Dear Professor Omta,

Thank you for reviewing our manuscript and providing insightful feedback. Below, we reply to your comments (marked as **bold**) and propose several changes to the manuscript motivated by your suggestions.

Overall assessment

I think this is a fundamentally solid study, but the writing still needs significant work. In particular, the title could be significantly sharper, some of the concepts are not well defined and some of the statements and arguments are not clearly explained. Overall, I recommend publication after a major revision.

Response: We are very grateful that you consider our work to be a fundamentally solid study that deserves publication.

General comments

Through simulations with the Verbitsky *et al.* (2018) model, the authors make the important point that the dominant period of the glacial cycles may depend on the initial values of the system's state variables. We found a qualitatively similar result with the periodically forced calcifier-alkalinity model (compare Figs. 3a and 3b in Omta *et al*. (2016)). A corollary of this is that the Mid-Pleistocene Transition may not have been due to a change in the Earth system but rather due to simple coincidence (as illustrated in Figs. 5 and 6 of the manuscript). Having said this, I think the title (and thus the overall framing) could be significantly stronger. An initial value problem is defined as a differential equation together with an initial condition (Wikipedia, 2025), which can be as simple as:

dx/dt = x, x(0) = 1

In other words, the title ”Milankovitch theory as an initial value problem” doesn't really convey much of a message at all. I think a title such as ”Sensitivity of glacial dynamics to initial values” would summarize the central point of the article much better. I would also suggest removing the essentially meaningless and potentially confusing words ”In this case the Milankovitch theory becomes an initial value problem” (l. 14/15) from the abstract.

Response: We definitely agree with you that a title of a paper is extremely important. It is particularly important in our case because we dedicate this paper to the memory of a pioneer of climate science Barry Saltzman. We admitted in the introduction that the title of our article alludes to Saltzman's (1962) landmark work "Finite amplitude free convection as an initial value problem" – to the paper that has become the cornerstone of the deterministic chaos theory. In both cases of Barry Saltzman's paper and of our paper "an initial value problem" conveys a message that for the physical phenomenon, being considered, initial values are of a pivotal importance. We explain what it means very unambiguously in line 15: "Depending on initial conditions, for the same orbital forcing and for the same balance between terrestrial positive and negative feedbacks, the historical glacial rhythmicity could have been dominated either by the eccentricity period of ~100 kyr, or by the doubled obliquity period of ~80 kyr, or by a combination of both”.

Action: Having said that, we, again, agree with you that the title can be more specific, and therefore, for the revised manuscript, we will use the following title: *“Milankovitch Theory as an Initial Value Problem: The Scaling Law and its Implications”.*

We are also grateful for bringing to our attention Omta *et al*. (2016) work. We will quote it appropriately.

Specific comments

• Point (b), p. 2, l. 72–75: The memory duration concept is not well defined. Is it the time it takes for the system to get within 1% or 10% (or another fraction) of its steady-state value?

Response: We agree. Indeed, for the introductory example (1) the memory duration M can be defined as the time needed by the system to get within a given distance $m \ll 1$ from the steady state:

$$\left(\frac{x_0}{a\tau} - 1\right) e^{-\frac{M}{\tau}} = m,$$

$$\frac{M}{\tau} = ln\left[\frac{1}{m}\left(\frac{x_0}{a\tau} - 1\right)\right]$$

Action: We will make it explicit in the revised text.

• Point (c), p. 2, l. 76: As their name indicates, short-memory systems have a short memory of their initial condition. Even so, it is not true that these systems are entirely initial-value independent. By definition, every dynamical system is dependent on its initial value, at least at *t* = 0.

Response: We respectfully disagree, though our disagreement is a bit of semantic nature. First, let us formally establish that the system (1) with $\tau \to 0$ is the short-memory system. This is the case because

$$\lim_{\tau\to 0} M = \lim_{\tau\to 0}\left\{\tau ln\left[\frac{1}{m}\left(\frac{x_0}{a\tau} - 1\right)\right]\right\} = 0.$$

Obviously, the memory duration $M = 0$ of this short-memory system is initial-values independent. Further, the time series of the short-memory system (1) $(\tau \to 0)$ are initial-value independent either, because the timescale $\tau \to 0$ and $M \to 0$ imply an “instant” process, and the governing ordinary differential equation (1) becomes an algebraic equation, $a - \frac{x}{\tau} = 0$, and $x = a\tau$.

Action: We will include this clarification in the revised text.

• Point (e), p. 2, 82–84: "A significant portion ... is adjusted accordingly." This is quite a convoluted statement, which I suggest simplifying to: "A long-memory system (low *a*, high *τ*) may have the same steady state *aτ* as a short-memory system (high *a*, low *τ*)."

Response: The goal of this paragraph was not to establish this simple property of the system (1), i.e., "A long-memory system (low a, high τ) may have the same steady state $a\tau$ as a short-memory system (high a, low τ)", but to recognize and preview a fundamental attribution challenge. Therefore, for generality, we tried to use the appropriate language.

Action: Having said this, we will do our best to combine some sophistication with the simple language you recommend.

• p. 6, l. 209–211: I don't think Fig. 4 provides a clear illustration of the point that the dominant period of the oscillations can be sensitive to the initial conditions. Looking at Fig. 4a, it appears to me that all three simulations converge to the same oscillation.

Response: Here, we hasten to point out that we are focusing on the time domain within system's memory duration where dominant periods are strikingly different.

Action: To make it more visual, we suggest a better format for Figs. 4 and 5 (below).

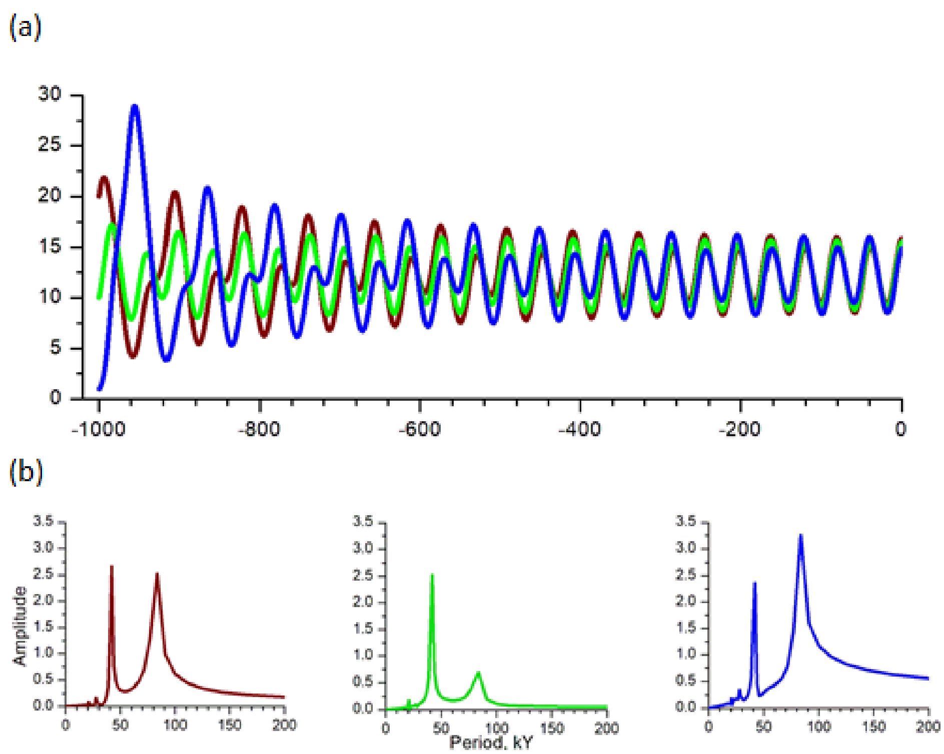

Figure 4. The dynamical system response to pure obliquity forcing, $F = 0.7\, sin\left(\frac{2\pi t}{41}\right)$.

Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** are the corresponding spectral diagrams; **brown** is for *S*(0) = 20 10^6 km^2; **green** is for *S*(0) = 10 10^6 km^2; **blue** is for *S*(0) = 1 10^6 km^2.

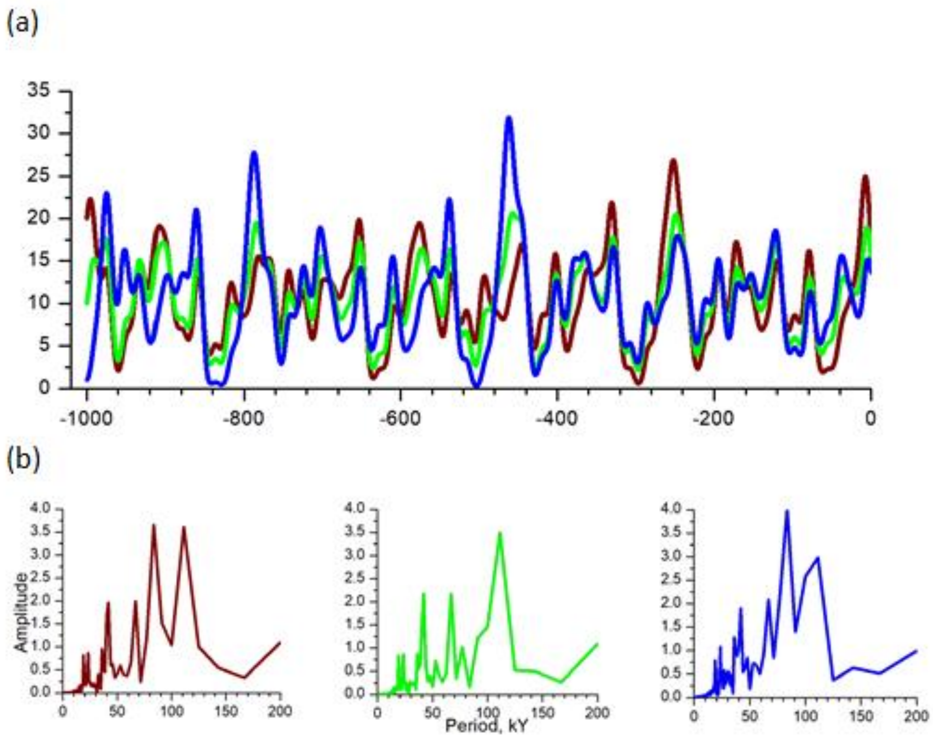

Figure 5. The dynamical system response to orbital forcing, $F = 0.7\left[sin\left(\frac{2\pi t}{19}\right) + sin\left(\frac{2\pi t}{23}\right)\right] + 0.7sin\left(\frac{2\pi t}{41}\right)$**.**

Here **(a)** is the time series (kyr before present) of the area of glaciation in (10^6 km^2) and **(b)** are the corresponding spectral diagrams; **brown** is for $S(0) = 20\ 10^6$ km^2; **green** is for $S(0) = 10\ 10^6$ km^2; **blue** is for $S(0) = 1\ 10^6$ km^2.

• p. 6, l. 224/225: To me, it is not immediately clear why a glaciation area of 12.3 10^6 km^2 would be equivalent to a ten-fold reduction of the terrestrial mass influx. Could you elaborate, perhaps by deriving this from eqs. (3–5)?

Response: In the absence of the external forcing, the equilibrium glaciation area $< S >$ has been derived as equation (30) of Verbitsky *et al* (2018). Using notations of equations (3-5) of the current manuscript, it can be presented as following:

$$< S >= S_0 + \frac{\hat{a}}{\beta c(1-V)}$$

For reference values of the model parameters, $< S >$ = 15 10^6 km^2, because S_0 = 12 10^6 km^2, and $\frac{\hat{a}}{\beta c(1-V)}$ = 3 10^6 km^2. The 10-fold reduction of the mass influx $\hat{a}$ shifts equilibrium glaciation area to $< S >$=12.3 10^6 km^2, i.e. S_0=12 10^6 km^2, and $\frac{\hat{a}}{\beta c(1-V)}$ = 0.3 10^6 km^2. Indeed, the actual mass influx, the right-hand side of the equation (3), is formed as an output of sophisticated interplay of the mass influx $\hat{a}$, external forcing, and positive and negative feedbacks. Still, changes of the equilibrium glaciation area may serve as a qualitative but still insightful indicator of the surface mass-balance changes.

Action: We will elaborate these details in the revised paper.

• p. 9, l. 319–322: "A dynamical bifurcation, i.e., abrupt changes of dynamical properties of the system as a result of small changes of governing parameters, should always be accompanied by an incomplete similarity of corresponding similarity parameters." This is an interesting point. Would it be correct to say that the bifurcation occurs at the end of the parameter range where the incomplete similarity applies?

Response: There are two types of bifurcations we describe in the manuscript. When we say (and you quote) "A dynamical bifurcation, i.e., abrupt changes of dynamical properties of the system as a result of small changes of governing parameters, should always be accompanied by an incomplete similarity of corresponding similarity parameters", we refer to the emergence of the initial-value dependence or to the emergence of the similarity parameter that defines system' sensitivity to initial values. This, indeed, occurs at the boundary of the parameter space where incomplete similarity applies, and your observation is absolutely correct. Other types of bifurcations are presented in Figs. 4 - 9, where changes of initial conditions lead to drastic changes of glacial rhythmicity. In all Figs. 4 – 9 cases, the dominant-period trajectories are controlled by the conglomerate similarity parameter (i.e., within the parameter space where the incomplete similarity applies).

Action: To avoid confusion, we will reformulate the quoted statement as following: "An abrupt change of dynamical properties of the system, i.e., an emergence of initial-values sensitivity as a result of small changes of similarity parameters, should always be accompanied by an incomplete similarity of these similarity parameters".

• p. 10, l. 374: What do you mean by "hardly predictable"? Is that unpredictable or difficult to predict?

Response: It is inheritably (fundamentally) difficult to predict.

Action: It will be clarified.

• p. 14, l. 500: "the timescale of vertical advection in ice sheets is defined by the same mass balance". I would say that it is the other way around: the mass balance is defined by (among other things) the advection timescale. Could you explain what you mean here?

Response: The evolution of the ice sheet is described by its *total mass balance*:

$$\frac{dV}{dt} = AS$$

Here V is ice volume, A is the *surface mass balance*, i.e., accumulation-minus-ablation, and S is the ice-sheet area. In other words, the changes of ice volume are defined by the *surface mass* *balance* accumulated over its area. Equation (3) of the manuscript is the more elaborated form of it.

H/A is the vertical advection timescale, where *H* is ice thickness. The ratio of this timescale to the orbital period makes the similarity parameter *C*, defining system's sensitivity to initial values. We are referring to equation (3) when we say "…the timing and amplitude of glacial variability are defined by poorly resolved ice-sheet mass balance" (meaning *A*). We are referring to *H/A* when we say "…since the timescale of vertical advection in ice sheets is defined by the same mass balance (meaning *A*), ice sheets' memory and periodicity can be sensitive to initial values".

Action: In the revised text, we will clarify that we are talking about *surface mass balance A* (accumulation-minus-ablation).

Minor corrections

Action: All minor corrections will be (gratefully) accepted.

References

Verbitsky, M. Y., Crucifix, M., and Volobuev, D. M.: A theory of Pleistocene glacial rhythmicity, Earth Syst. Dynam., 9, 1025–1043, https://doi.org/10.5194/esd-9-1025-2018, 2018.