terms of engagement of negative and positive feedbacks is not symmetrical. In the VCV18 model, this may be observed as a shift of the time-mean glaciation area that is dependent on the effective mass influx. For example, for the reference values of model parameters and without astronomical forcing, the equilibrium glaciation area S = 15 10^6 km^2. The obliquity forcing of a relatively small amplitude ($\varepsilon_o = 0.5$, Figure 3) does not engage sufficient positive feedback (due to vertical temperature advection that is period-sensitive and behaves like a lower-frequency filter) and the dominant negative feedbacks shift mean glaciation area to S = 13.5 10^6 km^2 that in the VCV18 model is equivalent to 50% reduction of mean terrestrial mass influx. The obliquity forcing of a large amplitude ($\varepsilon_o = 1$, Figure 2) administers strong positive feedbacks and shifts mean glaciation area to S = 17.7 10^6 km^2 that is equivalent to almost doubled mean terrestrial mass influx. The obliquity forcing of an intermediate amplitude ($\varepsilon_o = 0.7$, Figure 4) shifts mean glaciation area to 12.3 10^6 km^2 that is equivalent to the significant, ten-fold, reduction of terrestrial mass influx.

The timescale of the advection processes in the "thin" (relative to its horizontal size) ice sheet can be estimated as $\tau = \frac{H}{\tilde{A}}$, where H is the characteristic ice thickness and $\tilde{A}$ is the mean terrestrial mass influx. Since $H \sim S^{1/4}$, the timescale in the described experiments is mostly defined by the effective mass influx, and the ten-fold reduction of it means ten-fold longer vertical-advection timescale and, consequently, ten-fold longer system's memory of its initial conditions. Without astronomical forcing, the relaxation process in the VCV18 dynamical system takes about 100 kyr. The ten-time extension of it implies that the initial conditions of the ice-climate system may be remembered through the entire Late Pleistocene.

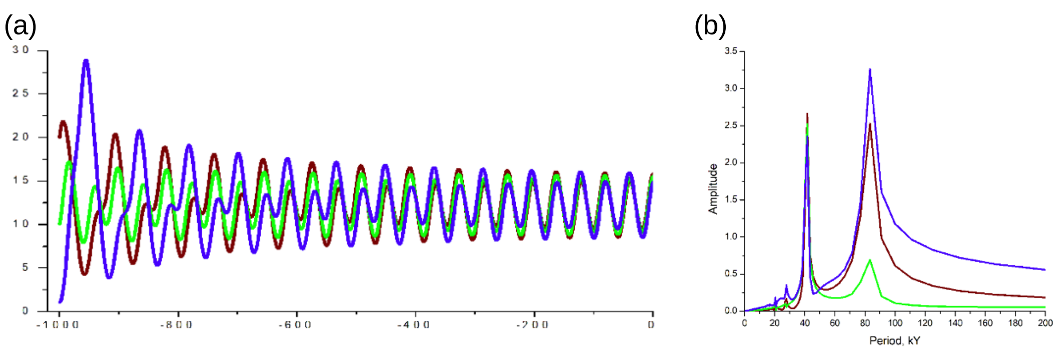

Figure 4. The dynamical system response to pure obliquity forcing, $F = 0.7\, sin\left(\frac{2\pi t}{41}\right)$. Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** is the corresponding spectral diagram; **brown** is for $S(0)$ = 20 10^6 km^2; **green** is for $S(0)$ = 10 10^6 km^2; **blue** is for $S(0)$ = 1 10^6 km^2.

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2.3 Eccentricity-modulated precession and obliquity forcing of intermediate intensity, $F = 0.7\left[sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right] + 0.7sin\left(\frac{2\pi t}{41}\right)$.

When the system is forced by a combination of eccentricity-modulated precession and obliquity, both with intermediate values of their amplitudes, $\varepsilon_p = \varepsilon_o = 0.7$, the system response, depending on initial conditions, may be dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both (Figure 5). Figure 6 shows the evolution of the ice volume wavelet spectra over the past 1,000 kyr based on Lisiecki and Raymo (2005) benthic foraminifera δ^{18}O data.

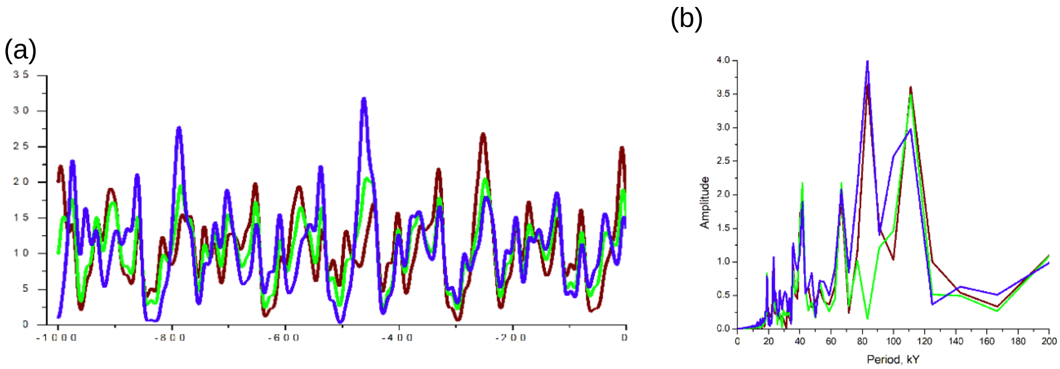

Figure 5. The dynamical system response to orbital forcing, $F = 0.7\left[sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right] + 0.7sin\left(\frac{2\pi t}{41}\right)$. Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** is the corresponding spectral diagram; **brown** is for $S(0)$ = 20 10^6 km^2; **green** is for $S(0)$ = 10 10^6 km^2; **blue** is for $S(0)$ = 1 10^6 km^2.

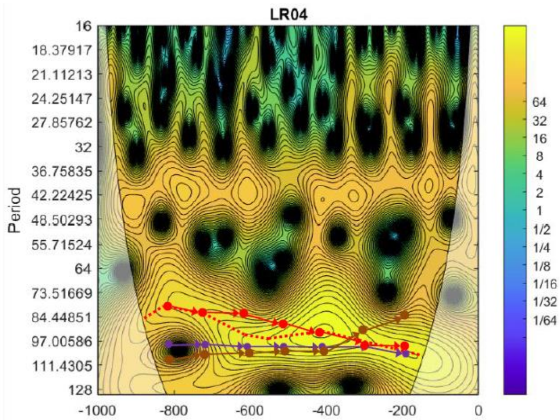

Figure 6. The evolution of the ice volume wavelet spectra over the past 1,000 kyr based on Lisiecki and Raymo (2005, "LR04" stack) benthic foraminifera δ^{18}O data. The color scale shows wavelet amplitude and the dotted red line displays the trajectory of the LR04 dominant period. The circled red line represents the trajectory of the dominant period for the long-memory-model solution shown in Figure 5 with $S(0)$ = 1 10^6 km^2; the circled purple line represents the trajectory of the dominant period for the model solution with $S(0)$ = 10 10^6 km^2; and the circled brown line represents the trajectory of the dominant period for the long-memory-model solution with $S(0)$ = 20 10^6 km^2.

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The trajectory of the dominant period of the ice-volume variations has a well-articulated slope and therefore infers that the Late Pleistocene glacial rhythmicity ***evolved from the dominant 80-kyr periods to about 100-kyr periodicity***. It is interesting that one of the ***long-memory-model*** trajectories (i.e., the one starting from the $S(0) = 1\ 10^6$ km^2 initial conditions, Figure 5) replicates reasonably well this evolving-rhythmicity slope. As we have already mentioned, all model parameters have been fixed to the VCV18 reference values, and therefore we didn't make any additional efforts to achieve a better fit to the empirical data. Nevertheless, the similarity of the empirical and one of the model-made dominant-period trajectories, strongly suggests that the records of the Late Pleistocene glacial rhythmicity could have been produced by a long-memory climate system.

3. The scaling law of the dominant-period trajectory: Why could ice ages be unpredictable?

We will now derive the scaling law that controls the dominant-period trajectory. Without orbital forcing and for fixed balance between positive and negative feedbacks ($V = const$) the memory duration of the system (3) – (5) depends on only two parameters, $\hat{a}$ and ζ. This should not come as a surprise. Indeed, these are the parameters that define the timescale of vertical advection, i.e., parameter ζ defines the thickness of an ice sheet for given area S ($H = \zeta S^{1/4}$) and $\hat{a}$ is the snowfall rate. Thus, the memory duration M, measured in units of time, can be described as

$$M = \varphi(\zeta, \hat{a}, V) \tag{8}$$

Here V is adimensional, and parameters ζ(m$^{1/2}$), $\hat{a}$ (m/s) have independent dimensions; therefore after applying π-theorem (Buckingham, 1914) to equation (8), we can see that the memory duration M of our system, without orbital forcing, is fully described by the timescale $\frac{\zeta^2}{\hat{a}}$ and the V-number:

$$M = \frac{\zeta^2}{\hat{a}} \Phi_M(V) \tag{9}$$

Remarkably, without astronomical forcing, the memory duration of our dynamical system does not depend on initial conditions $S(0)$, or speaking formally, the system has complete similarity in similarity parameter $C = \frac{\zeta^4}{S(0)} \sim 10^{-13}$ (Barenblatt, 2003). It is important to note that in this case, without orbital forcing, the memory still may be long, and the time series will depend on the initial conditions; ***it is the memory duration that is independent of initial values.***

As we have already observed, the orbital forcing may affect system's memory. Moreover, we observed that the ***memory duration may become sensitive to initial conditions***. Therefore the memory duration M of the full system (3) – (5) will depend on the parameters that define its internal memory span, as well as on, generally speaking, initial conditions $S(0)$ and on the amplitude and the period of the astronomical forcing:

$$M = \varphi\left[V, \frac{\zeta^2}{\hat{a}}, \varepsilon, T, S(0)\right] \tag{10}$$

If we choose ε(m/s), T (s) as parameters with independent dimensions, then, after applying π-theorem, we can state that:

$$\frac{M}{T} = \Phi_M\left[V, \frac{\zeta^2}{\hat{a}T}, \frac{S(0)}{\varepsilon^2 T^2}\right] \tag{11}$$

As we have already learned, a weakened mass influx affects memory. It means that terrestrial mass influx $\hat{a}$ may also play a role in appearance and disappearance of M-span sensitivity to the initial conditions.

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Indeed, this is the case. In Figure 7 we show the dynamical system response to the pure obliquity forcing, when $\varepsilon_o = 1$ and $\hat{a} = 0.035$ km/kyr instead of VCV18 reference value $\hat{a} = 0.065$ km/kyr. It can be observed that the memory duration M in this case is sensitive to initial conditions. Therefore, parameters $\hat{a}$ and $S(0)$ should be the components of the same similarity parameter. This may only happen if similarity parameters $\frac{\zeta^2}{\hat{a}T}$ and $\frac{S(0)}{\varepsilon^2 T^2}$ form a conglomerate similarity parameter C, such as:

$$C = \left(\frac{\zeta^2}{\hat{a}T}\right)^{\mu} \left[\frac{S(0)}{\varepsilon^2 T^2}\right]^{\nu} \qquad (12)$$

Here the power degrees μ and ν are empirical constants. In other words, the system has incomplete similarity in similarity parameters $\frac{\zeta^2}{\hat{a}T}$ and $\frac{S(0)}{\varepsilon^2 T^2}$ (Barenblatt, 2003). This is the remarkable observation: ***A dynamical bifurcation, i.e., abrupt changes of dynamical properties of the system as a result of small changes of governing parameters, should always be accompanied by an incomplete similarity of corresponding similarity parameters***. Indeed, neither changes of ε_o from $\varepsilon_o = 1$ to $\varepsilon_o = 0.7$, nor changes of $\hat{a}$ from $\hat{a} = 0.065$ km/kyr to $\hat{a} = 0.035$ km/kyr do not change significantly neither $\frac{S(0)}{\varepsilon^2 T^2} \gg 1$, nor $\frac{\zeta^2}{\hat{a}T} \ll 1$. A "sudden" sensitivity of the memory duration to the initial conditions emerges as the result of colliding two (big and small) similarity parameters into one conglomerate similarity parameter $C \sim 1$.

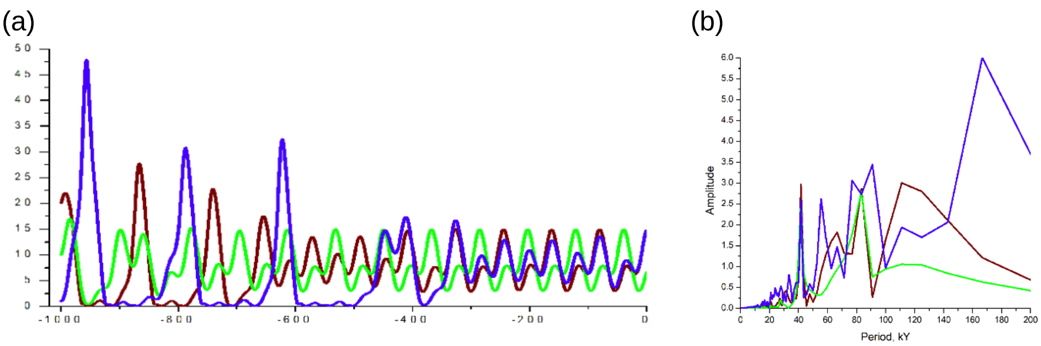

Figure 7. The dynamical system response to pure obliquity forcing, $F = 1.\sin\left(\frac{2\pi t}{41}\right)$, $\hat{a} = 0.035$. Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** is the corresponding spectral diagram; **brown** is for $S(0) = 20\ 10^6$ km^2; **green** is for $S(0) = 10\ 10^6$ km^2; **blue** is for $S(0) = 1\ 10^6$ km^2.

Actual values of parameters μ and ν reflect relative importance of the governing parameters involved in the C-number formulation, but just to clarify the physical meaning of the C-number, let us choose for a moment $\mu = 1/2$ and $\nu = 1/4$. Then the C-number becomes $C = \frac{\widehat{H}}{\hat{a}^{1/2}\varepsilon^{1/2}T}$ (here $\widehat{H} = \zeta S(0)^{1/4}$). The physical implication of the C-number is now very transparent: As we have already intuitively suspected, it is the ratio of the advection timescale $\frac{\widehat{H}}{\hat{a}^{1/2}\varepsilon^{1/2}}$ and the orbital period T. The term $\hat{a}^{1/2}\varepsilon^{1/2}$ tells us that both terrestrial mass influx and intensity of the orbital forcing are important for the initial-value sensitivity.

Finally, the scaling law for the memory duration can be written as

$$\frac{M}{T} = \Phi_M[C, V] \qquad (13)$$

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We can observe further that the dominant period of the long-memory-system response *P* is the function of time (making the dominant-period trajectory) and of the *M*-span:

$$P = \varphi(t, M) \tag{14}$$

Since both *t* and *M* are measured in units of time, then according to *π*-theorem:

$$\frac{P}{M} = \Phi_P\left(\frac{t}{M}\right) \tag{15}$$

and the scaling law of the dominant period trajectory can be expressed as:

$$P \sim T\Phi_M(C, V)\Phi_P\left[\frac{t}{T\Phi_M(C, V)}\right] \tag{16}$$

Since the memory duration M depends on initial conditions, the trajectory of the dominant period P is also sensitive to the initial values. For example, the model produced dominant-period trajectory that fits reasonably well the data in Figure 6, is a combination of the initial-value-independent eccentricity period and the obliquity period evolving according to the law (16) with the memory duration provided by the initial conditions $S(0) = 1\ 10^6$ km^2 (Figure 5). More generally, we may suggest that ***the slopes in empirical dominant-period trajectories are signatures of a long-memory initial-value-dependent system.***

Finding the exact functions Φ_M and Φ_P as well as precise values of power degrees *μ* and *ν* is outside of the scope of our current presentation, and until this is done the laws (13) and (16) are qualitative statements that, nevertheless, allow us to answer the question raised about a decade ago (Crucifix, 2013): Why could ice ages be unpredictable? In this specific regard, the results are very eloquent: (a) when system's memory is short, the period of its response to astronomical forcing is fully defined by the ratio of the orbital forcing amplitude to the amplitude of the terrestrial mass influx, $\frac{\varepsilon}{\hat{a}}$, and by the *V*-number that is the ratio of terrestrial positive-to-negative feedbacks amplitudes (equation (6), Figures 2 and 3, Verbitsky and Crucifix, 2020); (b) weaker (but not very weak) astronomical forcing (Figure 4) or weaker terrestrial influx (Figure 7) may make memory longer, but, more importantly, they make the memory duration to become dependent on initial conditions. This dramatically changes the ice-age periodicity. The conglomerate *C*-number, emerging as a result of incomplete similarity property, is in the center of the process. Its critical dependence on poorly defined accumulation-minus-ablation mass balance as well as its dependence on initial values makes ice ages to be hardly predictable.

4. The "holy grail" of ice-age studies: Obliquity or Eccentricity? The attribution challenge.

Discovering the property of incomplete similarity in a system is always insightful. Simply speaking, the conglomerate similarity parameters tell us that a physical phenomenon may be produced by physically unsimilar processes. For example, our learning that the dynamical properties of the ice-climate system (3) – (5) are largely described by the conglomerate similarity parameter, the *V*-number, which is the ratio of amplitudes of positive and negative feedbacks led us to the attribution challenge – we demonstrated that major events of the past, like the middle-Pleistocene transition, could have been produced by multiple physically unsimilar scenarios. Some of these scenarios were based on the strengthening of positive feedbacks, and some scenarios described the middle-Pleistocene transition as the result of weakened negative feedbacks (Verbitsky, 2022).

The conglomerate similarity parameter, the *C*-number, that defines the memory-duration sensitivity to initial values, is the ratio of the timescale of vertical advection and the period of orbital forcing. It implies that though we have discovered this phenomenon using the obliquity periods, there is nothing unique about it, and the response to the eccentricity-modulated precession should not be immune from the initial-

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value sensitivity either. Indeed, since the eccentricity period is longer than the obliquity period, it takes a consistently longer advection timescale to obtain the initial-value-sensitive memory. In Figure 8, we present the dynamical system response to the pure eccentricity-modulated precession forcing, when $\varepsilon_o = 0$, but $\varepsilon_p = 0.7$ combined with $\hat{a} = 0.0325$ km/kyr. It can be observed that the memory duration *M* and the dominant-period trajectories in this case are also sensitive to initial conditions.

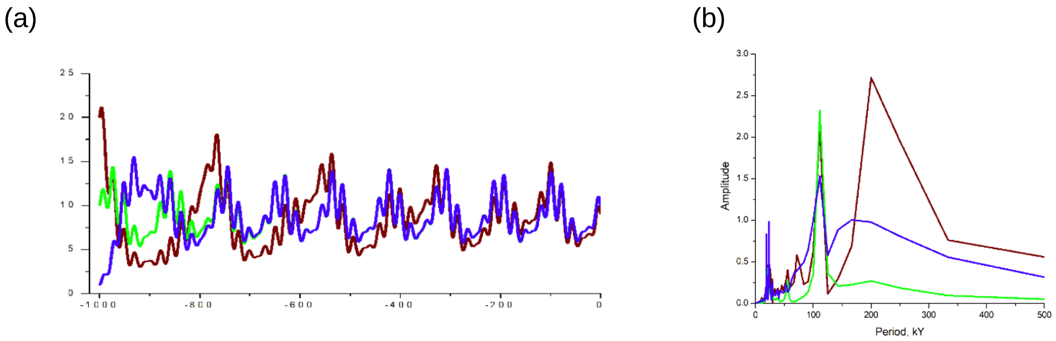

Figure 8. The dynamical system response to pure eccentricity-modulated precession, $\boldsymbol{F = 0.7\left[sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right]}$**,** $\boldsymbol{\hat{a} = 0.0325}$**.** Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** is the corresponding spectral diagram; **brown** is for *S*(0) = 20 10^6 km^2; **green** is for *S*(0) = 10 10^6 km^2; **blue** is for *S*(0) = 1 10^6 km^2.

Further, since the dominant-period trajectory of the system response to the astronomical forcing (16), for a given balance of positive and negative feedbacks ($V = const$), is also determined by the *C*-number, there should be initial values that would allow generating the "eccentricity" period *P* with the obliquity forcing *T*. Figure 7 may be a good example of such a response. It is pure obliquity forcing but the spectral diagram clearly shows the "eccentricity" pick without real eccentricity being involved. Figure 9 is even more remarkable. It is, again, the same pure obliquity forcing and only initial conditions are different from Figure 7: *S*(0) = 12.0666 10^6 km^2 instead of *S*(0) = 20 10^6 km^2. The time series show classical asymmetrical ice-age variability of dominant 100-kyr ***obliquity-forced*** period. Interestingly, the self-repeating pattern of the model time series is quite reminiscent, with Pearson correlation of 0.72, of the last 500 kyr of the LR04 data, including the interglacial of 400 kyr ago (marine isotopic stage 11) that is usually a challenge to reproduce. Most amazingly, no special efforts have been taken, i.e., none of the model parameters have been changed, in order to get this pattern, only the initial value has been adjusted, *S*(0) = 12.0666 10^6 km^2, four decimal digits being illustrative about model's sensitivity.

Thus, the eccentricity-like periods in the observational time series could have been produced either

(a) by the ***eccentricity modulated precession forcing*** in a short-memory system that is not sensitive to initial conditions due to the strong amplitude of the eccentricity modulated precession forcing (Figure 1); or
(b) by the ***eccentricity modulated precession forcing*** in a long-memory system that is sensitive to initial conditions due to weakened amplitude of the eccentricity modulated precession forcing and weakened terrestrial mass influx, under favorable initial conditions (Figure 8); or
(c) by the ***obliquity forcing*** in a long-memory system that is sensitive to initial conditions due to weakened terrestrial mass influx, under favorable initial conditions (Figures 7 and 9).

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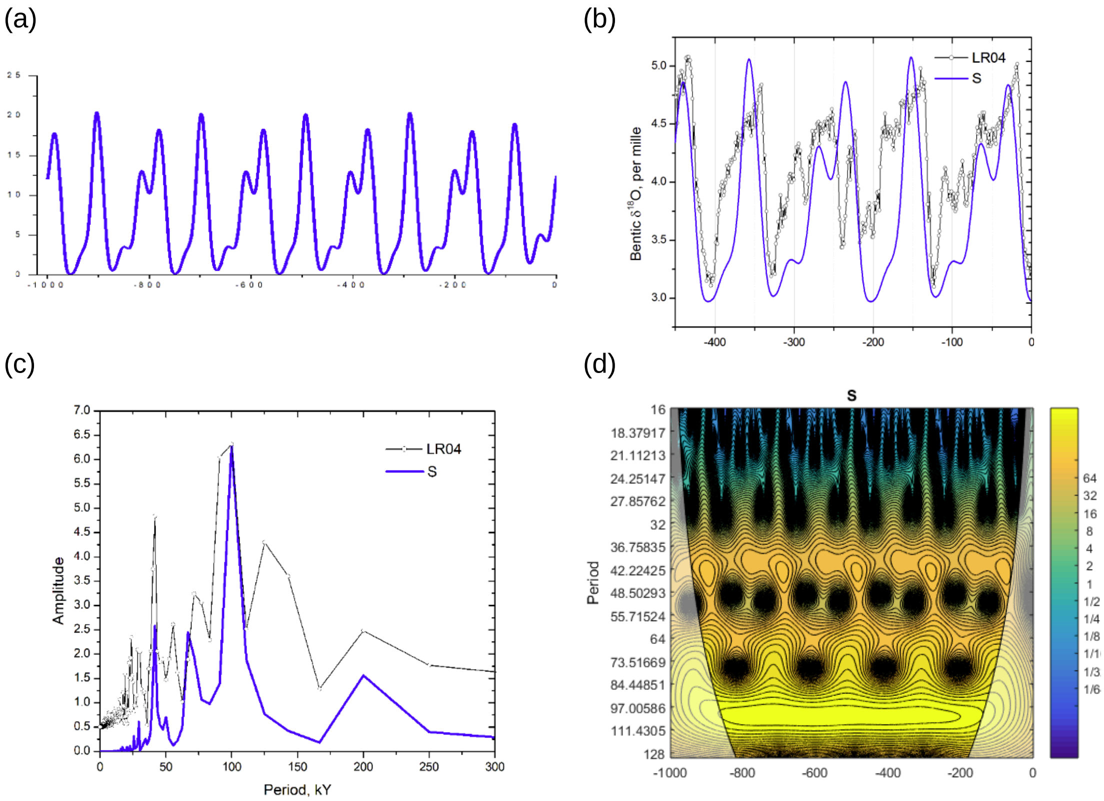

Figure 9. The dynamical system response to pure obliquity forcing, $F = 1.sin\left(\frac{2\pi t}{41}\right)$, $\hat{a} = 0.035$. Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2); **(b)** last 500 kyr of the Lisiecki and Raymo (2005, "LR04" stack) benthic foraminifera $\delta^{18}O$ data against the self-repeating pattern of the model time series; **(c-d)** are the spectral diagram and the evolution of the wavelet spectra for model glaciation area time series; all for $S(0) = 12.0666\ 10^6$ km^2.

Generally speaking, while our previous studies have highlighted multiple scenarios of the equation-(6)-driven middle-Pleistocene transition (Verbitsky and Crucifix, 2020, Verbitsky, 2022), equation (16) opens an opportunity (some people would say a "Pandora box") to envision even more scenarios. For example, we may suggest that the early-Pleistocene ice-climate system is the short-memory system that is, due to intensive terrestrial mass influx, evolves with 40-kyr ***obliquity driven*** periodicity according to Equation (6). Figure 3 may serve as an illustration (since a response of the short-memory system is defined by the ratio of the orbital forcing amplitude to the amplitude of the terrestrial mass influx $\frac{\varepsilon}{\hat{a}}$, ε_o= 0.5 of Figure 3 is equivalent to doubled $\hat{a}$). Further reduction of the terrestrial mass influx leads to the 80-kyr ***obliquity-period doubling*** (Equation (6), Figure 2). Additional reduction of the terrestrial mass influx and consequently longer vertical advection timescale leads to the long-memory initial-value-sensitive system that under favorable initial conditions produces ***obliquity driven*** 100-kyr periods (Figure 9). This sequence of events would produce the middle-Pleistocene transition and the observed late Pleistocene dominant-period slope from 80 kyr to 100 kyr.

Obviously, this scenario is different from what we have invoked earlier to illustrate the dominant-period slope in the LR04 data, i.e., a combination of the initial-value-independent eccentricity period and the obliquity period evolving according to the law (16) with the initial conditions $S(0) = 1\ 10^6$ km^2 (Figure

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5). It is also different from multiple scenarios described by Verbitsky (2022). Moreover, there may be (most likely, there must be) an interplay between the *V*-number scenarios of Verbitsky (2022) and the *C*-number scenarios of this presentation. So far, in this study, we did not change model parameters involved in the *V*-number conglomerate. We should be mindful however that vertical advection also affects the thickness of the ice-sheet basal boundary layer and parameter β that defines the intensity of basal sliding, i.e., intensity of the negative feedback, may change in concert with vertical advection.

This is the essence of the fundamental attribution challenge. We warned about it in the Introduction when we entertained a simple dynamical system (1), but it may be repeated here almost without changes: Any claim that the nature is just like a model because that model has successfully reproduced a sample time series, should be taken cautiously unless our knowledge about mass balance and internal dynamics is unambiguous. As we already know, this is not the case.

5. Discussion and Conclusions.

The interpretation of the Milankovitch theory that we present in this study is very simple and it is based on very explicit physics:

(1) The dynamics of large ice sheets is defined by the advection of mass and temperature;
(2) The timescale of ice advection depends mostly on the surface total mass influx;
(3) Because of the ice-climate system's nonlinearity, its response to the orbital forcing in terms of engagement of negative and positive feedbacks is not symmetrical. This may change the effective mass influx and the resultant advection timescale;
(4) Specifically, the orbital forcing of relatively weak amplitude may make the internal ice-sheet advection timescale significantly longer. It means that the ice-climate system may remember its initial conditions through the entire Late Pleistocene, and for the same orbital forcing and for the same balance between terrestrial positive and negative feedbacks, the historical glacial rhythmicity could have been dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both;
(5) In fact, empirical records demonstrate that the dominant period of the Late Pleistocene ice ages ***evolved*** from ~80-kyr to ~100-kyr rhythmicity. The quantitative similarity of this dominant-period trajectory and the one, made by the long-memory model, suggests that the records of the Late Pleistocene glacial rhythmicity could have been produced by an initial-value-dependent climate system, or, in other words, the slopes in empirical dominant-period trajectories are signatures of a long memory.
(6) The scaling law of the dominant-period trajectory provides a theoretical insight into the discovered phenomenon. It reveals that this trajectory is dependent on ***memory duration that is sensitive to initial conditions***. The sensitivity of the memory duration to initial values emerges as the result of system's incomplete similarity in two similarity parameters colliding into one conglomerate similarity parameter that is the ratio of the advection timescale and the orbital period. The critical dependence of this similarity parameter on poorly defined accumulation-minus-ablation mass balance as well as its dependence on initial values makes ice ages to be hardly predictable and disambiguation of paleo-records to be extremely challenging.
(7) The ***quasi***-eccentricity periods produced by the long-memory system in response to ***pure obliquity*** forcing make a remarkable example of this challenge because in the time series they may be naively attributed to the eccentricity modulated precession forcing.

Barry Saltzman has been advocating for dynamical paleoclimatology because the timing and amplitude of glacial variability are defined by poorly resolved ice-sheet mass balance, and the dynamical models, though they are not able to provide an unambiguous solution, can nevertheless expose the scope of the challenge. With this study, we want to expand this scope a bit more and to demonstrate, that since

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the timescale of vertical advection in ice sheets is defined by the same mass balance, ice sheets' memory and periodicity can be sensitive to initial values. The implications of this sensitivity, as we discussed above, can be dramatic for our understanding of the past as well as for the future vision.

Though all our conclusions have been derived from the physics-based model (3) – (5), other models would indeed be desirable for a comprehensive test. Obviously, this may be easier said than done. A hypothetical model that would be appropriate for such testing should be able to explicitly account for all the above physics. Therefore, all phenomenological models, obtained either from the fitting to the empirical data or by emulating the behavior of more comprehensive models, will be of little help because they may not have physical similarity with the Nature or even with the comprehensive models they mimic (Verbitsky and Crucifix, 2023). On the other hand, three-dimensional and intermediate-complexity models do have, indeed, all the physics needed, but, as we have already discussed in the Introduction, they may not be able to resolve the mass influx that may be responsible for a timescale of about few hundreds of thousands years (if a characteristic thickness of ice is a few thousands meters, then we are talking about snow accumulation rate of the order of few centimeters per year).

The ability to reproduce empirical time series is certainly tempting and self-gratifying. Even though it may be a natural first step in the studies, we nevertheless believe that the scientific community has been at this stage long enough and it is time to recognize further challenges. We therefore hope, that, other than VCV18, simple but physics-based models can be designed to support (or reject) the Milankovitch theory formulated here as an initial value problem.

Competing interests: The authors have declared that there are no competing interests.

Author contributions: MYV identified the phenomena, developed the formalism, and wrote the first draft of the manuscript. DV digitized the model and produced the graphics. The authors jointly discussed the findings and contributed equally to the editing of the manuscript.

References

Abe-Ouchi, A., Saito, F., Kawamura, K., Raymo, M. E., Okuno, J. I., Takahashi, K., and Blatter, H.: Insolation-driven 100,000-year glacial cycles and hysteresis of ice-sheet volume, Nature, 500, 190–194, 2013.

Bahr, D. B., Pfeffer, W. T., and Kaser, G.: A review of volume-area scaling of glaciers, Rev. Geophys., 53,95–140, doi:10.1002/2014RG000470, 2015.

Barenblatt, G. I.: Scaling, Cambridge University Press, Cambridge, ISBN 0 521 53394 5, 2003.

Buckingham, E.: On physically similar systems; illustrations of the use of dimensional equations, Phys. Rev., 4, 345–376, 1914.

Crucifix, M.: Why could ice ages be unpredictable?, Clim. Past, 9, 2253–2267, https://doi.org/10.5194/cp-9-2253-2013, 2013.

Ganopolski, A. and Calov, R.: The role of orbital forcing, carbon dioxide and regolith in 100 kyr glacial cycles, Clim. Past., 7, 1415–1425, https://doi.org/10.5194/cp-7-1415-2011, 2011.

https://doi.org/10.5194/egusphere-2024-1255
Preprint. Discussion started: 30 April 2024
© Author(s) 2024. CC BY 4.0 License.

Gowan, E.J.: Bayesian analysis and paleo ice sheet modelling: a commentary on the proposal by Tarasov and Goldstein. https://egusphere.copernicus.org/preprints/2023/egusphere-2022-1410/egusphere-2022-1410-RC2-supplement.pdf, 2023.

Lisiecki, L. E. and Raymo, M. E.: A Pliocene-Pleistocene stack of 57 globally distributed benthic δ18O records, Paleoceanography, 20, PA1003, https://doi.org/10.1029/2004PA001071, 2005.

Saltzman, B.: Finite amplitude free convection as an initial value problem, Journal of atmospheric sciences, 19, 4, 329-341, 1962.

Saltzman, B.: Dynamical paleoclimatology: generalized theory of global climate change, in: Vol. 80, Academic Press, San Diego, CA, ISBN 0126173311, 2002.

Verbitsky, M.Y.: Equilibrium ice sheet scaling in climate modeling, Climate Dynamics, 7, 105–110, https://doi.org/10.1007/BF00209611, 1992.

Verbitsky, M. Y.: Inarticulate past: similarity properties of the ice–climate system and their implications for paleo-record attribution, Earth Syst. Dynam., 13, 879–884, https://doi.org/10.5194/esd-13-879-2022, 2022.

Verbitsky, M. Y. and Crucifix, M.: π-theorem generalization of the ice-age theory, Earth Syst. Dynam., 11, 281–289, https://doi.org/10.5194/esd-11-281-2020, 2020.

Verbitsky, M. Y. and Crucifix, M.: Do phenomenological dynamical paleoclimate models have physical similarity with Nature? Seemingly, not all of them do, Clim. Past, 19, 1793–1803, https://doi.org/10.5194/cp-19-1793-2023, 2023.

Verbitsky, M. Y., Crucifix, M., and Volobuev, D. M.: A theory of Pleistocene glacial rhythmicity, Earth Syst. Dynam., 9, 1025–1043, https://doi.org/10.5194/esd-9-1025-2018, 2018.