Well-posedness of the isostatic boundary value problem

In the following, we use the notation from the preprint [4]. Without loss of generality, the coefficients in the bilinear forms

$$
\begin{aligned}
a(w,v) &= \int_{\mathcal{A}} D\big(\nu \Delta w \Delta v + (1-\nu)\nabla^2 w : \nabla^2 v\big)\,\mathrm{dA}, \\
b(w,v) &= \int_{\mathcal{A}} (\varrho_{\mathrm{m}} - \varrho_{\mathrm{r}}) g w v \,\mathrm{dA},
\end{aligned}
\tag{1}
$$

can be assumed to be real numbers with $D > 0$, $0 \leq \nu \leq 0.5$, $g > 0$, and $\varrho_{\mathrm{m}} > \varrho_{\mathrm{r}}$. If the coefficients are variable in space, we only require that $D(1-\nu)$ and $(\varrho_{\mathrm{m}} - \varrho_{\mathrm{r}})g$ are bounded from below by a positive number.

Claim: Let $\mathcal{A} \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then there exists a constant $\alpha > 0$ such that

$$
a(w,w) + b(w,w) \geq \alpha \|w\|^2_{\mathrm{H}^2(\mathcal{A})} \tag{2}
$$

for all $w \in \mathrm{H}^2(\mathcal{A})$.

Proof: Let $\mathrm{C}^\infty(\mathcal{A})$ denote the space of smooth real-valued functions on $\mathcal{A}$. Since $\mathrm{C}^\infty(\mathcal{A})$ is dense in $\mathrm{H}^2(\mathcal{A})$ with respect to the H^2 norm, it suffices to establish the inequality for $w \in \mathrm{C}^\infty(\mathcal{A})$. By definition of the bilinear forms, we have that

$$
\begin{aligned}
a(w,w) + b(w,w) &= \int_{\mathcal{A}} D\big(\nu|\Delta w|^2 + (1-\nu)|\nabla^2 w|^2\big)\,\mathrm{dA} + \int_{\mathcal{A}} (\varrho_{\mathrm{m}} - \varrho_{\mathrm{r}}) g |w|^2 \,\mathrm{dA} \\
&\geq D(1-\nu)\|\nabla^2 w\|^2_{\mathrm{L}^2(\mathcal{A})} + (\varrho_{\mathrm{m}} - \varrho_{\mathrm{r}}) g \|w\|^2_{\mathrm{L}^2(\mathcal{A})} \\
&\geq C\left(\|\nabla^2 w\|^2_{\mathrm{L}^2(\mathcal{A})} + \|w\|^2_{\mathrm{L}^2(\mathcal{A})}\right)
\end{aligned}
\tag{3}
$$

with $C = \min\{D(1-\nu), (\varrho_{\mathrm{m}} - \varrho_{\mathrm{r}})g\} > 0$. According to the Ehrling–Gagliardo–Nirenberg interpolation inequality in [1, Theorem 5.2] or the equivalence of norms in [3, Theorem 1.8], there exists a constant $K > 0$ such that

$$
\|\nabla w\|^2_{\mathrm{L}^2(\mathcal{A})} \leq K\left(\|\nabla^2 w\|^2_{\mathrm{L}^2(\mathcal{A})} + \|w\|^2_{\mathrm{L}^2(\mathcal{A})}\right) \tag{4}
$$

in the case of a bounded Lipschitz domain $\mathcal{A}$. Applying (4) to (3) yields

$$
\begin{aligned}
a(w,w) + b(w,w) &\geq \frac{C}{2}\left(\|\nabla^2 w\|^2_{\mathrm{L}^2(\mathcal{A})} + \|w\|^2_{\mathrm{L}^2(\mathcal{A})}\right) + \frac{C}{2K}\|\nabla w\|^2_{\mathrm{L}^2(\mathcal{A})} \\
&\geq \alpha \|w\|^2_{\mathrm{H}^2(\mathcal{A})}
\end{aligned}
\tag{5}
$$

with the coercivity constant $\alpha = \min\{C/2, C/(2K)\} > 0$. □

The above shows that $a + b \colon \mathrm{H}^2(\mathcal{A}) \times \mathrm{H}^2(\mathcal{A}) \to \mathbb{R}$ is a coercive bilinear form. Furthermore, symmetry and continuity of $a + b$ are easily verified. The well-posedness of the isostatic boundary value problem with Neumann boundary conditions then follows from a standard argument using the Lax–Milgram theorem [2, Chapter II, Section 2–3].

References

[1] Robert A. Adams and John J. F. Fournier. *Sobolev Spaces*. Academic Press, 2003.

[2] Dietrich Braess. *Finite Elements: Theory, Fast Solvers, and Applications in Solid Mechanics*. Cambridge University Press, 2007.

[3] Jindřich Nečas. *Direct Methods in the Theory of Elliptic Equations*. Springer, 2012.

[4] Rozan Rosandi, Yudi Rosandi, and Bernd Simeon. Isogeometric analysis of the lithosphere under topographic loading: Igalith v1.0.0. *EGUsphere [preprint]*, 2024.